The statement $\text{AN}$ is equivalent to the statement $n(\beta \omega \setminus \omega) > c$ (Logic and the Foundations of Mathematics).

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The statement $AN$ is equivalent to the statement $n(\beta\omega \setminus \omega) > c$

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1. Introduction and results. A filter $\mathcal{F}$ on $\omega$ is said to be ample, if there is an infinite subset $a$ of $\omega$ such that, whenever $x \in \mathcal{F}$, $a \setminus x$ is finite. A filter $\mathcal{F}$ on $\omega$ is said to be weakly ample, if for each free ultrafilter (uf) $\mathcal{U}$ on $\omega$, there is a function $f$ on $\omega$ such that $f(\mathcal{U}) \supset \mathcal{F}$. Let us denote by $AN$ the statement: "every free weakly ample filter on $\omega$ is ample."

In [2], we showed

PROPOSITION 1.

(i) $AN$ implies the existence of $c^+$ Ramsey ufs on $\omega$, where $c$ denotes the cardinality of $2^\omega$.

(ii) The existence of $c^+$ Ramsey ufs on $\omega$ does not imply $AN$.

(iii) $P$ implies $AN$, where $P$ denotes the statement: "every free filter on $\omega$ generated by a set cardinality less than $c$ is ample."

It seems to be interesting to consider how strong the statement $AN$ is. As to this, we first show

THEOREM 1. $AN$ is equivalent to the statement that $\beta\omega \setminus \omega$ can not be covered by a family of $c$ nowhere dense sets, where $\beta\omega$ denotes the Čech-Stone compactification of $\omega$. 
Let us denote by \( n(\beta w \setminus w) \) the Baire number of \( \beta w \setminus w \) (i.e. the minimal cardinal of a family of nowhere dense sets covering \( \beta w \setminus w \)). As to the Baire number of \( \beta w \setminus w \), the systematic estimation was given and several consistencies were shown in [1]. In [1], it is shown

**PROPOSITION 2 (5.2.V in [1])** The statement \( n(\beta w \setminus w) > c \) does not imply that \( \forall \kappa < c \ ( |2^\kappa| < c ) \).

Since \( \mathcal{P} \) implies that \( \forall \kappa < c \ ( |2^\kappa| < c ) \) ([3]), by Theorem 1 and Proposition 2, \( \text{AN} \) does not imply \( \mathcal{P} \).

Define the pseudo-ordering \( <^* \) on \( \omega^\omega \) by \( f <^* g \) iff \( \lim_{n \to \infty} ( g(n) - f(n) ) = \infty \). A family \( F \) of subsets of \( \omega^\omega \) is said to be unbounded, if there does not exist \( g \in \omega^\omega \) such that, whenever \( f \in F \), \( f <^* g \). Then, it holds

**PROPOSITION 3 (4.6 and 4.7 in [1])**

(i) The statement \( n(\beta w \setminus w) > c \) implies the statement \( \mathcal{D} \): "every unbounded family of subsets of \( \omega^\omega \) has the cardinality \( c \)."

(ii) \( \mathcal{D} \) does not imply that \( n(\beta w \setminus w) > c \).

By Propositions 1~3 and Theorem 1, the following diagram holds.

![Diagram](https://via.placeholder.com/150)

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The only interesting in the above diagram which is not mentioned is whether \( AN + \forall \kappa < c \ ( |2^\kappa| \leq c ) \) implies \( \mathbb{P} \) or not. As to this, we show

**THEOREM 2.** \( AN + \forall \kappa < c \ ( |2^\kappa| \leq c ) \) does not imply \( \mathbb{P} \).

We shall prove Theorem 1 in the following section and Theorem 2 in section 3.

2. Proof of Theorem 1. We first show that \( n(\beta \omega \setminus \omega) > c \) implies \( AN \). So, assume that \( n(\beta \omega \setminus \omega) > c \). Let \( \mathcal{F} \) be any weakly ample filter on \( \omega \). For each \( f \in \omega^\omega \), set

\[
D_f = \{ \mathcal{U} \in \beta \omega \setminus \omega \ ; \ f(\mathcal{U}) \supset \mathcal{F} \}.
\]

Since \( \mathcal{F} \) is weakly ample, it holds that

\[
\bigvee \{ D_f \ ; \ f \in \omega^\omega \} = \beta \omega \setminus \omega.
\]

So, there is some \( f \in \omega^\omega \) such that \( D_f \) is not nowhere dense in \( \beta \omega \setminus \omega \). Take an infinite subset \( a_0 \) of \( \omega \) such that

\[
(\ast) \quad \forall x \subset a_0 \ ( |x| = \omega \implies \exists \mathcal{U} \in D_f \ ( x \in \mathcal{U} ) )
\]

Set \( a_1 = f^a a_0 \). Then, by \( (\ast) \), it holds that \( a_1 \) is infinite and \( \forall x \in \mathcal{F} \ ( a_1 \setminus x \ \text{is finite} ) \). Hence, \( \mathcal{F} \) is ample.

Now, we shall prove that the inverse implication holds. The following fact which we shall use in the proof is well-known and easy.

**FACT 1.** There is a family \( W \) of subsets of \( \omega \) such that

1. \( |W| = c \),
2. \( \forall x \in W \ ( |x| = \omega ) \),
3. \( \forall x, y \in W \ ( x \neq y \implies x \cap y \ \text{is finite} ) \).
Assume that AN holds. Let $Q = \{ D_\alpha ; \alpha < c \}$ be any family of nowhere dense subsets of $\beta \omega \setminus \omega$. Let $\langle s_\alpha | \alpha < c \rangle$ be a monotone enumeration of a family $W$ of subsets of $\omega$ which satisfies (1) $\vee$ (3) in Fact 1. For each $\alpha < c$, take $f_\alpha \in w^\omega$ such that $f_\alpha : \omega \rightarrow a_\alpha$ one-to-one and onto. Define the filter $F$ on $\omega$ by

$$x \in F \text{ iff } \forall \alpha < c \forall U \in D_\alpha ( x \in f_\alpha(U) ).$$

Then, it is easy to see that $F$ is free and not ample. So, by AN, there is $U \in \beta \omega \setminus \omega$ such that

$$\forall g \in w^\omega ( g(U) \not\in F ).$$

Then, it holds that, for any $\alpha < c$, $U \not\in D_\alpha$, since $f_\alpha(U) \not\in F$. Hence, $U \not\in U \cup Q$.

3. Proof of Theorem 2. Let $M$ be a countable transitive model of ZFC + GCH. We shall show that a generic extension of $M$ on the poset $P \times Q$ which will be defined below satisfies that AN + $\forall \kappa < c ( |2^\kappa| < c ) + \neg \diamondsuit$. The poset $P \times Q$ is alike the poset used in 5.V of [1]. Let $P$ be the Solovay-Tennenbaum's poset used for the consistency of MA + $2^\omega = \omega_2$. Define the poset $Q \in M$ by, in $M$,

$$Q = \{ q ; \exists \alpha < \omega_1 ( q : \alpha \rightarrow 2 ) \}.$$

Let $G \times H$ be $m$-generic on $P \times Q$ and $\tilde{M} = M[G \times H]$. Then, similar arguments in [1] show that

$$\tilde{M} \models " |2^\omega| = |2^{\omega_1}| = \omega_2 + \text{AN}".$$

We shall show that $\tilde{M} \models \neg \diamondsuit$. Since CH holds in $M$, it holds that, in $M$, there is a dense embedding from $Q$ to $P(\omega)/\text{finite}$.
So, we may assume that $H$ is $\mathcal{M}$-generic on $(\mathcal{P}(\omega)/\text{finite})^\mathfrak{m}$.

Define $\mathcal{F} \in \tilde{\mathcal{M}}$ by

$$\tilde{\mathcal{M}} \models \{ x \in \omega \mid \exists \alpha / \text{finite} \in H ( \alpha \setminus x \text{ is finite} ) \}.$$  

Since $\tilde{\mathcal{M}} \models |H| = \omega_1$, it holds that

$$\tilde{\mathcal{M}} \models \{ \mathcal{F} \text{ is an } \omega_1\text{-generated free filter on } \omega \}.$$  

Moreover, since $H$ is not in $\mathcal{M}[G]$, we have that

$$\tilde{\mathcal{M}} \models \{ \mathcal{F} \text{ is not ample} \}.$$  

Hence, $\tilde{\mathcal{M}} \models \neg \mathcal{P}$.  

References

