The statement AN is equivalent to the statement \( n(\beta\omega \setminus \omega) > c \)

Shizuo Kamo (Univ. of Osaka Prefecture)

1. Introduction and results. A filter \( \mathcal{F} \) on \( \omega \) is said to be ample, if there is an infinite subset \( a \) of \( \omega \) such that, whenever \( x \in \mathcal{F} \), \( a \setminus x \) is finite. A filter \( \mathcal{F} \) on \( \omega \) is said to be weakly ample, if for each free ultrafilter \( (uf) \mathcal{U} \) on \( \omega \), there is a function \( f \) on \( \omega \) such that \( f(\mathcal{U}) \supseteq \mathcal{F} \). Let us denote by AN the statement: "every free weakly ample filter on \( \omega \) is ample."

In [2], we showed

PROPOSITION 1.

(i) AN implies the existence of \( c^+ \) Ramsey ufs on \( \omega \), where \( c \) denotes the cardinality of \( 2^\omega \).

(ii) The existence of \( c^+ \) Ramsey ufs on \( \omega \) does not imply AN.

(iii) \( \mathbb{P} \) implies AN, where \( \mathbb{P} \) denotes the statement: "every free filter on \( \omega \) generated by a set cardinality less than \( c \) is ample."

It seems to be interesting to consider how strong the statement AN is. As to this, we first show

THEOREM 1. AN is equivalent to the statement that \( \beta\omega \setminus \omega \) can not be covered by a family of \( c \) nowhere dense sets, where \( \beta\omega \) denotes the Čech-Stone compactification of \( \omega \).
Let us denote by \( n(\beta \omega \setminus \omega) \) the Baire number of \( \beta \omega \setminus \omega \) (i.e. the minimal cardinal of a family of nowhere dense sets covering \( \beta \omega \setminus \omega \)). As to the Baire number of \( \beta \omega \setminus \omega \), the systematical estimation was given and several consistencies were shown in [1]. In [1], it is shown

**PROPOSITION 2** (5.2.V in [1]) The statement \( n(\beta \omega \setminus \omega) > c \) does not imply that \( \forall \kappa < c \ ( |2^\kappa| \leq c \).

Since \( \mathfrak{P} \) implies that \( \forall \kappa < c \ ( |2^\kappa| \leq c \) ([3]), by Theorem 1 and Proposition 2, AN does not imply \( \mathfrak{P} \).

Define the pseudo-ordering \( <^* \) on \( \omega^\omega \) by \( f <^* g \) iff \( \lim_{n \to \infty} ( g(n) - f(n) ) = \infty \). A family \( F \) of subsets of \( \omega^\omega \) is said to be unbounded, if there does not exist \( g \in \omega^\omega \) such that, whenever \( f \in F \), \( f <^* g \). Then, it holds

**PROPOSITION 3** (4.6 and 4.7 in [1])

(i) The statement \( n(\beta \omega \setminus \omega) > c \) implies the statement \( \mathfrak{D} \) : "every unbounded family of subsets of \( \omega^\omega \) has the cardinality \( c \)."

(ii) \( \mathfrak{D} \) does not imply that \( n(\beta \omega \setminus \omega) > c \).

By Propositions 1~3 and Theorem 1, the following diagram holds.

\[
\begin{array}{cccc}
\mathfrak{P} & \uparrow & \mathfrak{AN} & \rightarrow & \forall \kappa < c \ ( |2^\kappa| \leq c ) \\
\downarrow & & \leftarrow \rightarrow & \exists c^+ \text{ Ramsey ufs on } & - 2 -
\end{array}
\]
The only interesting in the above diagram which is not mentioned is whether $AN + \forall \kappa < \omega ( |2^\kappa| \leq \omega )$ implies $\mathcal{P}$ or not. As to this, we show

**THEOREM 2.** $AN + \forall \kappa < \omega ( |2^\kappa| \leq \omega )$ does not imply $\mathcal{P}$.

We shall prove Theorem 1 in the following section and Theorem 2 in section 3.

2. Proof of Theorem 1. We first show that $n(\beta \omega \setminus \omega) > c$ implies $AN$. So, assume that $n(\beta \omega \setminus \omega) > c$. Let $\mathcal{F}$ be any weakly ample filter on $\omega$. For each $f \in \omega^\omega$, set $D_f = \{ U \in \beta \omega \setminus \omega ; f(U) \supseteq \mathcal{F} \}$. Since $\mathcal{F}$ is weakly ample, it holds that

$$\bigcup \{ D_f ; f \in \omega^\omega \} = \beta \omega \setminus \omega.$$ 

So, there is some $f \in \omega^\omega$ such that $D_f$ is not nowhere dense in $\beta \omega \setminus \omega$. Take an infinite subset $a_0$ of $\omega$ such that

$$(*) \quad \forall x \subseteq a_0 ( |x| = \omega \Rightarrow \exists U \in D_f ( x \in U ) ).$$

Set $a_1 = f^n a_0$. Then, by $(*)$, it holds that $a_1$ is infinite and $\forall x \in \mathcal{F} ( a_1 \setminus x$ is finite $)$. Hence, $\mathcal{F}$ is ample.

Now, we shall prove that the inverse implication holds. The following fact which we shall use in the proof is well-known and easy.

**FACT 1.** There is a family $W$ of subsets of $\omega$ such that

1. $|W| = c$,
2. $\forall x \in W ( |x| = \omega )$,
3. $\forall x, y \in W ( x \neq y \Rightarrow x \cap y$ is finite $)$. 
Assume that AN holds. Let \( Q = \{ D_\alpha : \alpha < \omega \} \) be any family of nowhere dense subsets of \( \beta \omega \setminus \omega \). Let \( \langle s_\alpha \mid \alpha < c \rangle \) be a monotone enumeration of a family \( W \) of subsets of \( \omega \) which satisfies (1) \( \vee (3) \) in Fact 1. For each \( \alpha < c \), take \( f_\alpha \in \omega^\omega \) such that \( f_\alpha : \omega \to a_\alpha \) one-to-one and onto. Define the filter \( F \) on \( \omega \) by

\[
x \in F \text{ iff } \forall \alpha < c \forall U \in D_\alpha ( x \notin f_\alpha(U) ) .
\]

Then, it is easy to see that \( F \) is free and not ample. So, by AN, there is \( U \in \beta \omega \setminus \omega \) such that

\[
\forall g \in \omega^\omega ( g(U) \not\in F ) .
\]

Then, it holds that, for any \( \alpha < c \), \( U \notin D_\alpha \), since \( f_\alpha(U) \notin F \). Hence, \( U \notin \omega F \).

3. Proof of Theorem 2. Let \( M \) be a countable transitive model of ZFC + GCH. We shall show that a generic extension of \( M \) on the poset \( P \times Q \) which will be defined below satisfies that AN + \( \forall \kappa < c ( |2^K| < c ) + \square \). The poset \( P \times Q \) is alike the poset used in 5.V of [1]. Let \( P \) be the Solovey-Tennenbaum's poset used for the consistency of \( MA + \omega^\omega = \omega_2 \). Define the poset \( Q \in M \) by, in \( M \),

\[
Q = \{ q : \exists \alpha < \omega_1 ( q : \alpha \to 2 ) \} .
\]

Let \( G \times H \) be \( m \)-generic on \( P \times Q \) and \( \tilde{\mathcal{M}} = M[G \times H] \). Then, similar arguments in [1] show that

\[
\tilde{\mathcal{M}} \models " \omega^\omega = \omega_2 + AN " .
\]

We shall show that \( \tilde{\mathcal{M}} \models \square \). Since CH holds in \( M \), it holds that, in \( \tilde{\mathcal{M}} \), there is a dense embedding from \( Q \) to \( P(\omega)/\text{finite} \).
So, we may assume that $H$ is $\mathcal{M}$-generic on $(P(\omega)/\text{finite})^{\mathcal{M}}$.

Define $\mathcal{F} \in \tilde{\mathcal{M}}$ by

$$\tilde{\mathcal{M}} \models \{ x \in \omega \mid \exists a/\text{finite} \in H ( a \setminus x \text{ is finite} ) \}.$$ 

Since $\tilde{\mathcal{M}} \models |H| = \omega_1$, it holds that

$$\tilde{\mathcal{M}} \models \mathcal{F} \text{ is an } \omega_1\text{-generated free filter on } \omega.$$ 

Moreover, since $H$ is not in $\mathcal{M}[G]$, we have that

$$\tilde{\mathcal{M}} \models \mathcal{F} \text{ is not ample}.$$ 

Hence, $\tilde{\mathcal{M}} \models \neg \mathcal{F}$.

References

