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Kyoto University
Topologically Extremal Real Surfaces in 
$\mathbb{P}^2 \times \mathbb{P}^1$ and $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

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From a general viewpoint we illustrate a method of construction of surfaces in $\mathbb{P}^2 \times \mathbb{P}^1$ and $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ defined over $\mathbb{R}$ having topologically extremal properties. Precisely we show that for each $d$, $e$ and $r$ there exists an $M$-surface $A$ in $\mathbb{P}^2 \times \mathbb{P}^1$ (resp. $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$) of degree $(d,r)$ (resp. $(d,e,r)$) such that the projection $A \rightarrow \mathbb{P}^1$ has the maximal number of real critical points. The construction of $M$-surfaces in $\mathbb{P}^3$ by O.Ya.Viro is also made more clear.

0. Introduction.

Harnack [H] pointed out that the number of components in the real locus of a curve in $\mathbb{P}^2$ of degree $d$ defined over $\mathbb{R}$ does not exceed $1 + (1/2)(d-1)(d-2)$ and, for each $d$, there exists a non-singular curve in $\mathbb{P}^2$ of degree $d$ defined over $\mathbb{R}$, the real locus of which has exactly $1 + (1/2)(d-1)(d-2)$ components.

Hilbert in his 16th problem proposed to investigate -1-
topological restrictions for hypersurfaces in $\mathbb{P}^n$ of fixed degree defined over $\mathbb{R}$.

One may regard an real algebraic function as an one-parameter family of hypersurfaces defined over $\mathbb{R}$, and it is natural to investigate topological restrictions for hypersurfaces in $\mathbb{P}^n \times \mathbb{P}^1$ of fixed degree defined over $\mathbb{R}$.

Let $A \subset \mathbb{P}^n \times \mathbb{P}^1$ be a real hypersurface of degree $(d, r)$, that is, the zero-locus of a polynomial $\sum_{0 \leq i \leq r} F_i (x_0, \ldots, x_n) \lambda^{-1} \mu^i$, where $F_i (0 \leq i \leq r)$ is a real homogeneous polynomial of degree $d$. Consider the projection $\varphi: A \rightarrow \mathbb{P}^1$. Our main object is the topology of real locus $A_\mathbb{R}$ of $A$ and singularities of the restriction $\varphi_\mathbb{R}: A_\mathbb{R} \rightarrow \mathbb{RP}^1$ of $\varphi$ to $A_\mathbb{R}$.

We denote by $P_t(X, K)$ the Poincaré series of a space $X$ over a field $K$ with indeterminate $t$, and by $s(f)$ the number of critical points of a function $f: X \rightarrow \mathbb{R}$ from a $n$-dimensional manifold to an one-dimensional manifold.

If $A \subset \mathbb{P}^n \times \mathbb{P}^1$ is non-singular, then the diffeomorphism type of $A$ is determined by $(d, r)$. For example,

$$P_t(A, K) = \begin{cases} \chi(A) & \text{(n:even)}, \\ 2(n+1) - \chi(A) & \text{(n:odd)}, \end{cases}$$

for any $K$,

$$\chi(A) = (n+1)(1-d)^n r + 2((1-d)^{n+1} - 1 + n+1),$$

(cf. 1.6).

We call $A$ generic if $A$ is non-singular and $\varphi: A \rightarrow \mathbb{P}^1$ has only non-degenerate critical points.
If $A$ is generic, then $s(\varphi) = (n+1)(d-1)^n r$ (cf. 1.6).

By Harnack-Thom's inequality ([G]), we have an uniform estimate:

\[
\begin{align*}
P_1(A_{\mathbb{R}}; \mathbb{Z}/2) & \leq P_1(A; \mathbb{Z}/2), \\
(s(\varphi_{\mathbb{R}})) & \leq s(\varphi).
\end{align*}
\]

(0.0)

In this note from a general viewpoint we show the following

**Theorem 0.1.** For $n = 1, 2$ and for each $(d,r)$, the estimate (0.0) is sharp, that is, there exists a generic real hypersurface of $\mathbb{P}^n \times \mathbb{P}^1$ of degree $(d,r)$ attaining both equalities in (0.0).

Notice that in the case $r = 1$ Theorem 0.1 is proved in [I]. A finer result is obtained in the case $n = 1$. For $A \subset \mathbb{P}^1 \times \mathbb{P}^1$, we denote by $\pi: A \rightarrow \mathbb{P}^1$ the projection to the first component.

**Proposition 0.2.** For non-singular real curves $A \subset \mathbb{P}^1 \times \mathbb{P}^1$ of degree $(d,e)$ such that both $\varphi$, $\pi$ have only non-degenerate critical points, there exists the sharp estimate:

\[
\begin{align*}
P_1(A_{\mathbb{R}}; \mathbb{Z}/2) & \leq 2 + 2(d-1)(e-1), \\
s(\varphi_{\mathbb{R}}) & \leq 2(d-1)e, \\
s(\pi_{\mathbb{R}}) & \leq 2d(e-1).
\end{align*}
\]

Now let us formulate a general theorem which implies Theorem 0.1.
Let $S$ be a real complex surface (cf. 2.1), $C \subset S$ be a real curve possibly with singularities. A non-singular component $E$ of $C_R^* \subset S_R^*$ is an oval (resp. an empty oval) if there exists an embedding $i : D^2 \rightarrow S_R^*$ such that $i(\overline{D^2}) = E$ (and that $i(\text{int } D^2) \cap C_R^*$ is empty).

Let $S$ be compact, $L$ a real holomorphic line bundle (cf. 2.6), $s_0, s_1$ M-sections of $L$ (cf. 2.7).

Consider the following condition (*):

(*1) The zero-loci $(s_0)_0$ and $(s_1)_0$ are both connected and of genus $g$.

(*ii) $(s_0)_0$ and $(s_1)_0$ intersect in $\langle c_1(L)^2, [S] \rangle$ points in $S_R^*$.

(*iii) The real locus of $(s_0 s_1)_0 = (s_0)_0 \cup (s_1)_0$ has $2g$ empty ovals.

We denote by $\mathbb{P}^1_L$ the real complex curve $(\mathbb{P}^1, \tau_1)$, where $\tau_1$ is the complex conjugation (cf. 2.3). Fix a pair of M-sections $\lambda, \mu$ of $\mathcal{O}_{\mathbb{P}^1_L}(1)$ such that $(\lambda)_0 \neq (\mu)_0$.

Denote by $\psi : S \times \mathbb{P}^1_L \rightarrow \mathbb{P}^1_L$, $\chi : S \times \mathbb{P}^1_L \rightarrow S$ the projections. For a transverse section $s$ of $\mathfrak{Z}^* L \otimes \mathcal{O}_{\mathbb{P}^1_L}(r)$ (cf. 1.3), denote by $\varphi : (s)_0 \rightarrow \mathbb{P}^1_L$, $\kappa : (s)_0 \rightarrow S$ the restrictions of projections. Then, associated to $s$, there is a natural section of $\text{Hom}(T(s)_0, \mathcal{O}_{\mathbb{P}^1_L})$ defined by the tangent map of $\varphi$. 

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Theorem 0.4. Let \( S \) be an \( M \)-surface with connected real part \( S_{\mathbb{R}} \), \( L \) be a real holomorphic line bundle with a pair \( s_0, s_1 \) of \( M \)-sections of \( L \) satisfying the condition (*)).

Then, for any \( r \), there exists an \( M \)-section \( s \) of \( \mathcal{I}^*L \otimes \mathcal{F}^*\mathcal{O}_{\mathbb{P}_1}^1(r) \) over \( S \times \mathbb{P}_1^1 \) near \( s_0 \otimes \lambda^r \), which associates an \( M \)-section of \( \text{Hom}(T(s), \mathcal{F}^*\mathcal{O}_{\mathbb{P}_1}^1) \) defined by the projection \( \varphi: (s)_0 \longrightarrow \mathbb{P}_1^1 \).

Explicitly, \( s \) can be taken in a form

\[
\sum_{0 \leq i \leq r} \xi_i s_1^i / \lambda^{r-i},
\]
where \( s_1 = s_0 \) (\( i \): even), \( s_1 = s_1 \) (\( i \): odd) and \( \xi_0, \xi_1, \ldots, \xi_r \) are real numbers with \( 1 = \xi_0 \gg |\xi_1| \gg \cdots \gg |\xi_r| > 0 \).

Remark 0.5. A sufficient condition for the existence of a pair of \( M \)-sections satisfying (*) is given in section 4. Theorem 0.4 with this sufficient condition implies immediately Theorem 0.1 in the case \( n = 2 \).

Putting \( S = \mathbb{P}_1^1 \times \mathbb{P}_1^1 \) (\( = \mathbb{P}_1^1 \times \mathbb{P}_1^1 \)) and \( L = \mathcal{O}_{\mathbb{P}_1^1}(d) \otimes \mathcal{O}_{\mathbb{P}_1^1}(r) \) over \( S \), we have

Corollary 0.6. For non-singular real surface \( A \subset \mathbb{P}_1^1 \times \mathbb{P}_1^1 \times \mathbb{P}_1^1 \) of degree \( (d,e,r) \) such that \( \varphi: A \longrightarrow \mathbb{P}_1^1 \) has only non-degenerate critical points, there exists the sharp estimate:

\[
\begin{align*}
P_1(A; \mathbb{Z}/2) & \leq 6de-4de-4er-4rd+4d+4e+4r, \\
s(\varphi) & \leq (6de-4d-4e+4)r.
\end{align*}
\]
From Theorem 0.4, it naturally arises the following general problem:

Problem 0.7. Let $E$ be a real holomorphic vector bundle over a real complex manifold. Give a criterion for the existence or the non-existence of $M$-sections of $E$.

Lastly we intend to clarify the construction of $M$-surfaces in $\mathbb{P}^3$ by Viro [V].

**Theorem 0.8.** (Viro) For non-singular real surfaces $A$ in $\mathbb{P}^3$ of degree $d$, there exists the sharp estimate:

$$P_1(A_\mathbb{R}; 3/2) \leq d^3 - 4d^2 + 6d.$$ 

Let $X_0, X_1, X_2, X_3$ be homogeneous coordinates of $\mathbb{P}^3$. Put $\mathbb{P}^2 = \{X_3 = 0\}$, $\mathbb{P}^1 = \{X_3 = X_3 = 0\}$ and $\mathfrak{l} = \{X_0 = X_1 = 0\}$.

Let $\phi: \mathbb{P}^3 - \mathfrak{l} \rightarrow \mathbb{P}^1$ be a projection. Fix a tubular neighborhood $U$ of $\mathfrak{l}$ in $\mathbb{P}^3$ such that $\overline{U} \cap \mathbb{P}^1$ is empty.

Observe that for each $d$ there exist $M$-sections $s_0, \ldots, s_d$ of $O_{\mathbb{P}^2}(0), \ldots, O_{\mathbb{P}^2}(d)$ near $X_2^0, \ldots, X_2^d$ respectively such that $(s_{i+1})_0$ and $(s_{i+1})_0$ intersect in $i(i+1)$ points in $\mathbb{RP}^2$, the real locus of $(s_{i+1})_0$ has $(1/2)(i-1)(i-2) + (1/2)i(i-1)$ empty ovals ($0 \leq i \leq d-1$) and $\phi((s_{i+1})_0$ has $(i-1)i$ real critical points ($0 \leq i \leq d$). Naturally each $s_i$ is extended to a section $\tilde{s}_1$ of $O_{\mathbb{P}^3}(1)$ ($0 \leq i \leq d$).
Put \( s = \sum_{0 \leq i \leq d} \xi_i x_2^i x_1^{d-i} \in H^0(\mathcal{O}^3, \mathcal{O}_3(d))_{\mathbb{R}}, \) and \( A = (s)_0. \)

Take real numbers \( \xi_0, \ldots, \xi_d \) to be \( 1 = \xi_0 \gg |\xi_1| \gg \cdots \gg |\xi_d| > 0 \)

and of appropriate signs.

\( \varphi_{\mathcal{R}}: A_{\mathbb{R}} \to \mathbb{R}^1 \) defines a vector field \( \mathfrak{j}' \) over \( A_{\mathbb{R}} \). \( \mathfrak{j}' \) is extended to a vector field \( \mathfrak{j} \) over \( A_{\mathbb{R}} \) with finite singularities.

Denote by \( s^+(\mathfrak{j}) \) (resp. \( s^-(\mathfrak{j}) \)) the sum of positive (resp. negative) indices of singular points of \( \mathfrak{j} \), and put

\[ t_1 = \dim H_1(A_{\mathbb{R}}; \mathbb{Z}/2) \quad (i=1,2,3). \]

Then we see

\[
s^+(\mathfrak{j}) \geq d + (1/3)d(d-1)(d-2),
\]

\[
s^-(\mathfrak{j}) \geq (1/3)(d+1)d(d-1) + (1/3)d(d-1)(d-2).
\]

Thus \( \chi(A_{\mathbb{R}}) = s^+(\mathfrak{j}) - s^-(\mathfrak{j}) \geq d - (1/3)(d+1)d(d-1) \). On the other hand \( t_0 + t_1 \geq 2 + (1/3)(d-1)(d-2)(d-3) \). Hence we have

\[
P_1(A_{\mathbb{R}}; \mathbb{Z}/2) = t_0 + t_1 + t_2
\]

\[
= 2(t_0 + t_2) - \chi(A_{\mathbb{R}})
\]

\[
\geq d^3 - 4d^2 + 6d \quad (= p_1(A; \mathbb{Z}/2)).
\]

By Harnack-Thom's inequality, all equalities are hold.

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1. Preliminary: Complex Topology.

(1.0) Let $X$ be a complex manifold, $\pi: E \longrightarrow X$ a holomorphic vector bundle and $s: X \longrightarrow E$ a holomorphic section. Put $(s)_0 = \{x \in X \mid s(x) = 0\}$.

We call $s$ transverse if $s$ is transverse to the zero section $\mathcal{Z} \subset E$, that is, for any $s \in (s)_0$, $s^* T_X \otimes T_s(x) \cong T_s(x)^\perp$.

If $s$ is transverse, then $(s)_0$ is a complex submanifold of $X$.

Denote by $H$ the complex vector space $H^0(X, E)$ of totality of holomorphic sections of $E$ over $X$, and by $PH$ the projectification of $H$.

Put $Z = \{(x, [s]) \in X \times PH \mid s(x) = 0\}$ and consider the projection $\bar{\Phi}: Z \longrightarrow PH$. Then $s$ is transverse if and only if $Z$ is non-singular along $\bar{\Phi}^{-1}[s]$ and $\bar{\Phi}$ is submersive over $[s]$.

In particular, for transverse sections $s, s' \in H$, $(s)_0$ and $(s')_0$ are diffeomorphic.

(1.1) Let $s \in H^0(X, E)$ be transverse. Put $Z = (s)_0$. Then we have an exact sequence

$$0 \longrightarrow TZ \longrightarrow TX|_Z \longrightarrow E|_Z \longrightarrow 0,$$

of complex vector bundles. Therefore $c_t(TX|_Z) = c_t(TZ)c_t(E|_Z)$ for Chern polynomials. The Chern classes of $TZ$ are calculated by the formula $c_t(TZ) = \frac{c_t(TX|_Z)}{c_t(E|_Z)}$ (cf. [F]).
(1.2) Let $L$ be a holomorphic line bundle over a complex manifold $V$ of dimension $n$. Let $Z$ be the zero-locus of a transverse section of $L$. Then by (1.1),

$$\chi(Z) = \langle \sum_{1+j=n+1} (-1)^j c_1(TV)(c_1(L))^j + 1, [V] \rangle.$$

For example, if $\dim V = 2$, then

$$\chi(Z) = \langle c_1(TV)c_1(L) - c_1(L)^2, [V] \rangle.$$

Furthermore, if $Z$ is connected, then

$$\chi(Z) = 1 + (1/2)\langle c_1(L)^2 - c_1(L)c_1(TV), [V] \rangle.$$

(1.3) Let $R$ be a non-singular curve of genus $g$. Denote by $\pi: V \times R \to V$ and $\psi: V \times R \to R$ the projections. Put $L' = \pi^*L \otimes \psi^*O_R(r)$ over $V \times R$ for each $r$. Let $A \subset V \times R$ be the zero-locus of a transverse section of $L'$.

Then $\chi(A) = \langle \psi, [V] \rangle$, where

$$\psi = r c_n(TV) + \sum_{1+j=n, j>0} ((j+1)r+2g-2)c_1(TV)(-c_1(L))^j,$$

as an element of $H^{2n}(V; \mathbb{Z})$.

For example, if $\dim V = 2$, then

$$\chi(A) = \langle rc_2(TV) - (2r+2g-2)c_1(TV)c_1(L) + (3r+2g-2)c_1(L)^2, [V] \rangle.$$

(1.4) Example. Let $C, C'$ and $C''$ be non-singular curves.
of genus $g, g'$ and $g''$ respectively. Put $X = C \times C' \times C''$, and denote projections by $p_1, p_2$ and $p_3$ to $C, C'$ and $C''$ respectively. Let $A \subset X$ be the zero-locus of a transverse section of $L' = p_1^*C_1(d) \oplus p_2^*C_1(d) \oplus p_3^*C_1(d'')$. Then $\chi(A)$ is equal to $6(d-1)(d'-1)(d''-1) + (2+4g')(d-1)(d'-1) + (2+4g)(d''-1) + (2+4g')(d''-1)(d-1) + (2+4g'g')(d''-1) + 6 - 4(g+g'+g'') + 4(gg'+g''+g')$.

(1.5) In (1.3), denote by $\varphi: A \longrightarrow \mathbb{P}$ the projection to $\mathbb{P}$. Put $\tilde{\varphi} = \text{Hom}(TA, \varphi^*T\mathbb{P})$. Then $\langle c_n(\tilde{\varphi}), [A] \rangle = \langle \gamma, [V] \rangle$, where

$$\gamma = (-1)^n r \sum_{1+j=n} (j+1)c_1(TV)(-c_1(L))^j,$$

as an element of $H^{2n}(V; \mathbb{Z})$.

For example, if $\dim V = 2$, then

$$\langle c_2(\tilde{\varphi}), [A] \rangle = r \langle c_2(TV) - 2c_1(TV)c_1(L) + 3c_1(L)^2, [V] \rangle.$$

(1.6) Let $A$ be a non-singular hypersurface of $\mathbb{P}^n \times \mathbb{P}^1$ of degree $(d, r)$. Then $\chi(A) = \langle c_n(TA), [A] \rangle$ is equal to

$$(n+1)(1-d)^n r + 2 \left( \frac{(1-d)^{n+1}-1}{d} + n+1 \right).$$

If $\varphi: A \longrightarrow \mathbb{P}^1$ has only isolated critical points, then $s(\varphi) = \langle c_n(\text{Hom}(TA, \ast T\mathbb{P}^1)), [A] \rangle$ is equal to $(n+1)(d-1)^n r$.

(1.7) Let $A$ be a non-singular irreducible projective variety of dimension $n$. Then $H_i(A; \mathbb{Z})$ is torsion free for all $i$, and rank $H_i(A; \mathbb{Z})$ is equal to

0 (if $n$, $i$: odd), 1 (if $n$, $i$: even), $n+1$- $\chi(A)$ (if $n$, $n$: odd), $\chi(A)-n$ (if $n$, $n$: even).
(1.8) If $A$ is a simply connected compact complex surface, then $P_t(A;K) = P_{-t}(A;K)$, and $P_1(A;K) = P_{-1}(A;K) = \chi(A)$ for any field $K$.

2. Preliminary: Real Topology.

(2.1) A real structure on a complex manifold $X$ is an anti-holomorphic involution $\tau: X \to X$. The pair $(X, \tau)$ is called a real complex manifold. Two real complex manifolds $(X, \tau)$, $(X', \tau')$ are isomorphic if there is an isomorphism $\sigma: X \to X'$ of complex manifolds satisfying $\sigma \circ \tau = \tau' \circ \sigma$ (cf. [S]).

(2.2) Let $(X, \tau)$ be a real complex manifold. We denote by $X^\tau$ the space of fixed points of $\tau$ in $X$, and call it the real locus of $X$ (with respect to $\tau$).

$(X, \tau)$ is a $M$-manifold if $P_1(X^\tau; \mathbb{Z}/2) = P_1(X; \mathbb{Z}/2)$ (cf. [G]). A $M$-manifold $(X, \tau)$ of dimension 1 (resp. 2) is called a $M$-curve (resp. $M$-surface).

(2.3) Example. The number of equivalence classes of real structures on $\mathbb{P}^n$ is one if $n$ is even and two if $n$ is odd (cf. [F], p. 240).

The anti-holomorphic involution $\tau': \mathbb{P}^{2m+1} \to \mathbb{P}^{2m+1}$ defined by $\tau'[X_0:X_1:...:X_{2i}:X_{2i+1}:...:X_{2m}:X_{2m+1}] = [-X_1:X_0:...:-X_{2i+1}:X_{2i}:...:-X_{2m+1}:X_{2m}]$ gives a real structure not equivalent to the usual real structure defined by the complex conjugation $(\mathbb{P}^{2m+1}, \tau_{2m+1})$. We often denote by $\mathbb{P}^{2m+1}_0 = (\mathbb{P}^{2m+1}, \tau')$, $\mathbb{P}^{2m+1}_1 = (\mathbb{P}^{2m+1}, \tau_{2m+1})$. 

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Then \( \mathbb{P}^{2m} \) and \( \mathbb{P}^{2m+1} \) are M-manifolds, but \( \mathbb{P}^{2m+1} \) is not a M-manifold.

(2.4) From properties of Poicaré series, we see

**Lemma.** Let \((X, \mathcal{U}), (X', \mathcal{U}')\) be M-manifolds. Then \((X \perp X', \mathcal{U} \perp \mathcal{U}')\) and \((X \times X', \mathcal{U} \times \mathcal{U}')\) are also M-manifolds.

(2.5) **Lemma.** Let \((X, \mathcal{U})\) be a M-surface with \(H_1(X; \mathbb{Z}/2) = 0\) and \(H_0(X^R; \mathbb{Z}/2) \cong \mathbb{Z}/2\). Then \(X(X) + X(X^R) = 4\).

**Proof.** 
\[P_1(X; \mathbb{Z}/2) = P_1(X; \mathbb{Z}/2) = P_1(X^R; \mathbb{Z}/2).\]
\[P_1(X^R; \mathbb{Z}/2) + P_1(X; \mathbb{Z}/2) = 2(\dim H_0(X^R; \mathbb{Z}/2) + \dim H_2(X^R; \mathbb{Z}/2)) = 4.\]

(2.6) Let \(\Pi: E \to X\) be a holomorphic vector bundle over a real complex manifold \((X, \mathcal{U})\). A real structure of \(\pi\) is a real structure \(\Pi: E \to E\) of \(E\) as a complex manifold (cf. 2.1) such that \(\Pi \circ T = \tau \circ \Pi\) and the restriction \(T_x: E_x \to E_{\tau(x)}\) to each fiber \((x \in X)\) is conjugate linear.

We call the triple \((\Pi, T, \tau)\) a **real holomorphic vector bundle** (cf. [A]). Notice that the restriction \(\Pi^R_x: E^R_x \to X^R_x\) to the real locus of \(\Pi\) is a real vector bundle.

A holomorphic section \(s \in H^0(X, E)\) of \(E\) is **real** if \(T \circ s \circ \tau^{-1} = s\), that is, \(s \in H^0(X, E)^R\) with respect to the anti-holomorphic involution \(s \mapsto T \circ s \circ \tau^{-1}\).
(2.7) **Definition.** A holomorphic section $s$ of a real holomorphic vector bundle over a real complex manifold $(X, \mathcal{L})$ is a **M-section** if $s$ is transverse, real and the zero-locus $(s)_0 \subset X$ with restricted $\mathcal{L}$ is a $M$-manifold.

(2.8) **Remark.** Two real holomorphic vector bundles are isomorphic as real holomorphic vector bundles if and only if they are isomorphic as holomorphic vector bundles.

On $\mathbb{R}^n$, any holomorphic line bundle has a structure of real holomorphic line bundle.

(2.9) **Poincaré-Hopf-Pugh formula (cf. [P]).**

Let $M$ be a compact $C^\infty$ manifold of dimension $n$ with boundary $\partial M$.

A tangent vector $\xi$ to $M$ at a point $x_0$ of $M$ is **external** if $df_{x_0}(\xi)$ is positive for some $C^\infty$ function $f$ defined in a neighborhood $U$ of $x_0$ such that $f^{-1}(0) = \partial M \cap U$, $f$ takes negative values in $(M - \partial M) \cap U$ and $df|_{\partial M \cap U}$ does not vanish (figure 1):

![Diagram](image)

external

Let $v: \partial M \to TM|\partial M$ be a $C^\infty$ section over $\partial M$ to the tangent bundle $TM$. 

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Assume that (a): for each $x_0 \in \partial M$, $\nu(x_0) \neq 0$.

First put $M_0 = M$. Next put

$$M_1' = \{x \in \partial M \mid \nu(x)\text{ is external}\},$$

and put $M_1 = \overline{M_1'}$, and $\partial M_1 = M_1 - M_1'$.

Inductively, if $M_k$ is a $C^\infty$ manifold with boundary $\partial M_k$ ($k \geq 0$), then put

$$M_{k+1}' = \{x \in \partial M_k \mid (v/\partial M_k)(x)\text{ is external w.r.t. } M_k\},$$

$M_{k+1} = \overline{M_{k+1}'}$ and $\partial M_{k+1} = M_{k+1} - M_{k+1}'$.

Assume that (b): $M_k$ is a $C^\infty$ manifold with boundary $\partial M_k$, $(k = 1, 2, \ldots, n-1)$.

Lemma. Let $\nu$ satisfy two assumptions (a), (b) stated above. Then for any $C^\infty$ extension $w: M \to TM$ with isolated singularities, we have

(c): \[ \text{ind } w = \sum_{i=0}^{n} (-1)^i \chi(M_i). \]

Remark. (0) We adopt the following definition of index of a vector field: Let $x_0 \in M$ be an isolated singular point of $w$. Take a system of coordinates $x_1, \ldots, x_n$ centered at $x_0$, and write locally

$$w(x) = a_1(x)(\partial/\partial x_1) + \ldots + a_n(x)(\partial/\partial x_n).$$

Define $\text{ind}_{x_0} w = \deg_0 (-a)$, where $a = (a_1, \ldots, a_n)$. 

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Then put \( \text{ind } w = \sum \text{ind}_{x_0} w \), where the sum runs over isolated singular points \( x_0 \) of \( w \).

(1) If \( \emptyset M \) is empty, then (c) is the Poincaré-Hopf's formula.

(2) For a \( C^\infty \) vector field \( w \) over \( M \) with only isolated singular points, there exists a non-negative \( C^\infty \) function \( f: U \to \mathbb{R} \) with the following properties:

(i) \( f^{-1}(0) = \emptyset M \). (ii) For any sufficiently small \( \varepsilon > 0 \), \( w \mid f^{-1}(\varepsilon) \) satisfies two assumptions (a), (b).

3. Non-linear systems of real sections.

In this section we prove Theorem 0.4.

In the situation of Theorem 0.4, put \( Z = (s_r)_0 \overset{\sim}{=}(s_1)_0 \) (0 \( \leq i \leq r \)), \( s(r) = \sum_{0 \leq i \leq r} \varepsilon_i s_i \lambda_i^{1/\mu-1} \) and \( A(r) = (s(r))_0 \). Denote by \( s_1^{(r)} \) (resp. \( t_1^{(r)} \)) (1 = 0, 1, 2) the number of real critical points of \( \varphi = \psi \mid A(r) \) of index 1 (resp. \( \dim H_1(A^{(r)}_\mathbb{R}; \mathbb{Z}/2) \)).

Identify \( H^4(S; \mathbb{Z}) \) with \( \mathbb{Z} \) by the fundamental class \([S]\).

(3.1) Proof of Theorem 0.4. By (1.2), \( g(Z) \) is equal to \( 1 + (1/2)(c_1(L)^2 - c_1(L)c_1(TS)) \).

Let \( N \) be \( S_\mathbb{R} \) minus the interiors of 2g(Z) empty ovals. Put \( M = \{ (x; \lambda, \mu) \in A^{(r)}_\mathbb{R} \mid |s^{(r-1)}(x; \lambda, \mu)| \geq \delta, x \in N \} \) for a positive number \( \delta \) with \( |\varepsilon_{r-1}| \gg \delta \gg |\varepsilon_r| > 0 \). Then \( M \) is a \( C^\infty \).
manifold with boundary such that $\chi(M) = \chi(S_R) - 2g(Z)$.

Set $w = \text{grad} \varphi | M$. Then, with respect to $w$, $\chi(M_1)$ is equal to $c_1(L)^2$ (cf. 2.9) and $M_2$ is empty. Thus we see

\[ \text{index } w = \chi(M) - \chi(M_1) = \chi(S_R) - 2g(Z) - c_1(L)^2. \]

Therefore on $M$, the number of critical point of $\varphi_R$ of index 1 is not less than $-\text{index } w = c_1(L)^2 + 2g(Z) - \chi(S_R)$.

Thus we have

\[ s_1(r) - s_1(r-1) \geq 2c_1(L)^2 - c_1(L)c_1(TS) - \chi(S_R) + 2, \]

\[ s_0(r) + s_2(r) - (s_0(r-1) + s_2(r-1)) \geq 2g(Z) = c_1(L)^2 - c_1(L)c_1(TS) + 2, \]

\[ s_0(0) = s_1(0) = s_2(0) = 0. \]

So we have

\[ s_1(r) \geq r(2c_1(L)^2 - c_1(L)c_1(TS) - \chi(S_R) + 2) \quad \ldots \quad (1), \]

\[ s_0(r) + s_2(r) \geq r(c_1(L)^2 - c_1(L)c_1(TS) + 2) \quad \ldots \quad (2). \]

By (2.5), $\chi(S) + \chi(S_R) = 4$. Hence we have

\[ s(\varphi_R) = s^{(r)} + s_1(r) + s_2(r) \]

\[ \geq r(3c_1(L)^2 - 2c_1(L)c_1(TS) + c_2(TS)) \quad \ldots \quad (3). \]

By (1.5), equalities in (1), (2) and (3) hold. Thus we have

\[ \chi(A_R) = s_0^{(r)} - s_1^{(r)} + s_2^{(r)} \]

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\[= r(-c_1(L)^2 - c_2(TS) + 4) \quad \ldots \ (4).\]

On the other hand, because of the existence of ovals, we have

\[t_0(r) + t_2(r) - (t_0(r-1) + t_2(r-1)) \geq 2g(Z),\]
\[t_0(1) + t_2(1) \geq 2.\]

Thus we have

\[t_0(r) + t_2(r) \geq 2g(Z)(r-1) + 2 \quad \ldots \ (5).\]

Therefore, by (4), (5) and (1.3), we have

\[P_1(A_R; \mathbb{Z}/2) = t_0(r) + t_1(r) + t_2(r)\]
\[= 2(t_0(r) + t_2(r)) - \chi(A_R)\]
\[\geq (3r-2)c_1(L)^2 - (2r-2)c_1(L)c_1(TS) + rc_2(TS)\]
\[= P_1(A; \mathbb{Z}/2) \quad \ldots \ (6).\]

By Harnack-Thom's inequality \(P_1(A_R; \mathbb{Z}/2) \leq P_1(A; \mathbb{Z}/2)\).

Hence equalities in (5) and (6) hold. This completes the proof of Theorem 0.4.

(3.2) Example. Let us consider the case \(S = \mathbb{P}^2\). Let \(A\) be a non-singular surface of \(\mathbb{P}^2 \times \mathbb{P}^1\) of degree \((d,r)\). Then

\[\chi(A) = P_1(A; \mathbb{Z}/2) = 3 + d^2 + 3(d-1)^2(r-1).\]

If \(\varphi: A \to \mathbb{P}^1\) has only isolated critical points, then

\[s(\varphi) = \sum_{x \in A} \mu_x(\varphi) = 3(d-1)^2r,\]

where \(\mu_x(\varphi)\) is the Milnor number of \(\varphi\) at \(x\).
**Proposition.** Let \( A \subset \mathbb{P}^2 \times \mathbb{P}^1 \) be a non-singular real surface of degree \((d,r)\) such that \( \varphi: A \rightarrow \mathbb{P}^1 \) has only isolated critical points. Then we have the sharp estimate

\[
P(A_R; \mathbb{P}^2/2) \leq 3 + d^2 + 3(d-1)^2(r-1),
\]

\[
(A_R) \leq 3(d-1)^2 r.
\]

**Example.** Let \( A = \left\{ [\lambda \mathcal{F} + \mu \mathcal{G}] / [\lambda ; \mu] \in \mathbb{P}^1 \right\} \) be a pencil of real plane curves in \( \mathbb{P}^2 \) of degree \( d \).

\( A = (\lambda \mathcal{F} + \mu \mathcal{G})_0 \subset \mathbb{P}^2 \times \mathbb{P}^1 \) is non-singular if and only if \( (\mathcal{F})_0 \) and \( (\mathcal{G})_0 \) intersect transversely in \( \mathbb{P}^2 \). If \( A \) is non-singular, then \( A \sim \mathbb{P}^2 - \mathbb{P}^2 - \cdots - \mathbb{P}^2 \). In this case, if \( (\mathcal{F})_0 \) and \( (\mathcal{G})_0 \) intersect in \( k \) points \((0 \leq k \leq d^2, k \equiv d \mod 2)\), then \( A_R \sim \# \mathbb{RP}^2 \). Thus \( A \) is an \( M \)-surface if and only if \( k = d^2 \).


Let \( S \) be a compact real complex surface, \( L, L' \) real holomorphic line bundles, \( s, s' \) \( M \)-sections of \( L, L' \) respectively.

Put \( C = (s)_0 \) and \( C' = (s')_0 \). Assume that \( C \) and \( C' \) are both rational and \( CC' = \langle c_1(L)c_1(L'), [S] \rangle \geq 0 \). (This assumption for \( S \) is rather restrictive (cf. [BPV], Proposition V.4.3)).

Consider the following condition:
(**) For any effective divisor $\alpha$ on $C$ of degree $CC'$ with $\text{supp} \alpha \subseteq C_{\mathbb{R}}$, there exists a real section $s'' \in H^0(S, L')_{\mathbb{R}}$ such that $(s'')_0|C = \alpha$.

**Theorem 4.0.** Under the condition (**), for any natural numbers $d$ and $e$, $L^d \otimes L^e$ has an M-section near $s^d \otimes s^e$ in $H^0(S, L^d \otimes L^e)_{\mathbb{R}}$. Furthermore, if $CC'$ is positive, then $L^d \otimes L^e$ has a pair of M-sections near $s^d \otimes s^e$ satisfying (*) (cf. Introduction).

**Corollary 4.1.** If $C^2 \geq 0$, then under the condition (**), for $C' = C$, for any natural number $d$, $L^d$ has an M-section near $s^d$. Furthermore, if $C^2$ is positive, then $L^d$ has a pair of M-sections near $s^d$ satisfying (*).

(4.2) Example. (1) $S = \mathbb{P}^2$, $L = L' = \mathcal{O}_{\mathbb{P}^2}(1)$ (This corresponds to the Harnack's method).

(2) $S = \mathbb{P}^2$, $L = L' = \mathcal{O}_{\mathbb{P}^2}(2)$, $C = C'$: a real ellipse with $C_{\mathbb{R}} \neq \emptyset$ (This corresponds to the Hilbert's method).

(3) $S = \mathbb{P}^1 \times \mathbb{P}^1$, $L = L' = p_1^* \mathcal{O}_{\mathbb{P}^1}(1) \otimes p_2^* \mathcal{O}_{\mathbb{P}^1}(1)$.

(4) $S = \mathbb{P}^1 \times \mathbb{P}^1$, $L = p_1^* \mathcal{O}_{\mathbb{P}^1}(1)$, $L' = p_2^* \mathcal{O}_{\mathbb{P}^1}(1)$ (This is used to show Proposition 0.2 and Corollary 0.6).
References:


(March.24.1986).