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Kyoto University
Algebraic surfaces for regular systems of weights

Dedicated to Professor Masayoshi NAGATA on the occasion of his sixtieth birthday

Kyoji Saito

RIMS, Kyoto University

ABSTRACT: We construct following families of surfaces by compactifying Milnor fibers.

i) 49 families of K3-surfaces with certain curve configurations, most of which admit elliptic fibrations over $\mathbb{P}^1$.

ii) 9 families of algebraic surfaces of $K = 1, q = 0, P_g = 1$ or 2 with elliptic fibrations over $\mathbb{P}^1$.

iii) 6 families of algebraic surfaces of general type satisfying the numerical equality $P_g = \lfloor c_1^2/2 \rfloor + 2$ for $c_1^2 = 1, 2, 2, 2, 3, 5$.

$(K$:Kodaira dimension, $P_g$=geometric genus, $q$=irregularity, $c_1^2$=second Chern number

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§2. The class having one negative exponent without 0 exponent

§3. The class having one negative exponent with 0 exponents

§4. The class for the smallest exponent $\epsilon$ equals to -2

§5. Weighted homogeneous singularity of dimension two

§1 Introduction

(1.1) Pinkham [20] gave an interpretation for the Arnold's strange duality [1], using compactifications of 14 triangle singularities of Dolgachev [5], where the compactifications are K3 surfaces with certain curve configurations. Looijenga studied such compactifications in details for triangle and Fuchsian singularities [15, 16], to describe possible singularities in the deformation of them.

Along similar idea, we study compactifications of some hypersurface singularities listed by regular systems of weights [24]. As a result we obtain 49 families of K3 surfaces with curve configurations for minimally elliptic singularities of Laufer, 9 families of elliptic surfaces of Kodaira dimension 1 and 6 families of surfaces
of general type with the equality $P \geq \lfloor \frac{c^2}{2} \rfloor + 2$. (See (1.6), (1.7), (1.8) and §2's 2.3, 4)

One motivation of this paper is an attempt to extend examples of period maps
associated to primitive forms (cf. (3.6), [18], [26]), which were well understood
only for simple and simple elliptic singularity cases.

(1.2) We briefly recall Pinkham's compactification $\overline{X}_1$ at a special point $1$ of the
moduli $\mathcal{S}$. A review on weighted homogeneous singularity of dim 2 and the construction of the family $\overline{Y}_c$ ($t \in \mathbb{C}^*$) of the surfaces for the singularity are given in §5,
which prepare notations and concepts for the paragraphs 2.3 and 4. Some readers
may be suggested to go directly to §5's 2.3 and 4 and refer to §5 for notations.

(1.3) Let positive integers $a, b, c$ and $h$ with $\text{GCD}(a, b, c, h) = 1$ called a reduced
system of weights, be given. The hypersurface $X_0 = \{ f(x, y, z, t) = 0 \}$
for a weighted homogeneous polynomial $f(x, y, z) = \sum_{\omega+j+k} c_{\omega+j+k} x^\omega y^j z^k$
with coefficients generic in $\mathbb{C}$ has an isolated singular point at the origin 0. If the following
rational function in $T$ does not have a pole on the unit circle $|T| = 1$ (cf [23]),
$$\chi(T) = \frac{\sum \left( T^{\omega} - T^{\omega-a} \right) \left( T^{\omega} - T^{\omega-b} \right) \left( T^{\omega} - T^{\omega-c} \right)}{(T^a - 1)(T^b - 1)(T^c - 1)}$$

We call such $(a, b, c, h)$ a regular system of weights. Then $\chi(T)$ can be developed in,
$$\chi(T) = T^{m_1} + T^{m_2} + \ldots + T^{m_p}$$
for some integers $m_1, \ldots, m_p$, called the exponents for $(a, b, c, h)$. This establish
a one to one correspondence between the hypersurface singularity $X_0$ with a $\mathbb{C}^*$-action
and the regular system of weights up to a suitable equivalence. Here $\mu := \frac{(h-a)(a-b)(b-c)}{abc}$
is the Milnor number of the singularity. The smallest exponent $a+b+c-h =: \varepsilon$ is
characterized by several means (for instance [8], [32], [23]), playing an important
role for $X_0$. For instance the singularity $X_0$ is a rational double point for $\varepsilon > 0$,
a simply elliptic singularity for $\varepsilon = 0$, and a Fuchsian singularity for $\varepsilon = -1$.

(1.4) For a regular system of weights $(a, b, c, h)$, let us consider the hypersurface
$\overline{X}_1 := \{ f(x, y, z, t) \in \mathbb{C}(a, b, c, 1) : f(x, y, z) = w^h \}$,
where $\mathbb{C}(a, b, c, 1) := (\mathbb{C}[x, y, z] / (a, b, c, h)) \otimes (\mathbb{C}, (t, x, y, z, t, w) \mid t \in \mathbb{C})$. $\overline{X}_1$ is a compactification of the Milnor fiber $X_1 := \{ f(x, y, z) \in \mathbb{C}[x, y, z] : f(x, y, z) = 1 \}$ by adding a curve at
infinity. Denote by $\overline{X}_1$ the surface of the minimal resolution of the singularities of
of $\tilde{X}_I$ at infinity. Put $D_\infty := \tilde{X}_I - X_I$ and call it the divisor at infinity, which defines a star forming dual graph with the central curve $E$.

For example, $\tilde{X}_I$ is a rational surface with $K^2 = 2$ for $\xi > 0$, $\tilde{X}_I - X_I$ is a Del Pezzo surface for $\xi = 0$, and $\tilde{X}_I$ is a K3 surface for $\xi = -1$ (See for instance [11][1][1[1]).

(1.5) After the above mentioned systems of weights $(a, b, c; h)$ with $\xi = 0$ or $\pm 1$, we are interested in the following three extremal boundary cases in the present paper.

i) $(a, b, c; h)$ having only one negative exponent $\xi$ without 0 exponent.

ii) $(a, b, c; h)$ having only one negative exponent $\xi$ with some 0 exponents.

iii) $(a, b, c; h)$ such that the smallest exponent $\xi := a + b + c - h$ is equal to -2.

(1.6) The surfaces $\tilde{X}_I$ for the first group (1.5) i) is studied in §2.

There are $49 = 22 + 7 + 8 + 7 + 3$ such reduced regular systems of weights according as $\xi = -1, -2, -3, -4, -5$ and -7 (See [24]). All these weights define minimally elliptic singularities $\tilde{X}_I$ in the sense of Laufer [14] (cf. (5.7) iv) b)).

This group includes 22 systems of weights with $\xi = -1$ for Fuchsian singularities, particularly 14 exceptional unimodular singularities. Including these Fuchsian cases, the surfaces $\tilde{X}_I$ for the group (1.5) i) have the following descriptions.

There is a maximal sub-configuration $D_1$ of $D_\infty$ which can be blow down to a smooth point. The blow down surface $\tilde{X}_I := \tilde{X}_I / D_1$ is a K3 surface with a curve configuration $D_\infty / D_1$. (Particularly $D_1 = \emptyset$ for Fuchsian singularities.)

There is a sub-configuration $D_2$ of $D_\infty / D_1$, whose linear system defines a fibration of $\tilde{X}_I$ over $\mathbb{P}^1$, most of which are elliptic fibrations.

The detailed descriptions of the divisor $D_\infty$ and the fibration are given in §2.

Note 1. Shioda's study on elliptic surfaces [29].

(1.7) The surfaces $\tilde{X}_I$ for the second group (1.5) ii) are studied in §3.

There are $12 = 9 + 2 + 1$ reduced regular systems weights according as $\xi = -1, -2$ or -3 for this group. The surface $\tilde{X}_I$ is already minimal whose Kodaira dimension $K$ is equal to 0 or 1 according as $\xi = -1$ or less. The geometric genus $P_g$ and the first Chern number $c_2$ of the surface are 1 and 0 respectively. The linear system $\{-\xi E_\infty\}$ defines an elliptic fibration which admits a global simple double or triple section according as $\xi = -1, -2$ or -3. The details will be described in §3.
(1.8) The surfaces $\tilde{S}_2$ for $\varepsilon = -2$ of the group (1.5) iii) are studied in §4.

There are 21 reduced regular systems of weights with $\varepsilon = -2$. In this case the
canonical divisor of the surface is given by $K_2 = E_\infty$, where $E_\infty$ is smooth of genus
$a_0$ and $E_{20} = a_0 - 1$. Here $a_0$ is the multiplicity of 0 exponents.

Therefore the surfaces $\tilde{S}_2$ are classified according to $a_0$ as follows.

i) $a_0 = 0$: There are 7 regular systems of weights of this class. They belong
to the class of (1.5) i) too, which are studied in §2. By blowing down the
curve $E_\infty$, one obtains a family of elliptic K3 surfaces as described there.

ii) $a_0 = 1$: There are 8 regular systems of weights of this class. Two of
them belong to the class (1.5) iii) studied in §3. The remainings are surfaces of
Kodaira dimension $K = 1$ with the irregularity $q = 0$ and $P_0 = 1$. The linear system
$\mathcal{I}E_\infty$ defines the elliptic fibration over $\mathbb{P}^1$ which has a global section.

iii) $a_0 > 1$: There are 6 regular systems of weights of this type. They give
families of surfaces of general type. The pair $(P_0, c_1^2)$ of the geometric genus
and the second Chern number of $\tilde{S}_2$ are $(4,5),(3,3),(3,2),(3,2),(2,1)$ and $(2,1)$,
which satisfy a relation $P_0 = [c_1^2/2] + 2$. The linear system $\mathcal{I}E_\infty$ defines either
a $g = 2$ fibration, a triple or double covering or an embedding as a quintic surface.

The more detailed description of the surfaces is given in §4.

Note 1. These 21 regular systems of weights are naturally corresponding
to co-compact subgroups $\Gamma'$ of $\text{SL}(2,\mathbb{R})$ satisfying $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)\Gamma' \subseteq \Gamma$ (cf. (5.3) Note 2.).

Note 2. In general an inequality $P_0 \geq [c_1^2/2] + 2$ holds. Those surfaces
with the equality are studied by several authors Enriques, Noether, Moishezon,
Horikawa, Todorov and others (cf [13],[32],[28]).

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expresses his gratitude to E. Brieskorn, E. Looijenga, I. Naruki, F. Sakai,
E. Sato, A. Todorov, M. Tomari and J. Wahl for their inspiring discussions.
§ 2 The class having one negative exponent without 0 exponent

In this paragraph, we study the surfaces for regular systems of weights which has one negative exponent but no 0 exponent. The main results formulated in (2.5), (2.6) show that the most of them give families of elliptic K3 surfaces.

(2.1) Systems of weights for minimally elliptic singularities.

Consider a weighted homogeneous hypersurface isolated singular point at 0 in $\mathbb{C}^2$.

\[ f(x,y,z) = \sum_{a \neq 0} c_{ijk} x^i y^j z^k \]  

where $(a,b,c;h)$ is a reduced regular system of weights (cf. (1.3), (5.5)).

The singularity $x_0$ is minimally elliptic (characterized as $p_g = 1$, Laufer [14]) iff there exists one non-positive exponent for $(a,b,c;h)(cf(5.7)ivb)$. The condition is equivalent that either one of the followings holds ((5.5.7), [24 (4.3)]):

(2.1.3) i) $\xi = -1$ and $\min(a,b,c) > -\xi + 1$,
ii) $\min(a,b,c) = -\xi + 1$.

The TABLE 1. is a recalling of the list of reduced regular systems of weights $(a,b,c;h)$ satisfying i) or ii) from [24]. (The 14 systems of $\xi = -1$ Type II in the table satisfy the inequality i) and all the remainings satisfy the equality ii).)

<table>
<thead>
<tr>
<th>$(a,b,c;h) \backslash \xi$</th>
<th>exponents</th>
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<tbody>
<tr>
<td>$\xi = 0$</td>
<td>0,1,1,2,2,3</td>
</tr>
<tr>
<td>(1,1,1;3)</td>
<td>0,1,1,2,2,3</td>
</tr>
<tr>
<td>(1,1,2;4)</td>
<td>0,1,1,2,2,3,3,4</td>
</tr>
<tr>
<td>(1,2,3;6)</td>
<td>0,1,2,2,3,3,4,4,5,6</td>
</tr>
<tr>
<td>$\xi = -1$ Type I</td>
<td></td>
</tr>
<tr>
<td>(2,2,3;8)</td>
<td>-1,1,1,2,3,3,4,5,5,6,7,7,9</td>
</tr>
<tr>
<td>(2,2,5;10)</td>
<td>-1,1,1,3,3,3,5,5,5,8,8,8,9,9,11</td>
</tr>
<tr>
<td>(2,3,3;9)</td>
<td>-1,1,2,2,3,4,4,5,5,6,7,7,8,10</td>
</tr>
<tr>
<td>(2,3,4;10)</td>
<td>-1,1,2,3,3,4,5,5,6,7,7,8,9,11</td>
</tr>
<tr>
<td>(2,3,6;12)</td>
<td>-1,1,2,3,4,5,5,6,7,7,8,9,10,11,13</td>
</tr>
<tr>
<td>(2,4,5;12)</td>
<td>-1,1,3,3,4,5,5,7,7,8,9,9,11,13</td>
</tr>
<tr>
<td>(2,4,7;14)</td>
<td>-1,1,3,3,5,5,7,7,7,9,9,11,13,15</td>
</tr>
<tr>
<td>(2,6,9;18)</td>
<td>-1,1,3,5,5,7,7,9,9,11,11,13,13,15,17,19</td>
</tr>
</tbody>
</table>
\( \xi = -1 \) Type II.

(3,4,4:12)  -1,2,3,3,5,6,6,7,9,9,10,13
(3,4,5:13)  -1,2,3,4,5,6,7,8,9,10,11,14
(4,5,6:16)  -1,3,4,5,7,8,9,11,12,13,17
(3,5,6:15)  -1,2,4,5,5,7,8,10,10,11,13,16
(4,6,7:18)  -1,3,5,6,7,9,11,12,13,15,19
(6,8,9:24)  -1,5,7,8,11,13,16,17,19,25
(3,4,8:16)  -1,2,3,5,6,7,8,9,10,11,13,14,17
(4,5,10:20) -1,3,4,7,8,9,11,12,13,16,17,21
(3,5,9:18)  -1,2,4,5,7,8,9,10,11,13,14,16,19
(4,6,11:22) -1,3,5,7,9,11,11,13,15,17,19,23
(6,8,15:30) -1,5,7,11,13,15,17,19,23,25,31
(3,8,12:24) -1,2,5,7,8,10,11,13,14,16,17,19,22,25
(4,10,15:30) -1,3,7,9,11,13,15,17,19,21,23,27,31
(6,14,21:42) -1,5,11,13,17,19,23,25,29,31,37,43

\( \xi = -2 \)

(3,3,4:12)  -2,1,1,2,4,4,4,5,5,7,7,8,8,10,11,11,14
(3,5,5:15)  -2,1,3,3,4,6,6,7,8,9,9,11,12,12,14,17
(3,5,7:17)  -2,1,3,4,5,6,7,8,9,10,11,12,13,14,16,19
(3,5,10:20) -2,1,3,4,6,7,8,9,10,11,12,13,14,16,17,19,22
(3,7,9:21)  -2,1,4,5,7,7,8,10,11,13,14,14,16,17,19,20,23
(3,7,12:24) -2,1,4,5,7,8,10,11,12,13,14,16,17,19,20,23,26
(3,10,15:30) -2,1,4,7,8,10,11,13,14,16,17,19,20,22,23,26,29,32

\( \xi = -3 \)

(4,5,7:19)  -3,1,2,4,5,6,7,8,9,10,11,12,13,14,15,18,22
(4,5,8:20)  -3,1,2,5,5,6,7,9,10,11,13,14,15,15,18,19,23
(4,5,12:24) -3,1,2,5,6,7,9,10,11,12,13,14,15,17,18,19,22,23,27
(4,7,10:24) -3,1,4,5,7,8,9,11,12,13,15,16,17,19,20,23,27,31
(4,7,14:28) -3,1,4,5,8,9,11,12,13,15,16,17,19,20,23,24,27,31
(4,10,13:30) -3,1,5,7,9,10,11,13,15,17,19,20,21,23,25,29,33
(4,10,17:34) -3,1,5,7,9,11,13,15,17,17,19,21,23,25,27,29,33,37
(4,14,21:42) -3,1,5,9,11,13,15,17,19,21,23,25,27,29,31,33,37,41,45
\( \xi = -4 \)

\[
\begin{align*}
(5, 6, 9: 24) & \quad -4, 1, 2, 5, 6, 7, 8, 10, 11, 12, 13, 14, 16, 17, 18, 19, 22, 23, 28 \\
(5, 6, 15: 30) & \quad -4, 1, 2, 6, 7, 8, 11, 12, 13, 14, 16, 17, 18, 19, 22, 23, 24, 28, 29, 34
\end{align*}
\]

\( \xi = -5 \)

\[
\begin{align*}
(6, 7, 9: 27) & \quad -5, 1, 2, 4, 7, 8, 9, 10, 11, 13, 14, 16, 17, 18, 19, 20, 23, 25, 26, 32 \\
(6, 8, 11: 30) & \quad -5, 1, 3, 6, 7, 9, 11, 12, 13, 15, 17, 18, 19, 21, 23, 24, 27, 29, 35 \\
(6, 8, 13: 32) & \quad -5, 1, 3, 7, 8, 9, 11, 13, 15, 16, 17, 19, 21, 23, 24, 25, 29, 31, 37 \\
(6, 8, 19: 38) & \quad -5, 1, 3, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31, 35, 37, 43 \\
(6, 16, 21: 48) & \quad -5, 1, 7, 11, 13, 16, 17, 19, 23, 25, 29, 31, 32, 35, 37, 41, 47, 53 \\
(6, 16, 27: 54) & \quad -5, 1, 7, 11, 13, 17, 19, 23, 25, 27, 29, 31, 35, 37, 41, 43, 47, 53, 59 \\
(6, 22, 33: 66) & \quad -5, 1, 7, 13, 17, 19, 23, 25, 29, 31, 35, 37, 41, 43, 47, 49, 53, 59, 65, 71
\end{align*}
\]

\( \xi = -7 \)

\[
\begin{align*}
(8, 9, 12: 36) & \quad -7, 1, 2, 5, 9, 10, 11, 13, 14, 17, 18, 19, 22, 23, 25, 26, 27, 31, 34, 35, 43 \\
(8, 10, 15: 40) & \quad -7, 1, 3, 8, 9, 11, 13, 16, 17, 19, 21, 24, 27, 29, 31, 32, 37, 39, 47 \\
(8, 10, 25: 50) & \quad -7, 1, 3, 9, 11, 13, 17, 19, 21, 23, 25, 27, 29, 31, 33, 37, 39, 41, 47, 49, 57
\end{align*}
\]

(2.2) The polynomial \( f(x,y,z,\lambda) \) and \( (m_+, m_0, m_-) \).

Let \( f(x,y,z) \) be a weighted homogeneous polynomial (2.1.2) having an isolated critical point at 0, for the system of weights \((a,b,c,h)\) of TABLE 1. (cf (1.3). Laufer [14, appendix] has already listed such polynomial equations for minimally elliptic singularities. Among them, 3 cases for \( \xi = 0 \) are simply elliptic singularities [ ] and 14 cases for \( \xi = -1 \) Type II. are exceptional unimodular singularities [ ]. In general, singularities for \( \xi = -1 \) are called Fuchsian [ ]).

In the TABLE 2, we recall and complete the list of polynomial \( f(x,y,z,\lambda) \) with \( m_- \)-number of parameters \( \lambda=(\lambda_1, \ldots, \lambda_{m_0}) \), where \( m_+ \), \( m_0 \) and \( m_- \) are dimensions of positive, zero and negative graded part of the universal unfolding of \( f \) respectively (5.7.2).

The polynomials are normalized for a later application (see (2.4) Note.).

TABLE 2.

| \( (a,b,c,h) \) | \( \mu \ | m_-, m_0, m_+ \) | polynomial |
|-----------------|-----------------|-----------------|
| \( (1,1,1:3) \) | 8 | 0, 1, 7 | \( x(x-y)(x-\lambda y) - yz \) | \( \lambda \neq 0, 1. \) |
| \( (1,1,2:4) \) | 9 | 0, 1, 8 | \( xy(x-y)(x-\lambda y) - z \) | \( \lambda \neq 0, 1. \) |
| \( (1,2,3:6) \) | 10 | 0, 1, 9 | \( y(x-y)(x-\lambda y) - z \) | \( \lambda \neq 0, 1. \) |
$E = -1$  Type 1.

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<tbody>
<tr>
<td>$(2,2,3:8)$</td>
<td>15</td>
<td>1.2.12</td>
<td>$x(x-y)(x-λy)(x-\overline{λ}y) + yz^2 \quad λ ≠ 0.1, \overline{λ} ≠ \overline{λ}$.</td>
</tr>
<tr>
<td>$(2,2,5:10)$</td>
<td>16</td>
<td>1.2.13</td>
<td>$xy(x-y)(x-λy)(x-\overline{λ}y) + z^2 \quad λ ≠ 0.1, \overline{λ} ≠ \overline{λ}$.</td>
</tr>
<tr>
<td>$(2,3,3:9)$</td>
<td>14</td>
<td>1.1.12</td>
<td>$x^3y + z(z-y)(z-λy) \quad λ ≠ 0.1$.</td>
</tr>
<tr>
<td>$(2,3,4:10)$</td>
<td>14</td>
<td>1.1.12</td>
<td>$x(x-λ^2)(x-λx^2) - yz \quad λ ≠ 0.1$.</td>
</tr>
<tr>
<td>$(2,3,6:12)$</td>
<td>15</td>
<td>1.1.13</td>
<td>$(y^2-x^2)(y^2-λx^2) + z^2 \quad λ ≠ 0.1$.</td>
</tr>
<tr>
<td>$(2,4,3:10)$</td>
<td>14</td>
<td>1.1.12</td>
<td>$y(y-x^2)(y-λx^2) - xz^2 \quad λ ≠ 0.1$.</td>
</tr>
<tr>
<td>$(2,4,7:14)$</td>
<td>15</td>
<td>1.1.13</td>
<td>$xy(y-x^2)(y-λx^2) - z^2 \quad λ ≠ 0.1$.</td>
</tr>
<tr>
<td>$(2,6,9:18)$</td>
<td>16</td>
<td>1.1.14</td>
<td>$y(y-x^2)(y-λx^2) - z^2 \quad λ ≠ 0.1$.</td>
</tr>
</tbody>
</table>

$E = -1$  Type 2.

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</thead>
<tbody>
<tr>
<td>$(3,4,4:12)$</td>
<td>12</td>
<td>1.0.11</td>
<td>$x^4 + yz(y-z)$.</td>
</tr>
<tr>
<td>$(3,4,5:13)$</td>
<td>12</td>
<td>1.0.11</td>
<td>$x^3y + y^2z + z^2x$.</td>
</tr>
<tr>
<td>$(4,5,6:16)$</td>
<td>11</td>
<td>1.0.10</td>
<td>$x^6 + y^2z + z^2x$.</td>
</tr>
<tr>
<td>$(3,5,6:15)$</td>
<td>12</td>
<td>1.0.11</td>
<td>$x^3z + y^2z + xz^2$.</td>
</tr>
<tr>
<td>$(4,6,7:18)$</td>
<td>11</td>
<td>1.0.10</td>
<td>$x^3y + y^3 + xz^2$.</td>
</tr>
<tr>
<td>$(6,8,9:24)$</td>
<td>10</td>
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<td>$x^4 + y^3 + xz^2$.</td>
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<tr>
<td>$(3,4,8:16)$</td>
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<td>$yx^4 + y^2z + z^2$.</td>
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<tr>
<td>$(4,5,10:20)$</td>
<td>12</td>
<td>1.0.11</td>
<td>$x^6 + y^2z + z^2$.</td>
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<tr>
<td>$(3,5,9:18)$</td>
<td>13</td>
<td>1.0.12</td>
<td>$x^3z + xy^2 + z^2$.</td>
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<tr>
<td>$(4,6,11:22)$</td>
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<tr>
<td>$(6,8,15:30)$</td>
<td>11</td>
<td>1.0.10</td>
<td>$x^6 + xy^2 + z^2$.</td>
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<tr>
<td>$(3,8,12:24)$</td>
<td>14</td>
<td>1.0.13</td>
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<tr>
<td>$(4,10,15:30)$</td>
<td>13</td>
<td>1.0.12</td>
<td>$xy^2 + y^3 + z^2$.</td>
</tr>
<tr>
<td>$(6,14,21:42)$</td>
<td>12</td>
<td>1.0.11</td>
<td>$x^7 + y^3 + z^2$.</td>
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$E = -2$

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<tr>
<td>$(3,3,4:12)$</td>
<td>18</td>
<td>3.1.14</td>
<td>$xy(x-y)(x-λy) + z^2 \quad λ ≠ 0.1$.</td>
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<tr>
<td>$(3,5,5:15)$</td>
<td>16</td>
<td>2.0.14</td>
<td>$x^6 + yz(y-z)$.</td>
</tr>
<tr>
<td>$(3,5,7:17)$</td>
<td>16</td>
<td>2.0.14</td>
<td>$x^8y + y^2z + z^2x$.</td>
</tr>
<tr>
<td>$(3,5,10:20)$</td>
<td>17</td>
<td>2.0.15</td>
<td>$x^8y + y^2z + z^2$.</td>
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<tr>
<td>$(3,7,9:21)$</td>
<td>16</td>
<td>2.0.14</td>
<td>$x^7z + y^3 + z^2x$.</td>
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<tr>
<td>$(3,7,12:24)$</td>
<td>17</td>
<td>2.0.15</td>
<td>$x^7z + xy^3 + z^2$.</td>
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<tr>
<td>$(3,10,15:30)$</td>
<td>18</td>
<td>2.0.16</td>
<td>$x^7z + y^3 + z^2$.</td>
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</table>
\[ \varepsilon = -3 \]

\((4, 5, 7:19)\) 18 3.0.15 \[ x^3z + y^2x + z^2y \]

\((4, 5, 8:20)\) 18 3.0.15 \[ x^3z + y^2x + z^2x \]

\((4, 5, 12:24)\) 19 3.0.16 \[ x^3z + y^2x + z^2y \]

\((4, 7, 10:24)\) 17 2.0.15 \[ x^6 + y^2z + z^2x \]

\((4, 7, 14:28)\) 18 2.0.16 \[ x^7 + y^2z + z^2 \]

\((4, 10, 13:30)\) 17 2.0.15 \[ x^7y + y^2 + z^2x \]

\((4, 10, 17:34)\) 18 2.0.16 \[ x^7y + y^2x + z^2 \]

\((4, 14, 21:42)\) 19 2.0.17 \[ x^7y + y^2 + z^2 \]

\[ \varepsilon = -4 \]

\((5, 6, 9:24)\) 19 3.0.16 \[ x^3z + y^2x + z^2y \]

\((5, 6, 15:30)\) 20 3.0.17 \[ x^3z + y^2x + z^2y \]

\[ \varepsilon = -5 \]

\((6, 7, 9:27)\) 20 4.0.16 \[ x^3z + y^3x + z^2 \]

\((6, 8, 11:30)\) 19 3.0.16 \[ x^6y + y^2x + z^2y \]

\((6, 8, 13:32)\) 19 3.0.16 \[ x^6y + y^2x + z^2x \]

\((6, 8, 19:38)\) 20 3.0.17 \[ x^6y + y^2x + z^2 \]

\((6, 16, 21:48)\) 18 2.0.16 \[ x^6y + y^2 + z^2 \]

\((6, 16, 27:54)\) 19 2.0.17 \[ x^6y + y^2 + z^2 \]

\((6, 22, 33:66)\) 20 2.0.18 \[ x^6y + y^2 + z^2 \]

\[ \varepsilon = -7 \]

\((8, 9, 12:36)\) 21 4.0.17 \[ x^3z + y^4 + z^3 \]

\((8, 10, 15:40)\) 20 3.0.17 \[ x^5 + y^4 + z^2y \]

\((8, 10, 25:50)\) 21 3.0.18 \[ x^5y + y^4 + z^2 \]

As a consequence of the table we see and it is not hard to prove the following.

**Assertion i)** \( m_\varepsilon = (e \in \{a, b, c\}: e < -2\varepsilon) + 1 \),

\( m_\varepsilon = (e \in \{a, b, c\}: e = -2\varepsilon) \).

**Assertion ii)** The polynomial \( f(x, y, z, \lambda) \) can be expressed as a sum of \( m_\varepsilon + 3 \) monomials in \( x, y \) and \( z \). Particularly if \( m_\varepsilon = 0 \) (which is most of the cases), the polynomial \( f(x, y, z) \) is unique up to automorphisms of the coordinate ring.

(Proof is a combination of (5.7, 2), (2.1, 3) and [23 (1.9, 1), (3.6)].)
(2.3) From now on in this paper, we consider only the 49 cases with $\varepsilon < 0$. Note that the intersection form for the middle homology group of the Milnor fiber for this class of singularities has signature $(\mu, \mu^*, \mu^*) = (2, 0, \mu^* - 2)$ (cf (5.7.4)).

(2.4) The minimal good resolution $\pi: \tilde{X}_0 \to X_0$ of the singularity $X_0$ (2.1.1) is described in (5.6) (cf [6],[14],[19],[21]). The exceptional set $\pi^*(0)$ defines a star shaped dual graph (5.6.1), whose central curve is denoted by $E_0$. The degree graph is numerically determined by the data: i) the genus of $E_0 = g(E_0)$, which is always 0 in this case ((5.6), 2) so that it will be omitted, ii) the self intersection number $E_0^2 = -(1 + \#(e \in (a, b, c): e = d+1))$ (cf (5.6.4)), iii) the set $A := \{p_1, \ldots, p_r\}$ of the orders of the cyclic isotropy subgroups of the Fuchsian group $\Gamma$ at the branching points on $E_0$ (5.6, 5), iv) the number $d := h - a - b - c = -\varepsilon$.

(The $p_i$'s for the 14 exceptional singularities are well known as Dolgachev numbers.)

Furthermore, the analytic data of the resolution is determined by the positions of the branching points on $E_0 \equiv \mathbb{P}^1$. Hence we give a rational parametrization:

$$\mathbb{P}^1 \cong E_0 = \{(x:y:z) \in \mathbb{P}^1(a,b,c): f(x,y,z,\lambda) = 0\}$$

$t \mapsto (x:y:z)$

of the central curve $E_0$.

We shall describe in the TABLE 3. the following data for every regular systems of weights $(a,b,c,h)$ of the TABLE 1.:

i) The set $A := \{p_1, \ldots, p_r\}$.

ii) Polynomial presentation $(x(t), y(t), z(t))$ of the parametrization: $\mathbb{P}^1 \to E_0$.

iii) The values $t_i$ of $t$ at the branching points $p_i$ on $E_0$.

iv) The order of zeros $(n_1, n_2, n_3)$ of $(x(t), y(t), z(t))$ at the branching point $p_i$.

v) The dual graph of the exceptional set $\pi^*(0)$.

(We used $p_i$'s $\in A$ as for the identification of the branching points on $E_0$.)

Note. In the TABLE 3. the polynomials $f(x,y,z,\lambda)$, $x(t)$, $y(t)$ and $z(t)$ are normalized as follows. (Recall that the branching points lie on the coordinate axis (5.6).)

i) The values of $t$ at branching points of $E_0$ = (the roots of $x(t)y(t)z(t)=0$) $\bigcup (\infty)$.

ii) $0 < n_4 \leq a$, $0 < n_2 \leq b$, and $0 < n_3 \leq c$ for $p_i \in A = \{p_1, \ldots, p_r\}$.

iii) $n_{2\varepsilon}$, $n_{3\varepsilon}$, and $n_{3\varepsilon}$ are defined by the following relation.

$$\sum_{i=1}^{r} \left[ \begin{array}{c} n_{2i} \\ n_{3i} \end{array} \right] = \left( m + 1 \right) \left[ \begin{array}{c} a \\ b \\ c \end{array} \right]$$ for $\varepsilon = -1$, or

$$= \left( m + 2 \right) \left[ \begin{array}{c} a \\ b \\ c \end{array} \right]$$ for $\varepsilon = -2$. 

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<table>
<thead>
<tr>
<th>(a,b,c,h)</th>
<th>A:=&lt;d_1,..,d_s&gt;</th>
<th>parametrization of E_6</th>
<th>dual graph</th>
</tr>
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<tbody>
<tr>
<td>(2,2,3,8)</td>
<td>2,2,2,2,3</td>
<td></td>
<td></td>
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<tr>
<td>t = 0.1, \lambda, \lambda_0,0 \in</td>
<td>x = -t^2(t-1)(t-\lambda_1)(t-\lambda_2),</td>
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<tr>
<td>n_x = 2 1 1 1 1</td>
<td>y = -t(t-1)(t-\lambda_1)(t-\lambda_2),</td>
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<td></td>
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<tr>
<td>n_y = 1 1 1 1 2</td>
<td>z = \pm t^2(t-1)^2(t-\lambda_1)^2(t-\lambda_2)^3.</td>
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<td></td>
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<tr>
<td>(2,2,5,10)</td>
<td>2,2,2,2,2</td>
<td></td>
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</tr>
<tr>
<td>t = 0.1,4,\lambda_0,0 \in</td>
<td>x = -t^2(t-1)(t-\lambda_1)(t-\lambda_2),</td>
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<tr>
<td>n_x = 2 1 1 1 1</td>
<td>y = -t(t-1)(t-\lambda_1)(t-\lambda_2),</td>
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<td>n_y = 1 1 1 1 2</td>
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<tr>
<td>(2,3,3,9)</td>
<td>2,3,3,3</td>
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</tr>
<tr>
<td>t = \omega,0.1,\lambda</td>
<td>x = -t(t-1)(t-\lambda),</td>
<td></td>
<td></td>
</tr>
<tr>
<td>n_x = 1 1 1 1</td>
<td>y = \pm t(t-1)(t-\lambda),</td>
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<tr>
<td>n_y = 1 1 1 1</td>
<td>z = \pm t^2(t-1)(t-\lambda)^2.</td>
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<td>(2,3,4,10)</td>
<td>2,3,3,4</td>
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<tr>
<td>t = 1,\lambda,0,0</td>
<td>x = -t(t-1)(t-\lambda),</td>
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<tr>
<td>n_x = 1 1 1 1</td>
<td>y = \pm t(t-1)(t-\lambda)^2,</td>
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<td>n_y = 2 2 1 1</td>
<td>z = t^2(t-1)^2(t-\lambda)^3.</td>
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<td>(2,3,6,12)</td>
<td>2,3,3,3</td>
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<td>t = \omega,0.1,\lambda</td>
<td>x = t(t-1)(t-\lambda),</td>
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<td>n_x = 1 1 1 1</td>
<td>y = \pm t(t-1)(t-\lambda),</td>
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<td>n_y = 2 2 1 1</td>
<td>z = t^2(t-1)^2(t-\lambda)^3.</td>
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<tr>
<td>(2,4,5,12)</td>
<td>2,2,2,5</td>
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<tr>
<td>t = 0.1,\lambda,0,0</td>
<td>x = -t(t-1)(t-\lambda),</td>
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<tr>
<td>n_x = 1 1 1 1</td>
<td>y = t^3(t-1)^2(t-\lambda)^2,</td>
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<tr>
<td>n_y = 3 2 2 1</td>
<td>z = \pm t^3(t-1)^3(t-\lambda)^3.</td>
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<td>n_x = 1 1 1 1</td>
<td>y = t^3(t-1)^2(t-\lambda)^2,</td>
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<td>(2,6,9,18)</td>
<td>2,2,2,3</td>
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<td>t = 0.1,\lambda,0,0</td>
<td>x = -t(t-1)(t-\lambda),</td>
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<tr>
<td>n_x = 1 1 1 1</td>
<td>y = t^3(t-1)^2(t-\lambda)^2,</td>
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<tr>
<td>n_y = 4 3 3 2</td>
<td>z = \pm t^3(t-1)^3(t-\lambda)^3.</td>
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<td>Type II.</td>
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<td>(3,4,4,12)</td>
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<tr>
<td>t = \omega,1,0</td>
<td>x = 2t(t-1),</td>
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<tr>
<td>n_x = 1 1 1</td>
<td>y = t^2(t-1),</td>
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<td>n_y = 1 1 2</td>
<td>z = -t(t-1)^2.</td>
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</table>
(3, 4, 5:13)
\[ t = \omega, 1, 0 \]
\[ n_x = 1 \quad 1 \quad 1 \]
\[ x = t (t-1) \]
\[ n_y = 2 \quad 1 \quad 1 \]
\[ y = -t (t-1) \]
\[ n_z = 2 \quad 2 \quad 1 \]
\[ z = \pm t (t-1)^2 \]

(4, 5, 6:16)
\[ t = \omega, 1, 0 \]
\[ n_x = 2 \quad 1 \quad 1 \]
\[ x = -t (t-1) \]
\[ n_y = 3 \quad 1 \quad 1 \]
\[ y = \pm t (t-1) \]
\[ n_z = 3 \quad 2 \quad 1 \]
\[ z = -t (t-1)^2 \]

(3, 5, 6:15)
\[ t = \omega, 1, 0 \]
\[ n_x = 1 \quad 1 \quad 1 \]
\[ x = \pm t (t-1) \]
\[ n_y = 2 \quad 2 \quad 1 \]
\[ y = \pm t (t-1)^2 \]
\[ n_z = 3 \quad 2 \quad 1 \]
\[ z = -t (t-1)^2 \]

(4, 6, 7:18)
\[ t = \omega, 1, 0 \]
\[ n_x = 2 \quad 1 \quad 1 \]
\[ x = t (t-1) \]
\[ n_y = 3 \quad 2 \quad 1 \]
\[ y = t (t-1)^2 \]
\[ n_z = 4 \quad 2 \quad 1 \]
\[ z = \pm t (t-1)^2 \]

(6, 8, 9:24)
\[ t = \omega, 1, 0 \]
\[ n_x = 3 \quad 2 \quad 1 \]
\[ x = -t (t-1)^2 \]
\[ n_y = 4 \quad 3 \quad 1 \]
\[ y = -t (t-1)^3 \]
\[ n_z = 5 \quad 3 \quad 1 \]
\[ z = \pm t (t-1)^3 \]

(3, 4, 8:16)
\[ t = \omega, 1, 0 \]
\[ n_x = 1 \quad 1 \quad 1 \]
\[ x = \pm t (t-1) \]
\[ n_y = 2 \quad 1 \quad 1 \]
\[ y = -t (t-1) \]
\[ n_z = 3 \quad 2 \quad 3 \]
\[ z = -t^2 (t-1)^2 \]

(4, 5, 10:20)
\[ t = \omega, 1, 0 \]
\[ n_x = 2 \quad 1 \quad 1 \]
\[ x = -t (t-1) \]
\[ n_y = 3 \quad 1 \quad 1 \]
\[ y = \pm t (t-1) \]
\[ n_z = 5 \quad 2 \quad 3 \]
\[ z = -t^3 (t-1)^2 \]

(3, 5, 9:18)
\[ t = \omega, 1, 0 \]
\[ n_x = 1 \quad 1 \quad 1 \]
\[ x = \pm t (t-1) \]
\[ n_y = 2 \quad 2 \quad 1 \]
\[ y = \pm t (t-1)^2 \]
\[ n_z = 4 \quad 3 \quad 2 \]
\[ z = \pm t^2 (t-1)^3 \]

(4, 6, 11:22)
\[ t = \omega, 1, 0 \]
\[ n_x = 2 \quad 1 \quad 1 \]
\[ x = -t (t-1) \]
\[ n_y = 3 \quad 2 \quad 1 \]
\[ y = -t (t-1)^2 \]
\[ n_z = 6 \quad 3 \quad 2 \]
\[ z = \pm t^2 (t-1)^3 \]

(6, 8, 15:30)
\[ t = \omega, 1, 0 \]
\[ n_x = 3 \quad 2 \quad 1 \]
\[ x = -t (t-1)^2 \]
\[ n_y = 4 \quad 3 \quad 1 \]
\[ y = -t (t-1)^2 \]
\[ n_z = 8 \quad 5 \quad 2 \]
\[ z = \pm t^2 (t-1)^3 \]
\( \varepsilon = -5 \)

\( 6.7:9:27 \) 7, 6, 3  
\( t = 0, 1, \infty \)
\( n_z = 3 5 4 \)
\( n_y = 3 6 5 \)
\( n_x = 4 8 6 \)
\( x = -t^5(t-1)^5 \)
\( y = t^7(t-1)^6 \)
\( z = t^9(t-1)^8 \)

\( 6.8:11:30 \) 11, 8, 2  
\( t = 00, 1, \infty \)
\( n_z = 5 4 3 \)
\( n_y = 7 5 4 \)
\( n_x = 9 7 6 \)
\( x = -t^3(t-1)^5 \)
\( y = t^8(t-1)^5 \)
\( z = t^9(t-1)^6 \)

\( 6.8:13:32 \) 13, 6, 2  
\( t = 0, 1, \infty \)
\( n_z = 4 5 3 \)
\( n_y = 5 7 4 \)
\( n_x = 8 11 7 \)
\( x = t^6(t-1)^5 \)
\( y = -t^3(t-1)^5 \)
\( z = t^9(t-1)^6 \)

\( 6.8:19:38 \) 8, 6, 2  
\( t = 0, 1, \infty \)
\( n_z = 4 5 3 \)
\( n_y = 5 7 4 \)
\( n_x = 12 16 10 \)
\( x = t^4(t-1)^5 \)
\( y = -t^3(t-1)^5 \)
\( z = t^9(t-1)^6 \)

\( 6.16:21:48 \) 21, 3, 2  
\( t = 0, 1, \infty \)
\( n_z = 5 4 3 \)
\( n_y = 13 11 8 \)
\( n_x = 17 14 11 \)
\( x = -t^5(t-1)^6 \)
\( y = -t^3(t-1)^5 \)
\( z = t^9(t-1)^6 \)

\( 6.16:27:54 \) 16, 3, 2  
\( t = 0, 1, \infty \)
\( n_z = 5 4 3 \)
\( n_y = 13 11 8 \)
\( n_x = 22 18 14 \)
\( x = -t^5(t-1)^6 \)
\( y = -t^3(t-1)^5 \)
\( z = t^9(t-1)^6 \)

\( 6.22:33:66 \) 11, 3, 2  
\( t = 0, 1, \infty \)
\( n_z = 5 4 3 \)
\( n_y = 18 15 11 \)
\( n_x = 27 22 17 \)
\( x = -t^5(t-1)^6 \)
\( y = t^3(t-1)^5 \)
\( z = t^9(t-1)^6 \)

\( \varepsilon = -7 \)

\( 8.9:12:36 \) 8, 4, 3  
\( t = 0, 1, \infty \)
\( n_z = 7 6 3 \)
\( n_y = 8 7 3 \)
\( n_x = 11 9 4 \)
\( x = -t^5(t-1)^6 \)
\( y = t^3(t-1)^5 \)
\( z = t^9(t-1)^6 \)

\( 8.10:15:40 \) 15, 5, 2  
\( t = 0, 1, \infty \)
\( n_z = 7 5 4 \)
\( n_y = 9 6 5 \)
\( n_x = 13 9 8 \)
\( x = -t^7(t-1)^5 \)
\( y = t^9(t-1)^6 \)
\( z = t^9(t-1)^6 \)

\( 8.10:25:50 \) 8, 5, 2  
\( t = 0, 1, \infty \)
\( n_z = 7 5 4 \)
\( n_y = 9 6 5 \)
\( n_x = 22 15 13 \)
\( x = -t^7(t-1)^5 \)
\( y = t^9(t-1)^6 \)
\( z = t^9(t-1)^6 \)
As a consequence of the above calculations, we obtain the following.

**Assertion** i) The number \( r \) of the branches of the resolution graph is given by

\[
r = m_0 + 3.
\]

ii) The coordinates of the branching points on the central curve \( E_0 \) can be chosen to be \( 0, 1, \infty, \lambda_1, \ldots, \lambda_N \) where \( (\lambda_1, \ldots, \lambda_N) \) is the coordinates for the \( S_0 \) (the degree 0 part of the universal unfolding of \( f \) (cf. (5.7) i)) used in the TABLE 2.

iii) \[
\det \begin{pmatrix}
a & n_x 0 & n_z 1 \\
b & n_y 0 & n_y 1 \\
c & n_x 0 & n_x 1
\end{pmatrix} = \pm 1
\]

iv) The shape of the resolution graph, forgetting about the self-intersections of the components, depends only on the integers \( m_\omega, m_0 \) and \( \xi := a + b + c - h \) (\( = -d \)).

v) The canonical divisor \( K_{E_0} \) (5.6.7) depends only on the shape of the graph.

In the following TABLE 4, we list the shape of the dual graph and the coefficients of the canonical divisor for the minimal good resolution above.

**TABLE 4.**

<table>
<thead>
<tr>
<th>( \xi )</th>
<th>( m_\omega )</th>
<th>( m_0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>-1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>-1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>-2</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>-2</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>-3</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>-3</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>-4</td>
<td>3</td>
<td>2</td>
</tr>
</tbody>
</table>
\[ \varepsilon = -5, \ m_0 = 0 \]

\[ \varepsilon = -7, \ m_0 = 0 \]

\[ \varepsilon = -7, \ m_0 = 3 \]

\[ \varepsilon = -7, \ m_0 = 4 \]

\[ \varepsilon = -5, \ m_0 = 3 \]

\[ \varepsilon = -5, \ m_0 = 2 \]

\[ \varepsilon = -5, \ m_0 = 4 \]

Note. Many of the above graphs have the figure of affine Coxeter graphs of types \( \tilde{D}_4, \tilde{C}_n \) (k=6,7,8). Such singularities (called Kodaira sing.) are studied in [10].

(2.5) Compactifications.

The compactification \( \tilde{X}_t \) of a Milnor fiber \( X_t \) for \( t \in S \) is described in (5.8).

Recall that \( \tilde{X}_t = \tilde{X}_t \cup D_{\infty} \) (5.8.3), where \( \tilde{X}_t \) is the minimal resolution of the affine variety \( X_t \) (5.7.3) and \( D_{\infty} \) is the divisor at infinity (5.8.4). The canonical divisor of \( \tilde{X}_t \) is \( K_{\tilde{X}_t} = K_{X_t} + \sum \nabla_{\tilde{X}_t} \) where \( K_{X_t} \) is the canonical divisor at infinity and \( K_{X_t} \) is the canonical divisor of the resolution \( \tilde{X}_t \rightarrow X_t \) of a singular point \( x \) on \( X_t \) (5.8.7).

In this paragraph in TABLES 5.6, we shall describe \( D_{\infty} \) and \( K_{X_t} \) explicitly.

Before giving the TABLES, we summarize some of their structures in the following Theorem, which implies that a minimal model \( \tilde{X}_t \) of \( \tilde{X}_t \) is a K3 surface for \( t \in S \).

Theorem Let \( (a,b,c,h) \) be a regular system of weights of TABLE I. Let \( (\tilde{X}_t, D_{\infty}) \) for \( t \in S \) be a pair of the compact smooth surface and its divisor at infinity for \( (a,b,c,h) \) as described in (5.8). Then the divisor \( D_{\infty} \) has the following decomposition.

\[ D_{\infty} = D_1 \cup D_2 \cup D_3 \]

with the following properties:

i) The divisor \( D_1 \) in \( \tilde{X}_t \) can be blow down to a smooth point. Let us denote by \( \pi: \tilde{X}_t \rightarrow \tilde{X}_t \) the blow down map, where \( \tilde{X}_t := \tilde{X}_t / D_1 \) is the smooth surface.
ii) The canonical divisor $K_{\omega}$ is equal to the canonical divisor of the map $\pi$. (i.e.,
$K_{\omega}=\text{div}(\Omega(\omega))$ for a nonvanishing holomorphic 2-form $\omega$ on $X_\omega$ near the point $\pi D_j$.)

This is equivalent to say that the canonical divisor $K_{\omega}$ of $X_\omega$ is given by

$$(2.5.2) \quad K_{\omega} = \sum_{x \in X_\omega} K_x,$$

where the sumation is over singularities of the affine surface $X_\omega$ (cf. (5.8.7)).

iii) Put $\tilde{D}_2 := \pi(D_2)$. Then $\tilde{D}_2$ is either one of the followings,

a) A system of smooth rational curves whose intersection diagram is $\tilde{D}_k \cup E_k$ (for $k=6, 7$);

b) Three smooth rational curves intersecting at a point normally each other.

c) Two smooth rational curves contacting at a point of order 2 or 3.

d) One rational curve with a cusp singular point of type $(2, 3), (2, 5)$ or $(3, 4)$.

(Here $(p, q)$-cusp is a plane curve singularity, locally given by a equation $x^p - y^q = 0$.)

(Complete)

iv) The linear system $|\tilde{D}_2|$ defines a fibration of $X_\omega$ over $\mathbb{P}^1$, most of which are elliptic fibrations. (For exact descriptions, see (2.6).)

v) $\tilde{D}_3 := \pi(D_3)$ is a union of smooth rational curves of selfintersections -2, whose connected components are of types either $A_1$, $A_2$, or $A_3$.

Corollary The surface $X_\omega$ is a K3 surface with a curve configuration $D_\omega D_1 = D_2 \cup D_3$ for $\omega \in \mathfrak{D}_f$ (the rational double point part of (cf. (5.7.ii))). Hence the middle homology group $H_2(X_\omega, \mathbb{Z})$ of a Milnor fiber of the polynomial of TABLE 2, is embedded in the lattice of the K3 surface as an orthogonal complement of the classes of $D_2 \cup D_3$.

$$(2.5.3) \quad H_2(X_\omega, \mathbb{Z}) \cong \mathbb{Z}[(\tilde{D}_2 \cup \tilde{D}_3)]^\perp.$$

A proof of the theorem is done, if we have explicitly determined the divisors $D_\omega$ and $K_\omega$. which will be done in the following TABLEs 3, 5 and 6. An explicit execution of the calculations is as described in §5 and is omitted from this paper.

For a proof of the Corollary, see (5.9).

The following TABLE 6. describes the dual graph of $D_\omega$ and its decomposition $D_1 \cup D_2 \cup D_3$ for each $(a, b, c, h)$ of TABLE 1. These data together with that of the position of branching points on $E_0 \cup E_0$ and $A := (p_1, ..., p_k)$ in the TABLE 3, completely determine the divisor $D_\omega$ at infinity.

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In the following TABLE 5, we summarize the data: $D_1 \cup D_2$ and $\pi$. Here $D_1$ (resp. $D_2$) is described by dotted (resp. real) lines.

**TABLE 5.**

$\xi = -1$. The configuration $D_1$ is always void. The configuration $D_2$ is a union of smooth rational curves whose intersection diagram is one of the affine Coxeter diagrams of type $D_2^*$ or $E_k$ ($k=6,7,8$) (cf TABLE 6.).

$\xi = -2$, $m_2 = 3$ and 2.

$$D_1 \cup D_2 = \begin{array}{ccc}
-3 & -3 & -3 \\
\end{array}$$

$K_{\infty} = E_{\infty}$.

Three smooth rational curves, intersecting transversally at a point.

$$\pi(D_2) = D_2^*$$

$E_2^* = 0$.

$\xi = -3$, $m_2 = 3$ and 2.

$$D_1 \cup D_2 = \begin{array}{ccc}
-4 & -2E_1 \\
\end{array}$$

$K_{\infty} = 2 E_{\infty} + E_1$

Two smooth rational curves, contacting at a point with order 2.

$$\pi(D_2) = D_2^*$$

$D_2^* = 0$.

$\xi = -4$, $m_2 = 3$.

$$D_1 \cup D_2 = \begin{array}{ccc}
-2E_2 & -2E_1 \\
\end{array}$$

$K_{\infty} = 3 E_{\infty} + 2 E_1 + E_2$

Two smooth rational curves, contacting at a point with order 3.

$\xi = -5$, $m_2 = 4, 3$ and 2.

$$D_1 \cup D_2 = \begin{array}{ccc}
-6 & -3E_2 & -2E_1 \\
\end{array}$$

$K_{\infty} = 4 E_{\infty} + 2 E_1 + E_2$

A rational curve with a $(2,3)$ cusp.

$\xi = -7$, $m_2 = 4$.

$$D_1 \cup D_2 = \begin{array}{ccc}
-8 & -4 \\
\end{array}$$

$K_{\infty} = 6 E_{\infty} + 4 E_1 + 2 E_2 + E_3$

A rational curve with a $(3,4)$ cusp.
\[ E = -7, \quad m_e = 3. \]

\[ D_1 \cup D_2 = \\
\begin{align*}
-2 & E_3 \quad -3 \\
\end{align*}
\]

\[ \pi(D_2) = \overset{\circ}{D}_2 \quad \overset{\circ}{D}_2^2 = 4. \]

TABLE 6.

(The subdiagrams surrounded by a real square describes \( D_1 \cup D_2 \) of \((2,4,1)\) and the subdiagrams surrounded by a dotted square describes \( D_1 \).)

<table>
<thead>
<tr>
<th>(a,b,c,h)</th>
<th>(m_e,m_o)</th>
<th>dual graph</th>
</tr>
</thead>
<tbody>
<tr>
<td>(2,2,3:8)</td>
<td>(1,2)</td>
<td><img src="#" alt="Diagram" /></td>
</tr>
<tr>
<td>(2,2,5:10)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(2,3,3:9)</td>
<td>(1,1)</td>
<td><img src="#" alt="Diagram" /></td>
</tr>
<tr>
<td>(2,3,4:10)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(2,3,6:12)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(2,4,5:12)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(2,4,7:14)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(2,6,9:18)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

14 systems of weights of type II

\[ E = -2 \]

| (3,3,4:12) | (3,1) | ![Diagram](#) |
| | | |
| (3,5,5:15) | (2,0) | ![Diagram](#) |
| | | |
| (3,5,7:17) | (2,0) | ![Diagram](#) |
| | | |
| (3,5,10:20) | (2,0) | ![Diagram](#) |
| | | |
| (3,7,9:21) | (2,0) | ![Diagram](#) |
| | | |
| (3,7,12:24) | (2,0) | ![Diagram](#) |
| | | |
| (3,10,15:30) | (2,0) | ![Diagram](#) |

2c
As a consequence of the above explicit description of the divisor $D_{\infty}$ at infinity, we have the following:

**Assertion**  Except for the case: $m_+ = 1$ and $m_0 = 0$ (corresponding to 14 exceptional singularities), the triple $(\xi, m_-, m_0)$ determines $D_1, D_2$ of $D_{\infty}$.

**Note 1.** It is curious to observe that the canonical divisor and the resolution graph of the singularity $\chi_0$ is also determined by the same triple $(\xi, m_-, m_0)$ (cf. (2.4) Assertion iv), v) and TABLE 4.). Since these numbers $\xi, m_-$ and $m_0$ are well defined for all Gorenstein singularity with a $\xi^2$-action, it may be reasonable to ask the following:

**Conjecture**  Let $\chi_0$ be a minimally elliptic singularity with $\xi^2$-action. Then a smoothing $\chi_t$ of $\chi_0$ over a positively graded part of the parameter, is naturally compactified by a K3 surface, whose structure such as described in (2.4) Assertion iv), v) and (2.5) Assertion depends only on the triple $(\xi, m_-, m_0)$.

**Note 2.** There are 9 more regular systems of weights with $\xi = -1$ besides those of the TABLE 1. The Milnor fibers are also compactified by K3 surfaces. In 6 cases of them, the divisor $D_{\infty}$ is a smooth elliptic curves with $D_{\infty}^2 = 0$. Hence the surface $\chi_t$ admits a structure of elliptic fibrations (cf §3).
§ 3 The classes having one negative exponent with 0 exponents

In this paragraph we study surfaces for regular system of weights \((a,b,c;h)\) which has \(E\) as the only negative exponent and 0 as an exponent. If \(E = -1\) then the corresponding singularities are Fuchsian and hence the corresponding surfaces are K3 as stated in the introduction. Otherwise we shall see that the surfaces are of Kodaira dim 1 with elliptic fibrations over \(\mathbb{P}^1\) (see (3.5), (3.6)).

(3.1) System of weights. There are \(9+2+1\) reduced regular systems of weights which have one negative exponent and some 0 exponents according as \(E = -1, -2\) or \(-3\), which are listed in the following TABLE 8. (The case \(E = -1\) is already treated in [23] so that we shall omit the case from the consideration in this paper.)

(Proof. For a system \((a,b,c;h)\) after the smallest exponent \(E\), the next small exponent is \(+\min(a,b,c)\). Hence the condition on the system implies \(E + \min(a,b,c) = 0\).

Further if \(E = -1\), then I must be an exponent for the system (cf (5.5), [24]), which implies \(-E + 1 \in \{a,b,c\}\). A calculation similar for the TABLE 1 shows the result.)

<table>
<thead>
<tr>
<th>((a,b,c;h))</th>
<th>exponents</th>
</tr>
</thead>
<tbody>
<tr>
<td>((2,3,5;12))</td>
<td>(-E = 2)</td>
</tr>
<tr>
<td>((2,3,7;14))</td>
<td>(-2, 0, 1, 2, 3, 4, 5, 6, 6, 6, 7, 8, 8, 9, 9, 10, 10, 11, 12, 14)</td>
</tr>
<tr>
<td>((3,4,5;15))</td>
<td>(-E = 3)</td>
</tr>
<tr>
<td>((3,4,5;15))</td>
<td>(-3, 0, 1, 2, 3, 4, 5, 5, 6, 6, 7, 8, 9, 9, 10, 10, 11, 12, 13, 14, 15, 18)</td>
</tr>
</tbody>
</table>

Note that the multiplicity \(a_0\) of zero exponents is 1 in all cases.

(3.2) Polynomial \(f(x,y,z,\lambda)\). For each system of weights \((a,b,c;h)\) of the TABLE 8., we associate: i) a weighted homogeneous polynomial \(f(x,y,z,\lambda)\) with a parameter \(\lambda\) for the weight (5.5.2), ii) the Milnor number \(\mu\) and the signature \((\mu_-, \mu_0, \mu_+\) of the Milnor fiber (5.7.4), iii) the dimensions \((m_-, m_0, m_+)\) of deformation of \(f\) (5.7.2).

<table>
<thead>
<tr>
<th>((a,b,c))</th>
<th>(\mu)</th>
<th>(\mu_-^c \mu_+)</th>
<th>(m_- m_0 m_+)</th>
<th>polynomial</th>
<th>restriction</th>
</tr>
</thead>
<tbody>
<tr>
<td>((2,3,5;12))</td>
<td>21</td>
<td>2.2.17</td>
<td>3.1.17</td>
<td>(x^6 + y^4 + xz^2 + \lambda x^2yz)</td>
<td>(\lambda^4-64\lambda^0)</td>
</tr>
<tr>
<td>((2,3,7;14))</td>
<td>22</td>
<td>2.2.18</td>
<td>3.1.18</td>
<td>(x^7 + xy^4 + z^2 + \lambda x^2yz)</td>
<td>(\lambda^4-64\lambda^0)</td>
</tr>
<tr>
<td>((3,4,5;15))</td>
<td>22</td>
<td>2.2.18</td>
<td>4.1.17</td>
<td>(x^6 + xy^3 + z^3 + \lambda x^2yz)</td>
<td>(\lambda^4+27\lambda^0)</td>
</tr>
</tbody>
</table>
Note that the number $m_0$ of the parameter $\lambda$ (=dimension of homogeneous deformation of $f$) is always 1. Another normal form will be given in § 4 TABLE 14.

(3.3) Resolution. The minimal good resolution of the singularity $y_0 := ((x,y,z) \in \mathbb{C}: f(x,y,z,\lambda) = 0)$ is described in (5.6). Numerically it is determined by the data: the genus $g(E_0)$ and the self-intersection number $E_0^2$ of the central curve $E_0$, the set $\Lambda$ of the order of cyclic groups and $d := -6$.

In the TABLE 10, we give such numerical data and the resolution graph with the coefficients of the canonical divisor near by for polynomials of TABLE 9.

<table>
<thead>
<tr>
<th>(a,b,c,h)</th>
<th>$g(E_0)$</th>
<th>$E_0^2$</th>
<th>$\Lambda$</th>
<th>resolution graph</th>
</tr>
</thead>
<tbody>
<tr>
<td>(2,3,5:12)</td>
<td>1</td>
<td>-1</td>
<td>5</td>
<td>$E_0$</td>
</tr>
<tr>
<td>(2,3,7:14)</td>
<td>1</td>
<td>-1</td>
<td>3</td>
<td>$E_0$</td>
</tr>
<tr>
<td>(3,4,5:15)</td>
<td>1</td>
<td>-1</td>
<td>4</td>
<td>$E_0$</td>
</tr>
</tbody>
</table>

Note. The shape of the dual graph and the canonical divisor depends only on the triple $(E, m_-, m_0)$. (Compare (2.4) Assertion iv), v).)

(3.4) The compactification. The unfolding of the polynomial $f$, the compactifications $\tilde{X}_\epsilon$ of their Milnor fiber $X_\epsilon$ for $\epsilon S$ (or $S_\epsilon$) are described in (5.7), (5.8). The surface $\tilde{X}_\epsilon$ is a union of the open part $\tilde{X}_\epsilon'$ (the resolution of the Milnor fiber $X_\epsilon$) and the divisor at infinity $D_\infty$. The canonical divisor of $\tilde{X}_\epsilon$ is a sum $K_\epsilon = \sum_{\gamma \in \Lambda} K_\gamma$, where $\text{supp}(K_\epsilon) \subset D_\infty$ and the second term $K_{\gamma}$ vanishes away for $\epsilon S_\epsilon$.

In the TABLE 11, we describe the dual graph of $D_\infty$ and the canonical divisor $K_\infty$.

<table>
<thead>
<tr>
<th>(a,b,c,h)</th>
<th>Dual graph</th>
<th>$K_\infty = E_\infty$</th>
<th>$E_\infty^2 = 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(2,3,5:12)</td>
<td>$E_1$</td>
<td>$E_1$</td>
<td></td>
</tr>
<tr>
<td>(2,3,7:14)</td>
<td>$E_1$</td>
<td>$E_1$</td>
<td></td>
</tr>
<tr>
<td>(3,4,5:15)</td>
<td>$E_1$</td>
<td>$2E_1$</td>
<td></td>
</tr>
</tbody>
</table>

Note. The vertex in the right terminal of the graphs denotes the curve $E_\infty$, which is an elliptic curve of self intersection zero. Note that the canonical
divisor $K_\infty$ is determined by the triple $(\xi, m_- m_0)$ (Compare (2.5) Assertion.)

(3.5) Now we have the following descriptions of the surface $\widetilde{X}_t$ for $t \in S_\ell$ (cf (5.7) ii)).

i) The surface $\widetilde{X}_t$ is minimal.

ii) The geometric genus $P_g(\widetilde{X}_t)$ is equal to 1. The second Chern number $c^2_t$ is equal to 0.

iii) The Kodaira dimension of the surface is equal to 1.

iv) The complete linear system $| - \xi E_0 |$ defines an elliptic fibration of $\widetilde{X}_t$ over $\mathbb{P}^1$ such that $-\xi E_0$ is a multiple fiber and $E_1$ (in the notation of the Table II) is a $-\xi$-ple section of the fibration. (See (3.6) for details.)

Proof. i) Since $K_\infty$ is an effective elliptic curve, the adjunction relation shows that $\widetilde{X}_t$ is minimal and that $\widetilde{X}_t$ is not a ruled surface.

ii) The first Chern number $c_1 = \text{Euler number of } X = 1 + \mu + \#(\text{irreducible components of } D_a \setminus E_0) = 24$ (TABLE's 9 and 11). The second Chern number $c^2_t = K^2_\infty + E^2_\infty = 0$.

Hence the Noether's formula $P^2_g + 1 = (c^2 + c_2)/12$ implies $P^2_g = 1$.

iii) $K^2_\infty = 0$ implies that $k \neq 2$. Since $\widetilde{X}_t$ is not ruled, $X$ is only possible to be 1.

(3.6) As was stated in (3.5) iv), we see in this section that:

The complete linear system $| - \xi E_0 |$ defines an elliptic fibration of $\widetilde{X}_t$ over $\mathbb{P}^1$.

First let us see that the $| (- \xi E_0) | = 2$ and $| - \xi E_0 |$ is spanned by the constant $1 = \bar{w}/w$ and $x/w^\xi$, where $(x:y:z:w)$ is the homogeneous coordinate for the ambient weighted projective space $\mathbb{P}(a,b,c,l)$ of $\widetilde{X}_t$. (Recall that $\widetilde{X}_t$ is the resolution of $\widetilde{X}_t$ and $E_0$ is the strict transform of the divisor in $\widetilde{X}_t$ defined by $w = 0$ (cf (5.8))). Since deg$(w) = 1$, homogeneous polynomials in $(x,y,z,w)$ of degree less or equal than $-\xi$ is either one of $1, \bar{w}, \ldots, \bar{w}^\xi, x$. Hence the complete linear system $| - \xi E_0 |$ is contained in the space spanned by $1$ and $x/w^\xi$. In fact we shall see by explicite calculations of each cases, the function $x/w^\xi$ is holomorphic on the exceptional set of the resolution $\widetilde{X}_t \rightarrow X_t$. Before we describe each individual cases, we summerize some generality of the fibration as a statement, which are verifified by case by case.

i) The rational function $x/w^\xi$ on $\widetilde{X}_t$ for $t \in S_\ell$ defines a flat morphism:

$$\pi = x/w^\xi : \widetilde{X}_t \rightarrow \mathbb{P}^1.$$
ii) The fiber \( \pi^{-1}(\infty) \) is \( -\varepsilon E_\infty \).

iii) The restriction of \( \pi \) on the curve \( E_1 \subset \mathbb{P}_2 \) defines a \(-\varepsilon\)-fold covering of \( \mathbb{P}^1 \) which is branching at \( \infty \) of order \(-\varepsilon\) and at some other points.

iv) The general fibers of \( \pi \) are elliptic curves.

v) In the following we figure the singular fibers of the fibrations \( \pi: \mathbb{X}_r \rightarrow \mathbb{P}^1 \) for \( r \in S \setminus \{0, \ell \} \) is the degree zero part of the parameterspace \( S \).

\[(2,3,5;12) \text{ equation: } x^5 + y^5 + xz^2 + z^2y - w^2 = 0.
\]

- **Case \( \lambda = 0 \)**
  - **Location**: singular fiber
  - \( x/w^2 = 0 \): a union of 5 smooth rational curves, intersecting in \( \tilde{D}_e \) diagram.
  - \( x/w^2 = 1 \): two smooth rational curves contacting at a point.
  - \( x/w^2 = \infty \): 2 multiple of the elliptic curve \( E_\infty \).

- **Case \( \lambda \neq 0 \)**
  - **Location**: singular fiber
  - \( x/w = 0 \): a union of 5 smooth rational curves, intersecting in \( \tilde{D}_e \) diagram.
  - \( (x/w^2)^2 = 1 \): a rational curve with a node.
  - \( (1 - \lambda^2)(x/w^2)^4 = 1 \): two smooth rational curves crossing at two points.
  - \( x/w = \infty \): 2 multiple of the elliptic curve \( E_\infty \).

\[(2,3,7;14) \text{ equation: } x^7 + x^2y^2 + z^2 + x^2yz - w^2 = 0.
\]

- **Case \( \lambda = 0 \)**
  - **Location**: singular fiber
  - \( x/w^2 = 0 \): two smooth rational curves contacting at 0 on \( E_1 \).
  - \( (x/w^2)^2 = 1 \): two smooth rational curves contacting at a point.
  - \( x/w^2 = \infty \): 2 multiple of the elliptic curve \( E_\infty \).
case $\lambda \neq 0$

location : singular fiber

$x/w = 0$ : two smooth rational curves contacting at 0 on $E_1$.

$(x/w)^2 = 1$ : a rational curve with a node.

$(1 - \frac{2^2}{2}) (x/w)^2 = 1$ : two smooth rational curve crossing at two points.

$x/w = \infty$ : 2 multiple of the elliptic curve $E_{\infty}$.

---

(3.4.5:15) equation: $x^5 + x^3 + z^3 + x^2 y z - w^4 = 0$.

case $\lambda = 0$

location : singular fiber

$x/w^3 = 0$ : three smooth rational curves crossing at 0 on $E_1$.

$(x/w^3)^2 = 1$ : three smooth rational curves crossing at a point.

$x/w^3 = \infty$ : 2 multiple of the elliptic curve $E_{\infty}$.

---

case $\lambda \neq 0$

location : singular fiber

$x/w^3 = 0$ : three smooth rational curves crossing at 0 on $E_1$.

$(x/w^3)^2 = 1$ : a rational curve with a node.

$(1 - \frac{2^2}{2}) (x/w^3)^2 = 1$ : three smooth rational curve forming a triangle.

$x/w^3 = \infty$ : 2 multiple of the elliptic curve $E_{\infty}$.
§4. The class for the smallest exponent $E$ equals to $-2$

In this paragraph we study surfaces for regular system of weights $(a,b,c;h)$ such that $E = a + b + c - h = -2$. According as the multiplicity $a_0$ of zero exponent is 0, 1 or >1, the surface is $K_3$, of Kodaira dim 1 or general type (see (4.5)).

(4.1) In the Table 13, we list up reduced regular system of weights with $E = -2$. (Due to the general inequality $-E + 1 \geq \min(a,b,c)$ (cf (5.5.7), (24)), we have only three cases $\min(a,b,c) = 1, 2$ or 3. Detailed calculations are cumbersome and omitted.)

According to the multiplicities of exponents, they are divided into groups.

<table>
<thead>
<tr>
<th>Table 13.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(a,b,c;h)$</td>
</tr>
<tr>
<td>------------</td>
</tr>
<tr>
<td>3, 10, 15:30</td>
</tr>
<tr>
<td>3, 7, 12:24</td>
</tr>
<tr>
<td>3, 7, 9:21</td>
</tr>
<tr>
<td>3, 5, 10:20</td>
</tr>
<tr>
<td>3, 5, 7:17</td>
</tr>
<tr>
<td>3, 5, 5:15</td>
</tr>
<tr>
<td>3, 4, 12</td>
</tr>
<tr>
<td>2, 3, 7:14</td>
</tr>
<tr>
<td>2, 3, 5:12</td>
</tr>
<tr>
<td>1, 6, 9:18</td>
</tr>
<tr>
<td>1, 5, 8:16</td>
</tr>
<tr>
<td>1, 5, 4:15</td>
</tr>
<tr>
<td>1, 3, 6:12</td>
</tr>
<tr>
<td>1, 3, 5:11</td>
</tr>
<tr>
<td>1, 3, 2:9</td>
</tr>
<tr>
<td>1, 2, 5:10</td>
</tr>
<tr>
<td>1, 2, 3:8</td>
</tr>
<tr>
<td>1, 1, 4:8</td>
</tr>
<tr>
<td>1, 1, 3:7</td>
</tr>
</tbody>
</table>
Here recall the convention that \( u \times v \) means \( u \ldots u \) (\( v \)-copies).

(4.2) The polynomial \( f(x, y, z, \lambda) \), \( (m_-, m_0, m_+) \) and \( (\mu_-, \mu_0, \mu_+) \)

Let \((a, b, c, h)\) be the system of weights of TABLE 13. In the TABLE 14., we shall give a weighted homogenous polynomial \( f(x, y, z, \lambda) \) with \( m_{\lambda} \)-number of parameters for the weights, where \( m_+, m_0 \) and \( m_- \) are the numbers of parameters of an universal unfolding of \( f \) with positive, zero and negative weights respectively(5.7.2).

The first 7 systems of TABLE 8, is already treated in TABLE 2, and are omitted.

**TABLE 14.**

<table>
<thead>
<tr>
<th>(a,b,c,h)</th>
<th>( \mu )</th>
<th>( \mu_+ ), ( \mu_0 ), ( \mu_- )</th>
<th>( m_- ), ( m_0 ), ( m_+ )</th>
<th>polynomial</th>
</tr>
</thead>
<tbody>
<tr>
<td>(2,3,7,14)</td>
<td>22</td>
<td>2,2,18</td>
<td>3,1,18</td>
<td>( x(x^2-y^2)(x^2-y^2) + z^2 ) ( \lambda \neq 0,1 )</td>
</tr>
<tr>
<td>(2,3,5,12)</td>
<td>21</td>
<td>2,2,17</td>
<td>3,1,17</td>
<td>( x(x^2-y^2)(x^3-y^2) + z^2x ) ( \lambda \neq 0,1 )</td>
</tr>
<tr>
<td>(1,6,9,18)</td>
<td>34</td>
<td>4,2,28</td>
<td>4,1,29</td>
<td>( y(x^3-y)(x^3-y^2) + z^2 ) ( \lambda \neq 0,1 )</td>
</tr>
<tr>
<td>(1,5,8,16)</td>
<td>33</td>
<td>4,2,27</td>
<td>4,1,28</td>
<td>( xy(x^3-y)(x^2-y^2) + z^2 ) ( \lambda \neq 0,1 )</td>
</tr>
<tr>
<td>(1,5,7,15)</td>
<td>32</td>
<td>4,2,26</td>
<td>4,1,27</td>
<td>( y(x^3-y)(x^3-y^2) + x^2z ) ( \lambda \neq 0,1 )</td>
</tr>
<tr>
<td>(1,3,6,12)</td>
<td>33</td>
<td>4,2,27</td>
<td>5,2,26</td>
<td>( z^2(x-y)(x^3-y)(x^3-y^2) + z^2 ) ( \lambda \neq 0,1 )</td>
</tr>
<tr>
<td>(1,3,5,11)</td>
<td>32</td>
<td>4,2,26</td>
<td>5,2,25</td>
<td>( (x^3-y)(x^3-y^2)(x^3-y^2) + y^2z + z^2 ) ( \lambda \neq 0,1 )</td>
</tr>
<tr>
<td>(1,3,3,9)</td>
<td>32</td>
<td>4,2,26</td>
<td>6,3,23</td>
<td>( x+y+y+z+(y+yz+z)x )</td>
</tr>
<tr>
<td>(1,2,5,10)</td>
<td>36</td>
<td>4,4,28</td>
<td>6,3,27</td>
<td>( y(x^2-y)(x^2-y^2)(x^2-y^2)(x^2-y^2) + z^2 ) ( \lambda \neq 0,1 )</td>
</tr>
<tr>
<td>(1,2,3,8)</td>
<td>35</td>
<td>4,4,27</td>
<td>7,4,24</td>
<td>( x^2z + \frac{y}{\lambda}(y-\lambda x^2) + z^2y ) ( \lambda \neq 0,1 )</td>
</tr>
<tr>
<td>(1,1,4,8)</td>
<td>49</td>
<td>6,6,37</td>
<td>10,5,34</td>
<td>( xy(x-y)(x-\lambda y)(x-\lambda y)(x-\lambda y) + z^2 ) ( \lambda \neq 0,1 )</td>
</tr>
<tr>
<td>(1,1,3,7)</td>
<td>48</td>
<td>6,6,36</td>
<td>11,6,31</td>
<td>( z^2x + g(x,y)z + h(x,y) ) where ( g,h ) are homogeneous of degree 4,7 respectively.</td>
</tr>
<tr>
<td>(1,1,2,6)</td>
<td>50</td>
<td>6,8,36</td>
<td>13,8,29</td>
<td>( z^2 + g(x,y)z + h(x,y) ) where ( g,h ) are homogeneous of degree 4,6 respectively.</td>
</tr>
<tr>
<td>(1,1,1,5)</td>
<td>64</td>
<td>8,12,44</td>
<td>20,12,32</td>
<td>( f(x,y,z) ): homogeneous of degree 5</td>
</tr>
</tbody>
</table>

(4.3) **Resolution.** The minimal good resolution of the singularity \( X^* = ((x,y,z) e^3) \): \( f(x,y,z,\lambda) = 0 \) is described in (5.6). It is numerically determined by the data:

the genus \( \gamma(X^*) \) and the self intersection number \( E_0^2 \) of the central curve \( E_0^2 \), the set \( A \)

\( (d_1, \ldots, d_\nu) \) of the order of cyclic groups and \( \mu := - \delta = 2 \). (See TABLE 15.)
For a system \((a,b,c,h)\) of Table 13, the set \(A\) consists of odd integers due to (5,6,5). Hence the dual graph for the minimal good resolution of the singularity and the coefficients of the canonical divisor \(K_D\) of the singularity are as follows:

\[
E_0 \quad \begin{array}{c}
\text{r-branches}, \\
\begin{array}{c}
-1 \\
-2 \\
-3
\end{array}
\end{array}
\quad \begin{array}{c}
k_i \\
-1
\end{array}
\]

where \(k_i = (p_i - 1)/2\) \((i = 1, \ldots, r)\), \(b = -E_0^2 = 1 + a - a_0\), and \(g = \text{genus}(E_0) = 1 + b - r\).

**Table 15.**

<table>
<thead>
<tr>
<th>((a,b,c,h))</th>
<th>(A)</th>
<th>Resolution graph</th>
<th>Dual graph of (D_{oo})</th>
</tr>
</thead>
<tbody>
<tr>
<td>((2,3,7,14))</td>
<td>3</td>
<td>(E_0) (g=1) (-1) 2 (-2)</td>
<td>(-3) (g=1) (E_\infty)</td>
</tr>
<tr>
<td>((2,3,5,12))</td>
<td>5</td>
<td>(E_0) (g=1) (-1) 2 (-3)</td>
<td>(-2) (g=1) (E_\infty)</td>
</tr>
<tr>
<td>((1,6,9,18))</td>
<td>3</td>
<td>(E_0) (g=1) (-1) (-2) (-2)</td>
<td>(-3) (g=1) (E_\infty)</td>
</tr>
<tr>
<td>((1,5,8,16))</td>
<td>5</td>
<td>(E_0) (g=1) (-1) (-2) (-3)</td>
<td>(-2) (g=1) (E_\infty)</td>
</tr>
<tr>
<td>((1,5,7,15))</td>
<td>7</td>
<td>(E_0) (g=1) (-1) (-2) (-4) (-2) (-2) (-3) (g=1)</td>
<td>(E_\infty)</td>
</tr>
<tr>
<td>((1,3,6,12))</td>
<td>3,3</td>
<td>(-2) (-2) (g=1) (-2) (-2) (-2)</td>
<td>(-2) (g=1) (E_\infty)</td>
</tr>
<tr>
<td>((1,3,5,11))</td>
<td>3,5</td>
<td>(-2) (-2) (g=1) (-2) (-2) (-3)</td>
<td>(-2) (g=1) (E_\infty)</td>
</tr>
<tr>
<td>((1,3,3,9))</td>
<td>3,3,3</td>
<td>(-2) (-2) (g=1) (-2) (-2) (-2)</td>
<td>(-2) (g=1) (E_\infty)</td>
</tr>
<tr>
<td>((1,2,5,10))</td>
<td></td>
<td>(E_0) (g=2) (-1)</td>
<td>(-1) (g=2) (E_\infty)</td>
</tr>
<tr>
<td>((1,2,3,8))</td>
<td>3</td>
<td>(E_0) (g=2) (-2) (-2) (-2)</td>
<td>(-2) (g=2) (E_\infty)</td>
</tr>
</tbody>
</table>
Note. The shape of the dual graph and the coefficients of canonical divisor are determined by the triple \((\xi,m_\infty,m_0)\) except for the pair \((1,3;3;9)\) and \((1,2;5;10)\), which are already distinguished by \(a_0\) (the multiplicity of zero exp.) (cf. (2.4) Ass.).

(4.4) Compactifications. The compactifications \(\tilde{X}_t\) of the Milnor fiber \(X_t\) \((t \in S_{cr,SF})\)
are described in (5.8). \(\tilde{X}_t\) is a union \(\tilde{X}_t \cup D_{oo}\) of the resolution \(\tilde{X}_t\) of the Milnor
fiber and the divisor \(D_{oo}\) at infinity. The canonical divisor of \(\tilde{X}_t\) is a sum \(K_{\tilde{X}} + \sum_{x \in X_t} K_x\n\)
such that \(\text{supp}(K_{\tilde{X}}) \subset D_{oo}\) and the second term \(K_X\) is zero for \(t \in SF\).

Let us describe more details for the case of \(\xi = -2\).

Assertion i) The dual graph of the divisor \(D_{oo}\) is the following (See TABLE 15.):

\[
\begin{array}{cccccc}
 & & & & & \\
 & & & & & \\
 & & & & & \\
 & & & & & 2 - \frac{3}{g-1} \quad E_\infty \quad \text{where} \quad k_1 := (p-1)/2 \quad \text{for} \quad \xi_{pl} \\
 & & & & & \\
 & & & & & \\
 & & & & & \\
 r \text{-branches} & \cdots & \cdots & \cdots & -2 & -3 & -2 \\
 & & & & & \\
 & & & & & \\
 & & & & & \\
 & & & & & \\
\end{array}
\]

ii) \(K_{oo} = E_\infty\) and \(K_0^\xi = E_{oo}^{\xi} = g^{-1} \), where \(g := g(E_{oo}) = g(E_0) = a_0\).

Proof. i) Since \(p \in A\) is an odd integer, it has the following continued fraction:

\[
p/(p-2) = 2 - \frac{1}{1} - \frac{1}{1} - \frac{1}{2 - \frac{2}{2} \cdots - \frac{2}{2} - 3}
\]

This gives the intersection numbers for the curves on the branches of \(D_{oo}\).

ii) Let us put \(K_{oo} = E_{oo} + K'\), where \(K'\) is a divisor with support on the branches. The adjunction formula \(K_{oo}E + E = 2g(E) - 2\) implies that \(K'E = 0\) for all curves \(E\) on the branches of \(D_{oo}\). Since the intersection matrices on branches are nondegenerate, \(K' = 0\) and hence \(K_{oo} = E_{oo}\). Again applying the adjunction formula \(2g(E_{oo}) - 2 = K_{oo}E_{oo} + E_{oo}\), we obtain ii). QED

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(4.5) Summarizing those calculations above, the surfaces $\widetilde{X}_t$ (tES$_I$) are as follows.

We distinguish three cases according to $a_1 = g(E_0) - 1$.

I. $g(E_0) - 1 < 0$.

In this case $K_{E_0}$ is an exceptional curve of the first kind. The canonical bundle of the blown down surface $\widetilde{X}_t = X_t/E_{\infty}$ is trivial for tES$_I$, so that $\widetilde{X}_t$ is a K3 surface with a configuration of three lines crossing normally at a point.

This case is already studied in §3, so that we omit further details.

II. $g(E_0) - 1 = 0$.

In this case $K_{E_0} = E_{\infty}$ is a smooth elliptic curve with self-intersection zero and hence the surface is minimal. $\widetilde{X}_t$ for tES$_I$ is of Kodaira dimension 1, which has a structure of elliptic fibration over $P^1$ with $E_{\infty}$ as a regular fiber.

(That $K_{E_0}$ is an elliptic curve implies $\widetilde{X}_t$ is minimal. Then $K_{E_0}^2 = 0$ implies that the Kodaira dimension of $\widetilde{X}_t$ can not be 2. Since $\widetilde{X}_t$ can not be a ruled surface (K$_{E_0}$ is effective), the Kodaira dimension of $\widetilde{X}_t$ is only possible to be 1. The fact the irregularity q of the surface is zero (5.9) implies that $\widetilde{X}_t$ has a structure of an elliptic fibration over $P^1$ according to the classification of surfaces [4]. QED)

An explicit description of the elliptic fibration is given in (4.8).

III. $g(E_0) - 1 > 0$.

In this case $K_{E_0} = E_{\infty}$ is a smooth curve of genus $> 1$, whose self-intersection number $K_{E_0}^2 = g(E_0) - 1$ is positive.

The surface $\widetilde{X}_t$ for tES$_I$ is minimal and of general type, which satisfy the numerical equality: $P_3 = c_1^2/2 + 2$ where $P_3$ is the geometric genus and $c_1^2$ is the second Chern number of the surface (cf (4.6.2)). For this class of the surface, we refer to [4].

(For the same reasons as II, $\widetilde{X}_t$ is minimal and cannot be ruled. Then the positivity $K_{E_0}^2 > 0$ implies that $\widetilde{X}_t$ is of general type due to classification of surfaces [4].)

The numerical invariants $P_3, c_1^2$ and $c_2$ of the surface $\widetilde{X}_t$ is calculated in (4.6).

(4.6) We calculate: the first Chern number $c_1$, the second Chern number $c_1^2 = K_{E_0}^2$ and the geometric genus $P_3 = h^2(\widetilde{X})$ for the surfaces $\widetilde{X}_t$ (tES$_I$). They are easily calculated by the following formula with the data in TABLE's 14, 15, 16.
\[ c_2 := \text{Euler } \# \text{ for } \overline{x_t} = (\text{Euler } \# \text{ for } \overline{x_t}) \ast (\text{Euler } \# \text{ for } D_{\nu}) \]
\[ = (1 + \mu) \ast (2 - 2g + \#(\text{irreducible components of } D - E)) \]
\[ c_2^2 := K_2^2 = g - 1 \]
\[ P_j^* + 1 = (c_1^* + c_2^* \ast 12 (\text{Noether's formula}) \]

The following TABLE 17 gives the invariants of the surfaces and the number of the weight \(a.b.c\) which is equal to 1 for an application in (4.7).

<table>
<thead>
<tr>
<th>system of weights</th>
<th>(c_1^*)</th>
<th>(c_2^*)</th>
<th>(P_j^*)</th>
<th>#(e \in {a,b,c}, e = 1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(2, 3, 7; 14)</td>
<td>24</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>(2, 3, 5; 12)</td>
<td>24</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>(1, 6, 9; 18)</td>
<td>36</td>
<td>0</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>(1, 5, 8; 16)</td>
<td>36</td>
<td>0</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>(1, 5, 7; 15)</td>
<td>36</td>
<td>0</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>(1, 3, 6; 12)</td>
<td>36</td>
<td>0</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>(1, 3, 5; 11)</td>
<td>36</td>
<td>0</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>(1, 3, 3; 9)</td>
<td>36</td>
<td>0</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>(1, 2, 5; 10)</td>
<td>35</td>
<td>1</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>(1, 2, 3; 8)</td>
<td>35</td>
<td>1</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>(1, 1, 4; 8)</td>
<td>46</td>
<td>2</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>(1, 1, 3; 7)</td>
<td>46</td>
<td>2</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>(1, 1, 2; 6)</td>
<td>45</td>
<td>3</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>(1, 1, 1; 5)</td>
<td>55</td>
<td>5</td>
<td>4</td>
<td>3</td>
</tr>
</tbody>
</table>

As a consequence of the above table, we get the following formula.

\[(4.6.1) \quad P_j^* (\overline{x_t}) = 1 + \#(e \in \{a,b,c\}, e = 1) \quad \text{for } t \in S_t^* \]

Another consequence of the table is the following equality:

\[(4.6.2) \quad P_j^* (\overline{x_t}) = \lfloor c_2^2/2 \rfloor + 2 \]

for the last group of 7 systems of weights satisfying the condition \(a_\nu > 1\).
The canonical linear system $|K_{\alpha}|$ for the surfaces $\tilde{X}_F$ $(\alpha \in \mathbb{C})$ are as follows.

**Assertion:** The module for the linear system $|K_{\alpha}|$ is spanned by $w$ and the coordinates $(x, y, z)$ such that the corresponding weight $\delta(a, b, c)$ is equal to 1.

**Proof:** Recalling $K_{\alpha} = E_{\alpha}$, we have $P_0 = h^0(\mathcal{O}_F) = h^0(\mathcal{O}(E_{\alpha})) = \dim(\text{the space of meromorphic function on } \tilde{X}_F \text{ which may have at most a simple pole along } E_{\alpha}).$

Let us show that if the weight $(a, b, c)$ of a coordinate $(x, y, z)$, say $x$, is 1, then the meromorphic function $x/w$ belongs to the space $H^0(\tilde{X}_F, \mathcal{O}(E_{\alpha}))$. In view of the equality (4.6.1), this proves the assertion. ($\tilde{X}_F$ is not linear.)

First recall that $\tilde{X}_F$ is a resolution of the surface $\tilde{X}_F$ in $\mathbb{P}(a, b, c, 1)$ by blowing up the cyclic quotient singularities on $\tilde{X}_F$, which appear at the coordinate axis $L_x \cup L_y \cup L_z$ in $\mathbb{P}(a, b, c) = (w=0) \subset \mathbb{P}(a, b, c, 1)$. Since $E_{\alpha}$ is the strict transform of the curve $\tilde{X}_F \cap \mathbb{P}(a, b, c)$ and hence $x/w$ has simple pole along $E_{\alpha}$, we have only to show that $x/w$ does not have poles on the exceptional set of the resolution $\tilde{X}_F \to \tilde{X}_F$. The assumption on the weight $\alpha = 1$ and the description of the points $\tilde{X}_F \cap (L_y \cup L_y \cup L_z)$ (5.6.5) implies the singular points of $\tilde{X}_F$ lie only on $L_x$. If, for instance, $z=0$ at a singular point, $\tilde{X}_F$ is locally at the point a quotient of smooth $Y: = \{(x, y, w) \in \mathbb{C}^2 : f(x, y, 1) = w^h\}$ by the action of $\zeta \in \mathbb{C}^*$, $(x, y, w) \mapsto (\zeta x, \zeta^b y, \zeta w)$.

Let $(v, w)$ be a local coordinate system of $Y$ at the fixed point, on which the action of $\zeta \in \mathbb{C}^*$ is $(\zeta v, \zeta w)$ (cf.). Let us develop $x$ into a power series $\sum_{i,j} a_{ij} v^iw^j$ in the local coordinates. Since $\zeta \in \mathbb{C}^*$ acts on $x$ as $\zeta x$, the power series is a sum over the indexes $(i, j) \in \mathbb{N}_0^2$ such that $-2i + j = 1 \mod(p)$. In case $j = 0$, the condition $2i + 1 = 0 \mod(p)$ implies $i = (p-1)/2 + np$ for some $n \in \mathbb{N}_0$. If we have shown that $v^{(p-1)/2}/w$ is holomorphic on the exceptional set of the resolution of the quotient singularity, we have also shown that $x/w$ is holomorphic on the exceptional set. Let us give a sharper form for a later use.

Let $E_1, \ldots, E_k$ be the exceptional set for the minimal resolution of the cyclic quotient singularity of the type $(p, -2)$ with $k = (p-1)/2$, which are intersecting as: $E_1^{\infty} \ldots , E_k^{\infty}$. Then the rational function $v^{\frac{1}{2}}/w$ defines a pole along $\omega = 0$.

1. a rational parametrization of $E_1$, and a zero function on $E_2 \cup E_1 \cup E_2^{\infty}$. (Proof omitted)

This completes a proof of the assertion. qed
We shall describe the canonical map $\tilde{X}_t \to \mathbb{P}^1$, for each system of weights.

The details of the calculations are omitted.

$P_1(\tilde{X}_t) = 1$ for these two cases. Hence the canonical maps are constants.

(Note that the multiple $-K_{\tilde{X}}$ defines elliptic fibration (3.6).)

$P_2(\tilde{X}_t) = 2$ and $H^0(\tilde{X}_t, O(K_0)) = [1 \times w]$ for these cases. The canonical map $\pi = (x:w) : \tilde{X}_t \to \mathbb{P}^1$ defines an elliptic fibration of $\tilde{X}_t$ as follows:

i) The map $\pi$ is a flat morphism.

ii) $\pi(\infty) = E_{\infty}$.

iii) The $-3$ curves of $D_0$ (in the TABLE 15) are global sections of the map $\pi$.

iv) The general fiber of $\pi$ is a smooth elliptic curve.

v) Singular fibers for $t \in \pi^{-1}(0_x T^n \vee 0_y)$ (the degree 0 subspace of $S$) are follows.

(1.6.9:18) equation: $Y(x^2 - Y)(x^2 - \lambda Y) + z^2 - w_0^2 = 0$.

\begin{itemize}
  \item **case**: $\lambda^2 - \lambda + 1 = 0$
  \item **location**: fiber
  \item $x/w = 0$: smooth elliptic curve.
  \item $(x/w)^3 = \gamma$: a rational curve with a $(2,3)$-cusp.
  \item $x/w = \infty$: $E_{\infty}$ (a smooth elliptic curve)
  \item $E$ smooth elliptic
  \item fiber
  \item $0$, $(x/w)^3 = 0$, $\infty$

\end{itemize}

\begin{itemize}
  \item **case**: $\lambda^2 - \lambda + 1 \neq 0$
  \item **location**: fiber
  \item $x/w = 0$: smooth elliptic curve.
  \item $(x/w)^3 = \gamma$: a rational curve with a node.
  \item $(x/w)^3 = \beta$: a rational curve with a node.
  \item $x/w = \infty$: $E$ (a smooth elliptic curve)
  \item $E$ smooth elliptic
  \item fiber
  \item $0$, $(x/w)^3 = \alpha$, $(x/w)^3 = \beta$, $\infty$

\end{itemize}

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(1, 5, 8; 16) equation: \( xy(y - 1)(x - 1) + z^2 - w^4 = 0 \).

**Case 1: \( \lambda = \lambda + 1 = 0 \)**

- Location: fiber
- \( x/w = 0 \): a union of 3 smooth rational curves intersecting at a point.
- \( (x/w)^{16} = \gamma \): a rational curve with a (2, 3)-cusp.
- \( x/w = \infty \): \( E_\infty \) (a smooth elliptic curve)

\[ E_{\infty} \text{ smooth elliptic} \]

**Case 2: \( \lambda = \lambda + 1 \neq 0 \)**

- Location: fiber
- \( x/w = 0 \): a union of 3 smooth rational curves intersecting at a point.
- \( (x/w)^{16} = \alpha \): a rational curve with a node.
- \( (x/w)^{16} = \beta \): a rational curve with a node.
- \( x/w = \infty \): \( E_\infty \) (a smooth elliptic curve)

\[ E_{\infty} \text{ smooth elliptic} \]

(1, 5, 7; 15) equation: \( y(x^5 - y)(x^5 - \lambda y) + z^2 - w^4 = 0 \).

**Case 1: \( \lambda = \lambda + 1 = 0 \)**

- Location: fiber
- \( x/w = 0 \): a union of 5 smooth rational curves intersecting in \( D \).
- \( (x/w)^{15} = \gamma \): a rational curve with a (2, 3)-cusp.
- \( x/w = \infty \): \( E_\infty \) (a smooth elliptic curve)

\[ E_{\infty} \text{ smooth elliptic} \]

**Case 2: \( \lambda = \lambda + 1 \neq 0 \)**

- Location: fiber
- \( x/w = 0 \): a union of 5 smooth rational curves intersecting in \( D \).
- \( (x/w)^{15} = \alpha \): a rational curve with a node.
- \( (x/w)^{15} = \beta \): a rational curve with a node.
- \( x/w = \infty \): \( E_\infty \) (a smooth elliptic curve)

\[ E_{\infty} \text{ smooth elliptic} \]
(13.6.12) equation: \( Y(x^2 - \lambda_1 y)(x^2 - \lambda_2 y)(x - \lambda_3 y) + z^2 - w^{12} = 0 \).

**Case Location**

**Fiber**

- \( x/w = 0 \)
  - a smooth elliptic curves.

- \((x/w)^2 = 1\)
  - two smooth rational curves contacting at a point.

- \( x/w = \infty \)
  - \( E_\infty \) (= a smooth elliptic curve).

**Case Location**

**Fiber**

- \( x/w = 0 \)
  - a smooth elliptic curves.

- \((x/w)^2 = 1\)
  - two smooth rational curves contacting at a point.

- \( x/w = \infty \)
  - \( E_\infty \) (= a smooth elliptic curve).

![Diagram](image)

(13.5.11) equation: \( x^2 y(x^3 - \lambda_1 y)(x^3 - \lambda_2 y) + y^2 z + xz^2 - w^{11} = 0 \).

**Case Location**

**Fiber**

- \( x/w = 0 \)
  - two smooth rational curves contacting at a point.

- \((x/w)^2 = 1\)
  - two smooth rational curves contacting at a point.

- \( x/w = \infty \)
  - \( E_\infty \) (= a smooth elliptic curve).

**Case Location**

**Fiber**

- \( x/w = 0 \)
  - two smooth rational curves contacting at a point.

- \((x/w)^2 = 1\)
  - two smooth rational curves contacting at a point.

- \( x/w = \infty \)
  - \( E_\infty \) (= a smooth elliptic curve).
case location fiber
\[ x/w = 0 \] a smooth elliptic curves.
\[ (x/w) = \]
\[ (x/w) = \]
\[ (x/w) = \]
\[ x/w = \] E (= a smooth elliptic curve).

(1.3.3:9) equation: \[ XY(X - Y)(X - Y)(X - Y) + Z - w = 0. \]

\[ C \]
\[ \infty \]
\[ E_{\infty} \]
\[ 0 \]
\[ \infty \]

case location fiber
\[ x/w = 0 \] a smooth elliptic curves.
\[ (x/w) = 1 \] three smooth rational curves crossing at a point.
\[ x/w = \infty \] \[ E_{\infty} (= \text{a smooth elliptic curve}). \]

case location fiber
\[ x/w = 0 \] a smooth elliptic curves.
\[ (x/w) = \]
\[ (x/w) = \]
\[ x/w = \infty \] \[ E_{\infty} (= \text{a smooth elliptic curve}). \]

case location fiber
\[ x/w = 0 \] a smooth elliptic curves.
\[ (x/w) = \]
\[ (x/w) = \]
\[ (x/w) = \]
\[ x/w = \infty \] E (= a smooth elliptic curve).

[Figure]

\[ 0 \]
\[ \infty \]

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(1,2,5:10),(1,2,3:8)

\( P(X) = 2 \) for these two cases. The linear system \( |K| \) has a fixed point on \( E \). By blowing up \( X \rightarrow \lambda \) at that fixed point, whose exceptional set will refer as \( E \), we obtain a fibration of \( \lambda : X \rightarrow \mathbb{P} \). The general fiber of is a genus 2 curve and the exceptional set \( E \) is a global section. The singular fibers for the special point \( \lambda \) is as follows.

(1,2,5:10)

---

(1,2,3:8)

---

(1,1,4:8),(1,1,3:7),(1,1,2:6)

\( P(X) = 3 \) for these 3 cases and \( H(X, (K)) \equiv 1, x/y, y/w \). The canonical map

\((x,y,w): X \rightarrow \mathbb{P}\) defines a covering, whose degree and discriminant are as follows:

(1,1,4:8) equation: \( Z + g(x,y,w)Z + h(x,y,w) = 0 \), where \( g \) is homogeneous of degree 8.

is a double covering branching along \( g = 0 \).

The discriminant \( : = -4g \) is homogeneous of degree 8.

(1,1,3:7) equation: \( XZ + g(x,y,w)Z + h(x,y,w) = 0 \), where \( g \) and \( h \) are homogeneous of degree 4 and 7 respectively.

is a double covering of \( \mathbb{P} \) branching along a degree 8 curve.

The discriminant \( := g - 4xh \) is homogeneous of degree 8.

(1,1,2:6) equation: \( Z + g(x,y,w)Z + h(x,y,w) = 0 \), where \( g \) and \( h \) are homogeneous of degree 4 and 6 respectively.

is a triple covering of \( \mathbb{P} \) branching along a degree 12 curve.

The discriminant \( := h - g \) is homogeneous of degree 12.

(1,1,1:5) equation \( f(x,y,z,w) = 0 \), where \( f \) is homogeneous of degree 5.

\( P(X) = 4 \) and \( H(X, (K)) \equiv 1, x/y, y/w, z/w \) for this case. The canonical map

\((x:y:z:w): X \rightarrow \mathbb{P}\) defines an embedding of \( X \) as a quintic surface in \( \mathbb{P} \).
§5 Weighted homogenous singularity of dimension two

(5.1) This § is a review on the weighted homogeneous singularities of dimension two, studied by V.I. Dolgachev, E. Looijenga, P. Orlik, H. Pinkham, P. Wagreich, J. Wahl and the author. We describe uniformization, resolution, compactification of Milnor fibers for mainly hypersurface cases in connection with regular system of weights to fix notations for §'s 2, 3 and 4. Many of the results are well-known or elementary so that we give only references or sketchy proofs.

(5.2) Cyclic extensions of $\mathrm{PSL}(2, \mathbb{R})$ and their action on $\mathcal{M}_d$.

In the following, we present a weighted homogeneous singularity $\chi_0$ as a quotient variety by a splitting factor for a cyclic extension of a Fuchsian group (5.4.1). This is a reformulation of a presentation of a quasi-homogeneous singularity by a use of automorphic forms by Dolgachev [7], Wagreich [35].

i) Let $\mathcal{H} := \{z \in \mathbb{C} : \text{Im}(z) > 0\}$ be the complex upper half plane. As usual $\text{Aut}(\mathcal{H})$ is isomorphic to $\mathrm{PSL}(2, \mathbb{R})/\{+1\}$ by $g(z) := (az+b)/(cz+d)$ for $z \in \mathcal{H}$ and $g^2 \equiv -1 \mod(\mathbb{Z})$.

ii) Since $\pi_1(\text{PSL}(2, \mathbb{R})) = \mathbb{Z}$, the universal covering map defines a cyclic extension.

\[
(5.2.1) \quad 1 \longrightarrow \mathbb{Z} \longrightarrow \tilde{\text{PSL}}(2, \mathbb{R}) \longrightarrow \text{PSL}(2, \mathbb{R}) \longrightarrow 1 \quad \text{(exact)}.
\]

An element $\tilde{g}$ of $\tilde{\text{PSL}}(2, \mathbb{R})$ is represented by a pair $(g, \varphi(z))$ of an element $g$ of $\text{PSL}(2, \mathbb{R})$ and a branch $\varphi(z)$ of the function $\log((cz+d)^2)/2\pi i$ on $\mathcal{H}$. The product is given by $\tilde{g} \tilde{h} = (gh, \varphi(z) \varphi(h(z)))$ for $\tilde{g} = (g, \varphi(z))$ and $\tilde{h} = (h, \varphi(z))$.

$\tilde{\text{PSL}}(2, \mathbb{R})$ acts on the infinite cyclic covering $\mathcal{M}_d$ of the canonical $\mathbb{C}^*$-bundle of $\mathcal{H}$.

\[
(5.2.2) \quad \tilde{g}(z, \lambda) = (g(z), \lambda + \varphi(z)) \quad \text{for} \quad (z, \lambda) \in \mathcal{M}_d / \mathbb{C} \times \mathbb{C}
\]

and $\tilde{g} = (g, \varphi(z)) \in \tilde{\text{PSL}}(2, \mathbb{R})$.

iii) For a positive integer $d$, (5.2.1) induces a finite cyclic extension.

\[
(5.2.3) \quad 1 \longrightarrow \mathbb{Z}/d \longrightarrow \tilde{\text{PSL}}(2, \mathbb{R})/d \longrightarrow \text{PSL}(2, \mathbb{R}) \longrightarrow 1 \quad \text{(exact)}.
\]

An element $\tilde{g}$ of $\tilde{\text{PSL}}(2, \mathbb{R})/d$ is represented by a pair $(g, \varphi(z))$ of an element $g$ of $\text{PSL}(2, \mathbb{R})$ and a branch $\varphi(z)$ of the function $(cz+d)^2/d$ on $\mathcal{H}$.

The product is $g \ast h = (gh, \varphi(z) \varphi(h(z)))$ for $\tilde{g} = (g, \varphi(z))$ and $\tilde{h} = (h, \varphi(z))$.

The group $\tilde{\text{PSL}}(2, \mathbb{R})/d$ acts on the $\mathbb{C}^*$-bundle $\mathcal{M}_d := \mathcal{M}_d / d$ over $\mathcal{H}$.
(5.2.4) \( \tilde{g}(z,v) = (g(z), v\tilde{g}(z)) \) for \((z,v) \in H_d \cong \mathbb{H} \times \mathbb{C}^* \) and \( \tilde{g} = (g, \tilde{g}(z)) \in \text{PSL}(2,\mathbb{R})/2d. \)

The action of \( \text{PSL}(2,\mathbb{R})/2d \) on \( H \) does not have a fixed point. (If \((z_0, v)\) were a fixed point of \((g, \tilde{g}(z))\), then \(z_0\) is an elliptic fixed point of \(g\) such that \( \tilde{g}(z_0) = 1\).)

Note. Recalling the fact \( dg(z)/dz = (cz + d)^2 \), it is easy to see that the \( d \)-th power of the \( \mathbb{C}^* \)-bundle \( H_d \) over \( H \) is the canonical \( \mathbb{C}^* \)-bundle of \( H \).

(5.3) A splitting factor for a finite cyclic extension of a Fuchsian group.

Let \( \Gamma \subset \text{PSL}(2,\mathbb{R}) \) be a co-compact Fuchsian group of the first kind.

Let \( \tilde{\Gamma}_2 \) be the inverse image of \( \Gamma \) in \( \text{PSL}(2,\mathbb{R})/2d \) by the map (5.2.3) so that

\[
\begin{align*}
1 &\longrightarrow \mathbb{Z}/2d \\ \longrightarrow &\tilde{\Gamma}_2 \\ \longrightarrow &\Gamma \\ \longrightarrow &1 \quad \text{(exact)}.
\end{align*}
\]

A splitting factor of the sequence (5.3.1) is a subgroup \( \Gamma \) of \( \text{PSL}(2,\mathbb{R})/2d \) which is bijective to its image \( \Gamma \). The projection map from \( \tilde{g} = (g, \tilde{g}(z)) \in \tilde{\Gamma} \) to its second factor \( \tilde{g}(z) \) defines an automorphic factor, discussed in [8], [35], (3.1.2).

Note 1. The sequence (5.3.1) does not split in general. Even it does split, the splitting is not unique, but depends on \( d \)-torsions of the Picard variety of \( H/\Gamma \).

Note 2. If \( d = 2 \), the sequence (5.2.3) and the \( \mathbb{C}^* \)-bundle \( H_d \) are rewritten as,

\[
\begin{align*}
1 &\longrightarrow (\mathbb{Z}) \\ \longrightarrow &\text{SL}(2,\mathbb{R}) \\ \longrightarrow &\text{PSL}(2,\mathbb{R}) \\ \longrightarrow &1 \quad \text{(exact)},
\end{align*}
\]

\[
\begin{align*}
\widetilde{H}_2 &\cong \mathbb{H} \times \mathbb{C}^* \\
\cong &\widetilde{H} := \{(u,v) \in \mathbb{C}^2 : \text{Im}(u/v) > 0 \} \\
(z,v) &\longmapsto (zv,v)
\end{align*}
\]

so that the linear action of \( \text{SL}(2,\mathbb{R}) \) on \( \widetilde{H} \) induces the action (5.2.4). Hence the splitting factor is nothing but a co-compact subgroup \( \Gamma \) of \( \text{SL}(2,\mathbb{R}) \) such that \( \Gamma \cong \mathbb{Z}/2d \).

(5.4) The Gorenstein singular point with good \( \mathbb{C}^* \)-action ([8], [21], [34]).

Let \( \Gamma \subset \text{PSL}(2,\mathbb{R})/2d \) be a splitting factor of (5.3.1), which acts on \( H \) proper and fixed point free so that \( H_d/\Gamma \) is a complex two manifold. By adding a point, put

\[
\begin{align*}
\chi_0 &:= (0) \cup H_d/\Gamma \cong \\
&
\end{align*}
\]

1. \( \chi_0 \) has naturally a structure of affine algebraic variety with an isolated normal singular point at 0 such that

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i) \( \chi_0 \) admits a good \( C^\ast \)-action (i.e., \( 0 \notin \chi_0 \) is in the closure of every orbit [19].)

ii) \( \chi_0 \) is normal Gorenstein variety so that there is no-where vanishing holomorphic 2-form \( \omega \) on \( \chi_0(0) \) such that the \( C^\ast \)-action induces,

\[
\tau^\ast(\omega) = \tau^{-d} \omega, \quad \text{for} \quad \tau \in \mathbb{C}^\ast(0).
\]

2. Conversely if \( \chi_0 \) is a two dimensional variety with an isolated singular point 0 satisfying the above i), ii) and \( d > 0 \), then it is expressed as (5.4.1) for a suitable Fuchsian group \( \Gamma \) and its splitting factor \( \Gamma^\ast \).

Proof. i) Let \( \Gamma' \) be a finite index normal subgroup of \( \Gamma \), which has no fixed point on \( \mathbb{H} \) (cf [3],[10]) and let \( \Gamma'^\ast \) be the corresponding subgroup of \( \Gamma^\ast \). Then \( \mathbb{H}_d/\Gamma'^\ast \) is a \( C^\ast \)-bundle over \( \mathbb{H}/\Gamma' \), whose associated line bundle \( (\mathbb{H}/\Gamma')^\prime \mathbb{U}(\mathbb{H}_d/\Gamma'^\ast) \) is negative, since its \( d \)-th power is the canonical bundle of the curve \( \mathbb{H}/\Gamma' \) (cf (5.2) Note.).

Hence the zero-section \( \mathbb{H}/\Gamma' \) of the bundle can be blow down to a point 0, to obtain an affine variety \( (0) \cup \mathbb{H}_d/\Gamma'^\ast \), on which still the finite group \( \Gamma/\Gamma'^\ast \) acts in a natural manner where 0 is the only fixed point of the action. Thus \( (0) \cup \mathbb{H}_d/\Gamma'^\ast = ((0) \cup \mathbb{H}_d/\Gamma'^\ast)/(\Gamma/\Gamma') \) naturally obtains a structure of an affine variety with an isolated singular point at 0, which is normal by definition.

ii) The \( C^\ast \)-action on the bundle \( \mathbb{H}_d/\Gamma^\ast \) naturally induces the \( \chi_0 \) \( C^\ast \)-action on \( \chi_0 \).

iii) The holomorphic two form on \( \mathbb{H}_d \) of the following form:

\[
\omega := dv^d/\nu^d
\]

is invariant by the action of \( \tilde{PSL}(2,\mathbb{R})/3d \) (5.2.4). Hence it induces a nowhere vanishing holomorphic two form on \( \chi_0(0) = \mathbb{H}_d/\Gamma^\ast \), denoted again by \( \omega \). Since the singularity \( \chi_0 \) is normal two dimensional, it is Macaulay. These imply that \( \chi_0 \) is Gorenstein. The (5.4.2) follows, since the form (5.4.3) satisfies the same formula.

The fact that exponent \(-d\) in (5.4.3) is \( = 1 \) implies that \( \chi_0 \) cannot be smooth.

2. Due to Pinkham [21] (compare also [4],[11]), there exists a finite covering \( X_0' \) of \( \chi_0 \) ramifying only at 0, s.t. \( X_0' \) is obtained by blowing down of the zero section section of a negative line bundle over a curve \( C \). \( X' \) is still Gorenstein and the existence of a non-vanishing holomorphic two form implies that a power of the line bundle is the canonical bundle of the curve \( C \) ([8,Prop.1], [23,(5. )]). That \( d > 0 \) implies that Euler number of \( C < 0 \). Uniformizing the curve \( C \) by \( \mathbb{H} \) gives the proof.

\( \text{q.e.d.} \)
(5.5) Hypersurface case.

i) The germ of \( \chi_0((5.4.1)) \) near at 0 can be analytically embedded in \( \mathbb{C}^3 \) iff \( \chi_0 \) is globally embedded in \( \mathbb{C}^3 \) as a hypersurface for a weighted homogeneous polynomial \( f \).

\[(5.5.1) \quad \chi_0 := \{(x,y,z) \in \mathbb{C}^3 : f(x,y,z) = 0\},
(5.5.2) \quad f(x,y,z) = \sum_{a+b+c=k} c_{ijk} x^i y^j z^k.\]

Here weights \( a, b, c \) and \( h \) are positive integers such that

\[(5.5.3) \quad 0 < a, b, c \leq h/2, \quad \text{GCD}(a, b, c, h) = 1 \quad \text{and} \quad d = h - a - b - c.

ii) Up to a constant factor, the form \( \omega \) (5.4.3) is identified with the form.

\[(5.5.4) \quad \omega = \text{Res}[dx dy dz/f(x,y,z)]\]

2. For given weights \( (a, b, c; h) \), there exist at least one polynomial (5.5.2) having an isolated critical point at 0, iff the following rational function \( \chi(T) \) may have poles only at \( T=0 \). Its Laurent expansion at \( T=0 \) has non-negative coefficients [23].

\[(5.5.5) \quad \chi(T) := T^{-h} \frac{(T^h - 1)^2(T^h - T^b)(T^h - T^c)}{(T^b - 1)(T^b - 1)(T^c - 1)}\]

Proof. 1. Suppose the germ \( (\chi_0, 0) \) is given by the hypersurface \( g=0 \) for a \( g \in C(x,y,z) \).

The existence of a \( C^* \)-action on \( \chi_0 \) implies that \( g \) belongs to the ideal \((2g/2x, 2g/2y, 2g/2z)\) in \( C(x,y,z) \). Then there exists a local coordinate change, which brings \( g \) to a polynomial of the form (5.5.2) ([25]). The local isomorphism of the surface \( \chi_0 \)

(5.4.1) and the hypersurface (5.5.1) extends to a global isomorphism since both surfaces admit unique good \( C^* \) actions. Since \( \chi_0 \) is normal, the proportion \( \text{Res}[dx dy dz/f(x,y,z)]/\omega \), which is holomorphic nowhere vanishing on \( \chi_0 - \{0\} \),

extends to a unit function on \( \chi_0 \). Hence \( a+b+c+d-h = 0 \).

Note. For a fixed \( (a, b, c; h) \), the set of polynomials having isolated critical point at 0 is Zariski open in the set of all polynomials of the form (5.5.2).

Definition [23] 1. A system of positive integers \( (a, b, c; h) \) with \( \max(a, b, c) \leq h \) is called regular if the function \( \chi(T) \) (5.5.5) may have poles at most at \( T = 0 \).

It is called reduced if \( \text{gcd}(a, b, c, h) = 1 \) except for the type \( A \) (cf [24, (4.5)]).

2. Let us develop \( \chi(T) \) in the finite Laurent series of the form.
\[ \chi(T) = \chi^m_1 + \chi^m_2 + \ldots + \chi^m_n = \sum_{m} a_m \chi^m. \]

We call \( m_1, \ldots, m_n \) the exponents for \( (a, b, c; h) \) and \( a_m \) the multiplicity of the exponent \( m \). We have \( \mu = \sum m_a \). The smallest exponent \( = a+b+c-h \) is denoted by \( \xi \). In case \( \xi < 0 \), we shall also use a notation \( d = -\xi = h-a-b-c \) (cf (5.5), 1, i)).

Let \( (a, b, c; h) \) be any reduced regular system of weights. Then there exists always an exponent either equal to 1 or -1 [24]. Hence if \( \xi \geq 1 \), we have an inequality
\[
d + 1 \geq \min(a, b, c).
\]

(5.6) Resolutions of the singularity.

The minimal good resolution \( \tilde{\chi}_0 \rightarrow \chi_0 \) of \( \chi_0 \) at 0 is described as follows [6], [19], [21].

1) let \( \Gamma \) and \( \Gamma^e \) be the Fuchsian group and the splitting factor for \( \chi_0 \) (5.3). There is a natural map from the quotient variety \( \tilde{\chi}_0 := (\mathbb{H} \cup \mathbb{H}_d) / / \Gamma^e = \mathbb{H} / / \Gamma \cup \mathbb{H}_d / / \Gamma^e \) to \( \chi_0 = (0) \mathbb{H} / / \Gamma^e \), which is the weighted blowing up of \( \chi_0 \) at 0 and \( \mathbb{H} / / \Gamma \) is its exceptional set. Then \( \tilde{\chi}_0 \) has a cyclic quotient singularity of type \( (p, d_x) \) at \( x \mathbb{H} / / \Gamma \mathbb{C} \tilde{\chi}_0 \), where \( x \) is a fixed point of \( \Gamma \) by an isotropy subgroup of order \( p \) and \( d \) is an integer s.t. \( d_x = d \mod (p) \) and \( 0 < d_x < p \).

By resolving such cyclic quotient singularities on \( \tilde{\chi}_0 \) minimally, we obtain the minimal good resolution \( \tilde{\chi}_0 \) of \( \chi_0 \). The strict transform of \( \mathbb{H} / / \Gamma \) in \( \tilde{\chi}_0 \) is denoted by \( E_0 \) and called the central curve. Let \( A := (a_1, \ldots, a_r) \) be the set of the orders of isotropy subgroups. Then the dual graph of the resolution (defined in [19]) is as follows.

Obviously the graph is branching at the fixed points on \( \mathbb{H} / / \Gamma \cong E_0 \).

\[
\begin{align*}
E_0 & \quad \begin{array}{c}
-\overline{b_1} \\
-\overline{b_2} \\
-\overline{b_3} \\
\ldots \\
-\overline{b_r}
\end{array} \\
& \quad \begin{array}{c}
-\overline{b_1} \\
-\overline{b_2} \\
-\overline{b_3} \\
\ldots \\
-\overline{b_r}
\end{array}
\end{align*}
\]

(continued fraction), \( i = 1, \ldots, r \).

In case \( \chi_0 \) is a hypersurface for the weights \( (a, b, c; h) \), \( E_0 \) is identified with the curve in \( \mathbb{P}(a, b, c) \) defined by the equation \( f = 0 \) and the branching points set is a subset of the intersection of \( E_0 \) with the coordinate axis of \( \mathbb{P}(a, b, c) \). Then,
\[
g(E_0) = a_0. \quad \text{(Here} \ g(E_0) \text{means the genus of} \ E_0).\]

(5.6.4) \[ -E_0 \cdot E_0 = a_1 - a_0 + 1. \quad \text{(Here} \ E_0^2 E_0 \text{means the self-intersection number of} \ E_0).\]

(5.6.5) \[ A = (a(\epsilon(a, b, c); e(h)) \cup (\gcd(e, f) \ast (N(e, f) - 1)) (e, f) \epsilon(a, b, c) - \text{diagonal subset}) \]

Here
\[
N(e, f) := \left( \frac{1}{h} \right) \left( \frac{2}{e_f} \right) \frac{1}{h! (1 - e_f) (1 - f_i) T=0} \quad \text{and} \quad \text{set} := t\text{-copies of} \ s.
\]

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(Exactly the set $A$ (5.6.5) presents the set of orders of isotropy groups at the point of $E_0$ (coordinate axis of $P(a,b,c)$). Hence 1 must be deleted from $A$ if it appears.)

The $\text{Vol}(\Gamma)/2\pi := 2g(E_0)-1 + \sum_{i=1}^r (1-1/p_i)$ of the fundamental domain for $\Gamma$ is given by

$$\text{Vol}(\Gamma)/2\pi := \frac{h}{abc}.$$  

(The formula is shown similarly to the case $d = \pm 1$ [23].)

ii) Let the canonical divisor $K_0$ on $\tilde{\Sigma}_0$ of the singularity $0 \in X_0$ be defined as,

$$K_0 := \text{div}(\pi^*(\omega)) := \text{the zeros minus poles of the lifted 2-form } \pi^*(\omega) \text{ on } \tilde{\Sigma}_0.$$  

In fact $\pi^*(\omega)$ does not have zeros for a minimal good resolution so that $-K_0$ is effective (Tomari, unpublished). The coefficients of $E_0$ in $K_0$ is equal to $\ell - 1$.

(5.7) The universal unfolding for $f(x,y,z)$ and the Milnor fiber.

i) The universal unfolding of $f(x,y,z)$ (Thom [4]) is defined as a polynomial

$$F(x,y,z,t_1,t_2,\ldots,t_{\ell})$$

such that $f(x,y,z) = F(x,y,z,0,\ldots,0)$ and the partial derivatives $\frac{\partial F}{\partial t_i}$ ($i=1,\ldots,\ell$) form a $C$-bases of the Jacobi ring $C[x,y,z]/(\partial f/\partial x, \partial f/\partial y, \partial f/\partial z)$.

Since the Jacobi ring is graded ring, whose Poincare polynomial is equal to $T^C(T)$, we may assume that $F$ is a weighted homogeneous polynomial of degree $h$ with respect to $\text{deg}(x)=a$, $\text{deg}(y)=b$, $\text{deg}(z)=c$ and $\text{deg}(t_i)=m_i + \epsilon$ ($i=1,\ldots,\ell$).

Denote by $m_- \cdot m_0$ and $m_+$ the number of parameters $t_i$, whose degree is negative, zero, and positive respectively. By definition,

$$m_- = \sum_{m < -\epsilon} a_m, \quad m_0 = a_{-\epsilon}, \quad m_+ = \sum_{m > -\epsilon} a_m \quad \text{and} \quad \mu = m_- + m_0 + m_+.$$  

The equation $F = 0$ defines a family of affine algebraic surfaces

$$(5.7.3) \quad \chi_t := \{(x,y,z) \in \mathbb{C}^3 : F(x,y,z,t) = 0\} \quad \text{for } t = (t_1,\ldots,t_{\ell}) \in \mathbb{C}^\mu.$$  

Particularly $(\chi_t,0)$ for $t \in \mathbb{C}^{m_-} \times \mathbb{C}^{m_0} \times \mathbb{C}^{m_+}$ defines a family of equisingularities. The family $\chi_t$ for $t \in \mathbb{C}^{m_0} \times \mathbb{C}^{m_+}$ is studied by many authors since the surfaces are naturally completed by adding a divisor at infinity as we see in (5.8).

ii) Let us denote by $S$ (resp. $S_f$) the Zariski open subset of $t \in \mathbb{C}^{m_0} \times \mathbb{C}^{m_+}$ consisting of points $t$ s.t. $\chi_t$ has at most finite number of (resp. rational) singularities.

$\xi \xi$
A smooth fiber $X_t$ over the curve $C$ is called a Milnor fiber, whose middle homology $H_2(X_t, \mathbb{Z})$ is a free abelian group of rank $\mu$ with the intersection form $I$ of sign $(\mu_+\mu_-, \mu_0)$. 

(5.7.4) \[ \mu_+ = 2 \sum_{m < 0} a_m = 2 \sum_{m > 0} a_m, \quad \mu_0 = 2a_0 = 2a_t, \quad \mu_-= \sum_{0 < m < k} a_m. \]

(iii) The geometric genus $\rho_g(X_t, 0)$ of $X_t$ at $0$ for $t \in \mathbb{C}, \mathbb{C}^0 \mathbb{C} \mathbb{C}^0 \mathbb{C} 0$ is defined as $h^1(X_t, \mathcal{O}_t)$ for a resolution $\tilde{X}_t \rightarrow X_t$ of the singular point 0. Then, we have a formula ([27],[9]).

(5.7.5) \[ \rho_g(X_t, 0) = (\mu_+ + \mu_-)/2 = \sum_{m \leq 0} a_m. \]

(iv) a) $X_t$ is rational, \( \iff \rho_g(X_t, 0) = 0 \iff \text{All exponents are positive.} \)

b) $X_t$ is minimally elliptic, \( \iff \rho_g(X_t, 0) = 1 \iff \xi$ is the only non-positive exponent.

(5.8) The family of compact surfaces over $S$.

i) Define the weighted homogeneous polynomial $G(x, y, z, w)$ of weights $(a, b, c, 1)$, and the compact hypersurface $\mathcal{X}_t$ in $\mathbb{P}(a, b, c, 1)$ with parameter $t \in S$.

(5.8.1) \[ G(x, y, z, w) = w^a F(x/\omega^a, y/\omega^b, z/\omega^c, 0, \ldots, 0, t_1, \ldots, t_\mu). \]

(5.8.2) \[ \mathcal{X}_t := \{(x: y: z: w) \in \mathbb{P}(a, b, c, 1) : G(x, y, z, w, t) = 0\} \text{ for } t \in S. \]

$\mathcal{X}_t$ is a $C^*$ equivariant compactification of $X_t$ such that the complement $E' := \mathcal{X}_t - X_t$ is a curve isomorphic to $E_0$. The surface $\mathcal{X}_t$ has cyclic quotient singularities of type $(p, p^{-d})$ for $p \in \mathbb{A}$ along $E'$. The family (5.8.2) is analytically trivial near $E'$ so that the singularities can be resolved simultaneously for $t \in S$.

ii) Denote by $\tilde{X}_t$ the smooth surface obtained by resolving the singular points of $\mathcal{X}_t$ minimally. Let us decompose $\tilde{X}_t$ as.

(5.8.3) \[ \tilde{X}_t = X_t \cup D_\infty. \]

Here $\mathcal{X}_t$ is the minimal resolution of the affine variety $X_t$ and $D_\infty := \mathcal{X}_t - \tilde{X}_t$, called the divisor at infinity. The strict transform of $E'$ in $\tilde{X}_t$ will be denoted by $E_\infty$ and called the central curve of $D_\infty$.

The dual graph of the divisor $D$ is as follows.

(5.8.4) \[ r\text{-branches} \quad \text{Graph} \quad E_\infty \]
(5.8.5) \[ \frac{d}{\text{d}z} \left( \frac{\text{d}z}{\text{d}t} \right) = c_{i,l} - \frac{1}{c_{i,l} - \frac{1}{c_{i,m} - \frac{1}{\cdots - c_{i,s}}}} \] (continued fraction). \( i = 1, \ldots, r \)

(5.8.6) \[ \epsilon_{\omega}^2 = r - a_i + a_0 - 1. \]

iii) The canonical divisor \( K_{\tilde{X}_t} \) of \( \tilde{X}_t \) is calculated as follows.

(5.8.7) \[ K_{\tilde{X}_t} = K_{\omega} + \sum_{x \in X_t} K_x, \]

where a) \( K_x \) is the canonical divisor of the singularity \( x \) of the affine surface \( X_t \).

b) \( K_{\omega} \) is the divisor having the support on \( D_{\omega} \), whose coefficients of \( E_{\omega} \) is \( d-1 \) satisfying the adjunction relation: \( 2g(E) - 2 = K_{\omega} \cdot E + E \) for the curves \( E \) on \( D_{\omega} \).

Particularly for \( t \in S_d \), the second term vanishes so that we obtain,

(5.8.8) \[ K_{\tilde{X}_t} = K_{\omega} \quad \text{for} \quad t \in S_d. \]

(Proof of iii). A canonical divisor \( K_{\tilde{X}_t} \) of \( \tilde{X}_t \) is given by the zeros and poles of

a two form on \( \tilde{X}_t \) induced from \[ \text{Res}_{\tilde{X}_t} \left( \frac{axdydz + bydxdz + czdxdy}{w + \epsilon_{\omega}(x,y,z,w,t)} \right) \]

which is regular and non-zero on \( X_t \) and is zero of order \( d-1 \) along \( E_{\omega} \).

(5.9) Middle homology groups of \( X_t \) and \( \tilde{X}_t \).

Let \( \tilde{X}_t \) be any smooth surface obtained by blowing down some exceptional curves contained in \( D_{\omega} \) and let us denote by \( \tilde{D} \) the blow down image of \( D_{\omega} \) in \( \tilde{X}_t \).

1. The surface \( \tilde{X}_t \) for \( t \in S_d \) is simply connected. Hence the first Betti number \( b_1 \) and the irregularity \( q := \dim H^1(\tilde{X}_t, G_{\omega}) \) of the surface are zero.

2. The natural inclusion \( \tilde{X}_t \subset \tilde{X}_t \) induces an isomorphism of lattices.

(5.9.1) \[ H_2(\tilde{X}_t, \mathbb{Z}) / \text{rad}(1) = (2[\tilde{D}^1])^1, \quad \text{for} \quad t \in S_d. \]

Here \( \text{rad}(1) := \{ e \in H_2(\tilde{X}_t, \mathbb{Z}) : l(e,x) = 0 \text{ for } x \in H_2(\tilde{X}_t, \mathbb{Z}) \} \).

\( 2[\tilde{D}^1] := \) the submodule of \( H_2(\tilde{X}_t, \mathbb{Z}) \) generated by the homology classes \([E_t]\) for irreducible components \( E_t \) of \( \tilde{D} \).

3. Homology classes for irreducible components of \( \tilde{D} \) are linearly independent.

(5.9.2) \[ \text{disc} 2[\tilde{D}^1] = \pm \text{disc} H_2(\tilde{X}_t, \mathbb{Z}) / \text{rad}(1), \]

(5.9.3) \[ \text{rank } H_2(\tilde{X}_t, \mathbb{Z}) = \mu - \mu_0 + \# \{ \text{irreducible components of } \tilde{D} \}. \]
Proof. 1. Due to a theorem of Brieskorn [2], the resolution $\widetilde{X}_t$ of rational double point is homeomorphic to a smooth fiber, say $X_t$. Hence we have only to prove for the case when $X_t$ is a smooth Milnor fiber. Since $\widetilde{S} - \widetilde{X}_t$ has real codimension 2 in $\widetilde{X}_t$, one has an epimorphism $\pi_1(\widetilde{X}_t) \rightarrow \pi_1(\widetilde{S})$. The Milnor fiber $X_t$ is simply connected.

2.3. We have only to consider the case $\widetilde{X}_t = \widetilde{S}$ due to the following:

Let $S$ be a smooth surface with an exceptional curve $E$ of the first kind.

Put $S = S/E$. Then we have isomorphisms $H_2(S, \mathbb{Z}) = (\mathbb{Z}E)^\perp$ of lattices.

The natural inclusion map $\chi_t \subset \widetilde{X}_t = \chi_t \cup D_0$ induces a homomorphism,

$$(5.9.4) \quad H_2(\chi_t, \mathbb{Z}) \rightarrow H_2(\widetilde{X}_t, \mathbb{Z})$$

which is a part of the following long exact sequence,

$$0 = H_1(\widetilde{X}_t, \mathbb{Z}) \rightarrow H_1(\chi_t, X_t, \mathbb{Z}) \rightarrow H_2(X_t, \mathbb{Z}) \rightarrow H_2(\widetilde{X}_t, \mathbb{Z}) \rightarrow H_2(\widetilde{X}_t, X_t, \mathbb{Z}) \rightarrow H_1(X_t, \mathbb{Z}) = 0.$$

Here $H_2(\chi_t, X_t, \mathbb{Z}) \cong H^1(D_0, \mathbb{Z}) \cong H^1(E, \mathbb{Z})$ and $H_2(\widetilde{X}_t, X_t, \mathbb{Z}) \cong H^2(D_0, \mathbb{Z}) = 2[D_0]$.

The map $H(E, \mathbb{Z}) \rightarrow H_2(X_t, \mathbb{Z})$ is obtained by associating to a cycle $c \in H(E, \mathbb{Z})$ the total space of a $S^1$-bundle $l(c)$ over $c$ ( = the boundary of the normal disc-bundle of $c$ in $\widetilde{X}_t$).

The map $H_2(\chi_t, \mathbb{Z}) \rightarrow H^2(D_0, \mathbb{Z}) = 2[D_0]$ is obtained by taking the cup products with the homology classes $[E]$ of the irreducible components $E$ of $D_0$. Hence the kernel of the map is $(2[D_0])^\perp$. The surjectivity of the map implies the linear independence of irreducible components of $D_0$ and hence $\text{rank}(2[D_0])^\perp = \text{rank} H_2(\widetilde{X}_t, \mathbb{Z}) - \# \text{irreducible components of } D_0$.

Since the map (5.9.4) is metric preserving so that its kernel $H_1(D_0, \mathbb{Z})$ is contained in $\text{rad}(1)$. Thus we obtain a surjection, $(2[D_0])^\perp \rightarrow H_2(X_t, \mathbb{Z})/\text{rad}(1)$.

The Euler number $c_2(\widetilde{X}_t)$ of the compact surface $\widetilde{X}_t$ is calculated as

$$c_2(\widetilde{X}_t) = \text{Euler number of } X_t + \text{Euler number of } D_0$$

$$= (1 + \mu^c) + (2 - 2g(E)) + \# \{ \text{irreducible components of } D_0 - E \}$$

Recalling $c_2(\widetilde{X}_t) = 2 + \text{the second Betti number of } \widetilde{X}_t$ and $g(E) = g(E_0)$, we get an equality $\text{rank} H_2(X_t, \mathbb{Z})/\text{rad}(1) = \text{rank}(2[D_0])^\perp$, which implies the isomorphism (5.9.1), qed.

Note. The above calculation shows also the bijection of the modules,

$$\text{rad}(1) = H_2(D_0, \mathbb{Z})$$.
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\[ g(E) \]
\[ E_0 \ E_0 \]
\[ E_f \ E_f \]

\[ \# C = 22 - \text{Milnor} \]

\[ d(1 + \# \text{infinity curves}) = \lambda + r - 2 \]