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<tr>
<td>Author(s)</td>
<td>Oka, Mutsuo</td>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (1986), 595: 99-111</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1986-07</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/99526">http://hdl.handle.net/2433/99526</a></td>
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<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
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Kyoto University
Examples of Algebraic Surfaces with $q = 0$ and $p_g \leq 1$ which are Locally hypersurfaces

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§ 1. Introduction

Algebraic surfaces with $q = p_g = 0$ have been studied through pluri-canonical mappings in various papers ([3, 5, 10, 11, 9, 12, 1, 2]). The purpose of this note is to give examples of algebraic surfaces with $q = 0$ and $p_g \leq 1$ from the viewpoint of the singularity theory.

Let $\overline{M}$ be a compactification of an affine surface $M$ which is defined by

\[(1.1) \quad g(w) = w_1^a w_3^b + w_2^c w_3^d + w_3^e + 1 = 0\]

where $a > b$, $c > d$ and

\[(1.2) \quad a + b \geq c + d \geq e > 0.\]

This simple class of algebraic surfaces contains many
interesting algebraic surfaces. The fundamental group \( \pi_1(\overline{M}) \) is always a finite cyclic group ([7]). In particular, the irregularity \( q(\overline{M}) \) is zero for such \( \overline{M} \). In our previous paper ([8]) we have studied rational or K3-surfaces which are exceptional divisors of the resolutions of three dimensional Brieskorn singularities. In this paper we give five minimal surfaces of the above type with \( p_g \leq 1 \) which are not either rational or K3-surfaces. Though most of them are known surfaces, our method gives a different approach to them.

In § 2, we study a canonical way of the compactification \( \overline{M} \) of \( M \) through the toroidal embedding theory.

In § 3, we study three algebraic surfaces \( \overline{M}_1, \overline{M}_2, \) and \( \overline{M}_3 \) with \( q = p_g = 0 \). \( \overline{M}_1 \) and \( \overline{M}_3 \) are known as an Enriques surface and a Godeaux surface. \( \overline{M}_2 \) is a minimal surface with \( \pi_1(\overline{M}_2) = \mathbb{Z}/3\mathbb{Z}, \ e = 12 \) and \( K^2 = 0 \) where \( K \) is a canonical divisor and \( e \) is the Euler characteristic.

In § 4, we study two minimal surfaces \( \overline{M}_4 \) and \( \overline{M}_5 \) with \( q = 0 \) and \( p_g = 1 \). \( \overline{M}_4 \) satisfies that \( K^2 = 2, e = 22 \) and \( \pi_1(\overline{M}_4) = \mathbb{Z}/2\mathbb{Z}. \overline{M}_5 \) is a simply connected surface with \( K^2 = 1 \) and \( e = 23 \). \( \overline{M}_3, \overline{M}_4 \) and \( \overline{M}_5 \) are surfaces of general type. There are systematical studies by Todorov for \( \overline{M}_4 \) and \( \overline{M}_5 \) ([11, 12]).

§2. Compactification
Unless otherwise stated, we use the same notations as in [7, 8] throughout this paper. Let \( f_\varepsilon(z) = \sum_{i=1}^{4} a_i z_1 \cdots z_4 \) be a homogeneous polynomial. We assume that \( A_i = (a_i, \ldots, a_i) \) (\( i = 1, \ldots, 4 \)) span a three-simplex \( \Xi \).

Let \( f(z) = f_\varepsilon(z) + \sum_{i=1}^{4} z_i^n \) for a sufficiently large \( N \) and let \( V = f^{-1}(0) \). Then \( V \) has an isolated singular point at the origin and the Newton boundary \( \Gamma(f) \) is non-degenerate. Let \( \Gamma^*(f) \) be the dual Newton diagram and let \( \Sigma^* \) be a simplicial subdivision. Let \( \pi : \tilde{V} \to V \) be the associated resolution of \( V \). For each strictly positive vertex \( Q \) of \( \Sigma^* \) with \( \dim \Delta(Q) \geq 1 \), there is a corresponding exceptional divisor \( E(Q) \) of the above resolution ([7]). Let \( P = t_{(1,1,1,1)} \).

Then \( \Delta(P) = \Xi \) and \( E(P) \) is the surface in which we are interested. The birational class of \( E(P) \) does not depend on either the choice of \( N \) or on \( \Sigma^* \) but depends only on \( f_\varepsilon(z) \).

Let \( P_1, \ldots, P_4 \) be the vertices of \( \Sigma^* \) which are adjacent to \( P \) and \( \dim \Delta(P_i) \geq 2 \). We assume that \( \Delta(P_i) \cap \Xi \) is the triangle with vertices \( A_j \) for \( j \neq i \). We also assume that \( \Sigma^* \) is canonical around \( P \) on each triangle \( \Delta(P_i, P_j, P_k) \) in the sense of [7]. The fundamental group \( \pi_1(E(P)) \) is a finite cyclic group by Theorem (7.3) of [7].

Let \( M \) be the affine algebraic surface in \( \mathbb{C}^3 \) which is defined by

\[
(2.1) \quad g(w) = w_1^a w_2^b + w_2^c w_3^d + w_3^e + 1 = 0
\]

where \( a > b \) and \( c > d \) and
(2.2) \[ a + b \geq c + d \geq e > 0. \]

As the homogeneous polynomial \( f_\Sigma(z) \), we take

(2.3) \[ f_\Sigma(z) = z_1^a z_2^b + z_2^c z_3^d z_4^e + z_3^e z_4^a + z_4^{a+b} \]

where

(2.4) \[ a + b = c + d + h = e + i. \]

We will show the following.

**Theorem (2.5).** The exceptional divisor \( E(P) \) is a smooth compactification of \( M \).

**Proof.** To prove the assertion, it suffices to show that there exists a three dimensional simplex \( \sigma = (P, Q_1, Q_2, Q_3) \) in \( \Sigma^* \) such that the defining equation of \( E(P) \) in \( C_0^3 \cap C_4^4 = \{ y_0 = 0 \} \) is equal to \( g(y_{Q_1}, y_{Q_2}, y_{Q_3}) = 0. \) Let \( P_1, \ldots, P_4 \) be the vertices of \( \Sigma \) which are adjacent to \( P \) and \( \dim \Delta(P_i) \geq 2 \) as before. It is easy to see that \( P_1 \equiv t(1,0,0,0) \) and \( P_2 \equiv t(0,1,0,0) \) modulo \( \mathbb{Z} <P> \). We assume that \( P_3 \equiv t(0,\alpha,\beta,\gamma) \) modulo \( \mathbb{Z} <P> \). By the definition, \( P_3 \) satisfies the following.

(2.6) \[ b\beta = c\alpha + d\beta + h\gamma = (a + b)\gamma < e\beta + i\gamma. \]

Note that

(2.7) \[ \det (P, P_1, P_2) = 1 \]

and
(2.8) \[ \det(P, P_1, P_2, P_3) = \beta - \gamma. \]

Here \( \beta - \gamma \) is strictly positive by the inequality of (2.6) and (2.4). Thus we can take \( Q_1 = P_1, \ Q_2 = P_2 \) and

\[ Q_3 = (P_3 + \delta P_1 + \epsilon P_2 + \theta P) / (\beta - \gamma) \]

where \( \delta, \epsilon \) and \( \theta \) are integers such that \( 0 \leq \delta, \epsilon, \theta < (\beta - \gamma) \) as in Lemma (3.8) of [7]. If we replace \( P_i \) by \( P_i' = P_i + n_i P \) for some integer \( n_i \), \( \delta \) and \( \epsilon \) do not change but only \( \theta \) changes in (2.9). Thus the defining equation of \( E(Q) \) in \( C^3 \) does not change. See also the argument below. Thus we may assume that \( P_1 = t(1, 0, 0, 0) \) and \( P_2 = t(0, 1, 0, 0) \) and \( P_3 = t(0, \alpha, \beta, \gamma) \). Then the integrity of \( Q_3 \) implies that

\[ (2.10) \delta + \theta = \epsilon + \alpha + \theta = \beta + \theta = 0 \text{ modulo } \beta - \gamma. \]

Let

\[ h(y_\sigma) = y_{\sigma 1}^a y_{\sigma 3}^b + y_{\sigma 2}^{c'} y_{\sigma 3}^{d'} + y_{\sigma 3}^{e'} + 1 = 0 \]

be the defining equation of \( E(P) \) in \( C^3 \). Then we have

\[ a' = P_1(A_1) - d(P_1) = a, \]
\[ b' = Q_3(A_1) - d(Q_3) = \delta a / (\beta - \gamma), \]
\[ c' = P_2(A_2) - d(P_2) = c, \]
\[ d' = Q_3(A_2) - d(Q_3) = \epsilon c / (\beta - \gamma), \]
\[ e' = Q_3(A_3) - d(Q_3) = (P_3(A_3) - d(P_3)) / (\beta - \gamma). \]
By (2.4) and (2.6), we have the following equalities.

\[(2.11) \quad b(\beta - \gamma) = a\gamma \quad \text{and} \quad \]
\[(2.12) \quad c(\gamma - \alpha) = d(\beta - \gamma). \quad \]

Therefore we have

\[b' = b = \frac{8a}{(\beta - \gamma)} \]
\[= \frac{\beta a}{(\beta - \gamma)} \quad \text{modulo } a \quad \text{by (2.10)} \]
\[= \frac{\gamma a}{(\beta - \gamma)} \quad \text{modulo } a \]
\[= b \quad \text{modulo } a \quad \text{by (2.11)}. \]

As \(0 \leq b' < a\) and \(b < a\) by the definition, this implies \(b' = b\). Similarly we have

\[d' = \frac{8c}{(\beta - \gamma)} \]
\[= \frac{(\beta - \alpha)c}{(\beta - \gamma)} \quad \text{modulo } c \quad \text{by (2.10)} \]
\[= \frac{c(\gamma - \alpha)}{(\beta - \gamma)} \quad \text{modulo } c \]
\[= d \quad \text{modulo } c \quad \text{by (2.12)}. \]

As \(0 \leq d' < c\) and \(d < c\), we have that \(d' = d\). Finally \(e' = (P_3(A_3) - d(P_3)) / (\beta - \gamma) = e\).

Thus we have shown that \(h(w) = g(w)\), which completes the proof.

Hereafter we denote \(E(P)\) by \(\tilde{M}\). In §3 and §4, we study
algebraic surfaces $\overline{M}$ with $p_g \leq 1$. The details of the calculation for $K^2$, $e(\overline{M})$ and $\pi_1(\overline{M})$, we refer to [7] and [8].

**Remark (2.13).** Let $E'$ be the simplex in $\mathbb{R}^3$ with vertices $(a,0,b)$, $(0,c,d)$, $(0,0,e)$ and $(0,0,0)$. Let $v^1, \ldots, v^k$ be the other possible integral points in $E'$. Let

$$g_t(w) = g(w) + \sum_{i=1}^{k} t_i w_i^i$$

and let $M_t$ be defined by $g_t(w) = 0$. Let $U$ be the Zariski open set which is defined by the union of $t \in \mathbb{C}^k$ such that $g_t(w)$ is globally non-degenerate in the sense of [6]. Then $\{M_t\}_{t \in U}$ can be compactified simultaneously with $M = M_0$ and the complex manifold $\overline{M}$ which is the union $U \bigcup_{t \in U} M_t$ gives a $k$-dimensional deformation of $\overline{M}$. We call $(w_i^i)$ the embedded monomials of $g(w)$. All the numerical calculations for $\overline{M}$ which follow in §3 and §4 remain true for $\overline{M}_t$.

§ 3. **Surfaces with $q = p_g = 0$.**

In this section, we will study three minimal algebraic surfaces $\overline{M}_1$, $\overline{M}_2$ and $\overline{M}_3$ with $q = p_g = 0$. $\overline{M}_1$ is known as an Enriques surface and $\overline{M}_3$ is a Godeaux surface. $\overline{M}_2$ is a minimal surface with $\pi_1(\overline{M}_2) \cong \mathbb{Z}/3\mathbb{Z}$, $e(\overline{M}_2) = 12$ and $K^2 = 0$. Here $K$ is a canonical divisor and $e(\overline{M}_2)$ is the Euler characteristic.

(I) Let $M_1 = \{ g_1(w) = 0 \}$ where
\[ g_1(w) = w_1^4 w_3^3 + w_2^4 w_3^2 + w_3 + 1. \]

Then \( f_A(z) = z_1^4 z_3^3 + z_2^4 z_2^2 z_4 + z_2^6 + z_4 \) is the corresponding homogeneous polynomial. We may take \( P_3 = t(0, 1, 7, 3) \) and \( P_4 = t(0, -1, -6, -2) \). As \( \det(P, P_1, P_3) = \det(P, P_2, P_4) = 2 \), we need two vertices \( T_{13} = (P + P_1 + P_3)/2 \) on \( T(P, P_1, P_3) \) and \( T_{24} = (P_2 + P_4)/2 \) on \( T(P, P_2, P_4) \) respectively. Here we are only considering vertices of \( \Sigma^* \) which are adjacent to \( P \).

We denote the divisor \( E(P) \cap E(P_i) \) in \( E(P) \) by \( C(P_i) \) etc.

Let \( \sigma = (P, P_1, P_2, R) \) be the fixed three-simplex of \( \Sigma^* \) where \( R = (3P_1 + P_2 + P_3 + P)/4 = t(1, 1, 2, 1) \). Let \( \omega \) be the meromorphic two form on \( \tilde{M}_1 = E(P) \) which is defined by

\[ \frac{dy_{\sigma_1} \wedge dy_{\sigma_2} \wedge dy_{\sigma_3}}{dg_1(y_{\sigma})} \]

on \( C_0^3 \) and \( K = (\omega) \). By § 9 of [7], we get

\[(3.1) \quad K = 2C(P_4) + C(T_{24}) - 2C(P_3) - C(T_{13}), \]

\[(3.2) \quad K^2 = 0, \quad e(\tilde{M}_1) = 12 \quad \text{and} \quad \pi(\tilde{M}_1) \cong \mathbb{Z}/2\mathbb{Z}. \]

Let \( p : \tilde{M}_1 \to \tilde{M}_1 \) be the universal covering and let \( \varphi_{34} \) be the rational function on \( \tilde{M}_1 \) which is defined by \( \pi^*(z_4 z_3^{-1}) \). Then we have that

\[(3.4) \quad (\varphi_{34}) = 2K \]

Thus there is a rational function \( \psi \) on \( \tilde{M}_1 \) such that \( \psi^2 = p^* \varphi_{34} \). Then it is easy to see that \( \psi^{-1} p^* \omega \) is a nowhere vanishing two-form on \( \tilde{M}_1 \). This implies that \( \tilde{M}_1 \) is a K3-surface and \( \tilde{M}_1 \) is called an Enriques surface. (See Griffiths
(4), P.541 for the standard way of the construction of a
Enriques surface.)

g_1(w) has 6 embedded monomials \( w^{v_i} \) where \( (v_i) \) (i=1,...,6)
are (0,1,1), (0,2,1), (1,0,1), (1,2,2), (2,0,2) and (2,1,2).

(II) Let \( M_2 = \{ g_2(w) = 0 \} \subset \mathbb{C}^3 \) where

\[
g_2(w) = w_1^9 w_3^6 + w_2^3 w_3^2 + w_3 + 1
\]

Then \( E_4(z) = z_1^9 z_3^6 + z_2^3 z_3^2 z_4^{10} + z_3^3 z_4^{14} + z_4^{15} \) and
\( \mathcal{P}_3 = \mathcal{P}(0,0,5,2) \) and \( \mathcal{P}_4 = \mathcal{P}(0,-2,-14,-5). \) As
\( \det(\mathcal{P}, \mathcal{P}_1, \mathcal{P}_4) = 3, \) we need a vertex \( T_{14} = (\mathcal{P}_4 + \mathcal{P}_1 + 2\mathcal{P}) / 3 \) on
\( T(\mathcal{P}, \mathcal{P}_1, \mathcal{P}_4). \) Let \( \sigma = (\mathcal{P}, \mathcal{P}_1, \mathcal{P}_2, \mathcal{R}) \) where
\( \mathcal{R} = (\mathcal{P}_3 + 2\mathcal{P}_1 + 2\mathcal{P}_2 + \mathcal{P}) / 3. \) Then we have

\[
K = 7C(\mathcal{P}_4) + 2C(T_{14}) - 2C(\mathcal{P}_3), \quad K^2 = 0,
\]

\[
e(\mathcal{H}_2) = 12 \quad \text{and} \quad \pi_1(\mathcal{H}_2) \cong \mathbb{Z}/3\mathbb{Z}.
\]

As \( (\phi_{34}) = 9C(\mathcal{P}_4) - 3C(\mathcal{P}_3) + 3C(T_{14}), \) \( 3K \) is linearly
equivalent to \( 3C(\mathcal{P}_4). \) This easily proves that \( \mathcal{H}_2 \) is
minimal.

g_2(w) has 10 embedded monomials \( w^{v_i} \) where \( (v_i) \) (i=1,...,10)
are (1,0,1), (2,0,2), (3,0,2), (4,0,3), (6,0,4),
(0,1,1), (2,1,2), (3,1,3), (5,1,4) and (1,2,2).

(III) Let \( M_3 = \{ g_3(w) = 0 \} \) where

\[
g_3(w) = w_1^5 w_3^3 + w_2^5 w_3^2 + w_3 + 1.
\]
Then $f_\Delta(z) = z_1^5 z_3^3 + z_2^5 z_3^2 z_4 + z_3 z_4^7 + z_4^8$ and $P_3 = t(0,1,8,3)$ and $P_4 = t(0,-1,-7,-2)$. Let $\sigma = (P, P_1, P_2, R)$ where $R = (P_3 + 3 P_1 + 2 P_2 + 2 P) / 5$. Then we have

(3.9) $K = 2C(P_4) - C(P_3')$, $K^2 = 1$.

(3.10) $e(\bar{M}_3) = 11$ and $\pi_1(\bar{M}_3) \cong \mathbb{Z}/5\mathbb{Z}$.

As $3K \sim C(P_4) + 2C(P_3)$, $\bar{M}_3$ is minimal by Lemma (4.23) of [8]. $\bar{M}_3$ is a Godeaux surface. See [10, 5]. $\bar{M}_3$ is isomorphic to the surface in Example (7.12) of [7].

g_3(w) has 8 embedded monomials $w^i$ where $(w^i)$ (i = 1, ..., 8) are $(1,0,1)$, $(3,0,2)$, $(0,1,1)$, $(1,1,1)$, $(2,1,2)$, $(0,2,1)$, $(2,2,2)$ and $(1,3,2)$. As 8 is the dimension of the moduli space of the Godeaux surface ([5]), it is possible that our deformation is complete. We do not discuss this in this paper.

§4. Surfaces with $q = 0$ and $p_g = 1$

In this section, we will study three minimal surfaces $\bar{M}_4$, $\bar{M}_5$ and $\bar{M}_6$ with $q = 0$ and $p_g = 1$.

(IV) Let $M_4 = (g_4(w) = 0)$ where

(4.1) $g_4(w) = w_1^8 w_3^3 + w_2^4 w_3^2 + w_3 + 1$.

Then $f_\Delta(z) = z_1^8 z_3^3 + z_2^4 z_3^2 z_4 + z_3 z_4^7 + z_4^8$ and $P_3 = t(0,-1,11,3)$ and $P_4 = t(0,0,-5,-1)$. We need three vertices $T_{13}^1$, $T_{13}^2$ and $T_{13}^3$ on $T(P, P_1, P_3)$ where $T_{13}^1 = (P_3 + 3 P_1 + P) / 4$ and etc.. Let $\sigma = (P, P_1, P_2, R)$ where
R = (P_3 + 3P_1 + 4P_2 + 5P) / 8. Then we have

\[(4.2) \quad K = C(P_4), \quad K^2 = 2.\]

\[(4.3) \quad e(\overline{M}_4) = 22 \quad \text{and} \quad \pi_1(\overline{M}_4) \cong \mathbb{Z}/2\mathbb{Z}.\]

Thus \(p_g = 1\) and \(\overline{M}_4\) is minimal. It is known that there is an algebraic surface \(S\) with \(q = p_g = 0\) and \(\pi_1(S) \cong \mathbb{Z}/4\mathbb{Z}\) ([10]). We do not know whether our surface \(\overline{M}_4\) is the double cover of such a surface \(S\) or not.

\(g_4(w)\) has 11 embedded monomials \(w^{v_i}\) where \((v_i) (i = 1, \ldots, 11)\) are \((1,0,1), (2,0,1), (4,0,2), (5,0,2), (0,1,1), (3,1,2), (4,1,2), (0,2,1), (2,2,2)\) and \((1,3,2)\).

(V) Let \(M_5 = \{ \text{ } g_5(w) = 0 \} \) where

\[(4.7) \quad g_5(w) = w_1^6 w_3^4 + w_2^3 + w_3^2 + 1.\]

Then \(f_4(z) = z_1^6 z_3^4 + z_2^3 z_4^2 + z_3^2 z_4^8 + z_4^{10}\) and \(P_3 = t(0,2,5,2)\) and \(P_4 = t(0,-3,-4,-1)\). We need two vertices \(T^1_{13}\) and \(T^2_{13}\) on \(T(P,P_1,P_3)\), where \(T^1_{13} = (P_3 + 2P_1 + P) / 3\). We take \(\sigma = (P,P_1,P_2,T^1_{13})\) and by an easy calculation, we have

\[(4.8) \quad K = C(P_4), \quad K^2 = 1,\]

\[(4.9) \quad e(\overline{M}_5) = 23 \quad \text{and} \quad \pi_1(\overline{M}_5) = \{1\}.\]

\(g_5(w)\) has 14 embedded monomials which correspond to \((0,0,1), (1,0,1), (1,0,2), (2,0,2), (3,0,2), (3,0,3), (4,0,3), (0,1,0), (0,1,1), (1,1,1), (2,1,2), (3,1,2), (0,2,0)\) and \((1,2,1)\). There are beautiful studies by Todorov for \(\overline{M}_4\) and
References


