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Kyoto University
Topologically Extremal Real Surfaces in
\( \mathbb{P}^2 \times \mathbb{P}^1 \) and \( \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \).

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From a general viewpoint we illustrate a method of construction of surfaces in \( \mathbb{P}^2 \times \mathbb{P}^1 \) and \( \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \) defined over \( \mathbb{R} \) having topologically extremal properties. Precisely we show that for each \( d, e \) and \( r \) there exists an M-surface \( A \) in \( \mathbb{P}^2 \times \mathbb{P}^1 \) (resp. \( \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \)) of degree \( (d,r) \) (resp. \( (d,e,r) \)) such that the projection \( A \to \mathbb{P}^1 \) has the maximal number of real critical points. The construction of M-surfaces in \( \mathbb{P}^3 \) by O.Ya.Viro is also made more clear.

0. Introduction.

Harnack [H] pointed out that the number of components in the real locus of a curve in \( \mathbb{P}^2 \) of degree \( d \) defined over \( \mathbb{R} \) does not exceed \( 1+(1/2)(d-1)(d-2) \) and, for each \( d \), there exists a non-singular curve in \( \mathbb{P}^2 \) of degree \( d \) defined over \( \mathbb{R} \), the real locus of which has exactly \( 1+(1/2)(d-1)(d-2) \) components.

Hilbert in his 16th problem proposed to investigate
topological restrictions for hypersurfaces in $\mathbb{P}^n$ of fixed degree defined over $\mathbb{R}$.

One may regard an real algebraic function as an one-parameter family of hypersurfaces defined over $\mathbb{R}$, and it is natural to investigate topological restrictions for hypersurfaces in $\mathbb{P}^n \times \mathbb{P}^1$ of fixed degree defined over $\mathbb{R}$.

Let $A \subset \mathbb{P}^n \times \mathbb{P}^1$ be a real hypersurface of degree $(d,r)$, that is, the zero-locus of a polynomial $\sum_{0 \leq i \leq r} F_i(x_0, \ldots, x_n)X^{r-i}Y^i$, where $F_i (0 \leq i \leq r)$ is a real homogeneous polynomial of degree $d$.

Consider the projection $\varphi: A \rightarrow \mathbb{P}^1$. Our main object is the topology of real locus $A_\mathbb{R}$ of $A$ and singularities of the restriction $\varphi_\mathbb{R}: A_\mathbb{R} \rightarrow \mathbb{R}\mathbb{P}^1$ of $\varphi$ to $A_\mathbb{R}$.

We denote by $P_t(X,K)$ the Poincare series of a space $X$ over a field $K$ with indeterminate $t$, and by $s(f)$ the number of critical points of a function $f: X \rightarrow \mathbb{R}$ from an $n$-dimensional manifold to an one-dimensional manifold.

If $A \subset \mathbb{P}^n \times \mathbb{P}^1$ is non-singular, then the diffeomorphism type of $A$ is determined by $(d,r)$. For example,

$$P_t(A,K) = \begin{cases} X(A) \quad & (n:\text{even}), \\ 2(n+1) - X(A) \quad & (n:\text{odd}), \end{cases}$$

for any $K$,

$$X(A) = (n+1)(1-d)^n + 2\left(\frac{(1-d)^{n+1} - 1}{d} + n+1\right), (\text{cf. 1.6}).$$

We call $A$ generic if $A$ is non-singular and $\varphi: A \rightarrow \mathbb{P}^1$ has only non-degenerate critical points.

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If \( A \) is generic, then \( s(\varphi) = (n+1)(d-1)^n r \) (cf. 1.6).

By Harnack-Thom's inequality ([G]), we have an uniform estimate:

\[
\begin{align*}
& P_1(A_R; \mathbb{Z}/2) \leq P_1(A; \mathbb{Z}/2), \\
& s(\varphi_R) \leq s(\varphi).
\end{align*}
\]

In this note from a general viewpoint we show the following

**Theorem 0.1.** For \( n = 1, 2 \) and for each \((d,r)\), the estimate \((0.0)\) is sharp, that is, there exists a generic real hypersurface of \( \mathbb{P}^n \times \mathbb{P}^1 \) of degree \((d,r)\) attaining both equalities in \((0.0)\).

Notice that in the case \( r = 1 \) Theorem 0.1 is proved in [I]. A finer result is obtained in the case \( n = 1 \). For \( A \subset \mathbb{P}^1 \times \mathbb{P}^1 \), we denote by \( \pi: A \to \mathbb{P}^1 \) the projection to the first component.

**Proposition 0.2.** For non-singular real curves \( A \subset \mathbb{P}^1 \times \mathbb{P}^1 \) of degree \((d,e)\) such that both \( \varphi, \pi \) have only non-degenerate critical points, there exists the sharp estimate:

\[
\begin{align*}
P_1(A_R; \mathbb{Z}/2) & \leq 2 + 2(d-1)(e-1), \\
s(\varphi_R) & \leq 2(d-1)e, \\
s(\pi_R) & \leq 2d(e-1).
\end{align*}
\]

Now let us formulate a general theorem which implies Theorem 0.1.
Let $S$ be a real complex surface (cf. 2.1), $C \subset S$ be a real curve possibly with singularities. A non-singular component $E$ of $C_R \subset S_R$ is an **oval** (resp. an **empty oval**) if there exists an embedding $i: D^2 \to S_R$ such that $i(D^2) = E$ (and that $i(\text{int} \ D^2) \cap C_R$ is empty).

Let $S$ be compact, $L$ a real holomorphic line bundle (cf. 2.6), $s_0, s_1$ M-sections of $L$ (cf. 2.7).

Consider the following condition (**):

(**i**) The zero-loci $(s_0)_0$ and $(s_1)_0$ are both connected and of genus $g$.

(**ii**) $(s_0)_0$ and $(s_1)_0$ intersect in $\langle c_1(L)^2, [S] \rangle$ points in $S_R$.

(**iii**) The real locus of $(s_0s_1)_0 = (s_0)_0 \cup (s_1)_0$ has $2g$ empty ovals.

We denote by $\mathbb{P}^1_\mathbb{R}$ the real complex curve $(\mathbb{P}^1, \tau_1)$, where $\tau_1$ is the complex conjugation (cf. 2.3). Fix a pair of M-sections $\lambda, \mu$ of $\mathcal{O}_{\mathbb{P}^1}(1)$ such that $\langle \lambda \rangle_0 \neq \langle \mu \rangle_0$.

Denote by $\varphi: S \times \mathbb{P}^1_\mathbb{R} \to \mathbb{P}^1_\mathbb{R}$, $\psi: S \times \mathbb{P}^1_\mathbb{R} \to S$ the projections.

For a transverse section $s$ of $\mathcal{O}_S$ (cf. 1.3), denote by $\varphi: (s)_0 \to \mathbb{P}^1_\mathbb{R}$, $\tau: (s)_0 \to S$ the restrictions of projections. Then, associated to $s$, there is a natural section of $\text{Hom}(T(s)_0, \mathcal{O}_\mathbb{P}^1_\mathbb{R})$ defined by the tangent map of $\varphi$. 

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Theorem 0.4. Let $S$ be an $M$-surface with connected real part $S_{|\mathbb{R}|}$, $L$ be a real holomorphic line bundle with a pair $s_0, s_1$ of $M$-sections of $L$ satisfying the condition $(\ast)$. Then, for any $r$, there exists an $M$-section $s$ of $\mathcal{O}_{\mathbb{P}_1}(r)$ over $S \times \mathbb{P}_1$ near $s_0 \otimes \lambda^r$, which associates an $M$-section of $\text{Hom}(T(s)_0, \mathcal{O}_{\mathbb{P}_1})$ defined by the projection $\mathcal{F}: (s)_0 \rightarrow \mathbb{P}_1$.

Explicitly, $s$ can be taken in a form
\[ \sum_{0 \leq i \leq r} \mathcal{E}_i s_i \lambda^i, \]
where $s_i = s_0$ (i: even), $s_i = s_1$ (i: odd) and $\mathcal{E}_0, \mathcal{E}_1, \ldots, \mathcal{E}_r$ are real numbers with $1 = \mathcal{E}_0 \gg |\mathcal{E}_1| \gg \ldots \gg |\mathcal{E}_r| > 0$.

Remark 0.5. A sufficient condition for the existence of a pair of $M$-sections satisfying $(\ast)$ is given in section 4. Theorem 0.4 with this sufficient condition implies immediately Theorem 0.1 in the case $n = 2$.

Putting $S = \mathbb{P}_1 \times \mathbb{P}_1 (= \mathbb{P}_1 \times \mathbb{P}_1)$ and $L = \mathcal{O}_{\mathbb{P}_1}(d) \otimes \mathcal{O}_{\mathbb{P}_1}(r)$ over $S$, we have

Corollary 0.6. For non-singular real surface $A \subset \mathbb{P}_1 \times \mathbb{P}_1 \times \mathbb{P}_1$ of degree $(d,e,r)$ such that $\mathcal{F}: A \rightarrow \mathbb{P}_1$ has only non-degenerate critical points, there exists the sharp estimate:

\[
\begin{cases}
\mathbb{P}_1(A_{\mathbb{R}}; 2/2) \leq 6d(e-r-4e)d+4d+4e+4r, \\
\mathcal{S}(\mathcal{F}_{\mathbb{R}}) \leq (6de-4d-4e+4)r.
\end{cases}
\]
From Theorem 0.4, it naturally arises the following general problem:

**Problem 0.7.** Let $E$ be a real holomorphic vector bundle over a real complex manifold. Give a criterion for the existence or the non-existence of $M$-sections of $E$.

Lastly we intend to clarify the construction of $M$-surfaces in $\mathbb{P}^3$ by Viro [V].

**Theorem 0.8.** (Viro) For non-singular real surfaces $A$ in $\mathbb{P}^3$ of degree $d$, there exists the sharp estimate:

$$P_1(A_{\mathbb{R};\mathbb{Z}/2}) \leq d^3 - 4d^2 + 6d.$$

Let $X_0, X_1, X_2, X_3$ be homogeneous coordinates of $\mathbb{P}^3$. Put $\mathbb{P}^2 = \{X_3 = 0\}$, $\mathbb{P}^1 = \{X_3 = X_3 = 0\}$ and $\mathcal{L} = \{X_0 = X_1 = 0\}$.

Let $\varphi: \mathbb{P}^3 - \mathcal{L} \rightarrow \mathbb{P}^1$ be a projection. Fix a tubular neighborhood $U$ of $\mathcal{L}$ in $\mathbb{P}^3$ such that $\bar{U} \cap \mathbb{P}^1$ is empty.

Observe that for each $d$ there exist $M$-sections $s_0, ..., s_d$ of $\mathcal{O}_{\mathbb{P}^2}(0), ..., \mathcal{O}_{\mathbb{P}^2}(d)$ near $X_2^0, ..., X_2^d$ respectively such that $(s_1)_0$ and $(s_{i+1})_0$ intersect in $i(i+1)$ points in $\mathbb{R}\mathbb{P}^2$, the real locus of $(s_1s_{i+1})_0$ has $(1/2)(i-1)(i-2) + (1/2)i(i-1)$ empty ovals $(0 \leq i \leq d-1)$ and $\varphi|(s_1)_0$ has $(i-1)1$ real critical points $(0 \leq i \leq d)$. Naturally each $s_i$ is extended to a section $\tilde{s}_i$ of $\mathcal{O}_{\mathbb{P}^3}(1) (0 \leq i \leq d)$. 

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Put \( s = \sum_{0 \leq i \leq d} \xi_i X_2 \overset{d-1}{\sim} \xi_1 \in H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}^{(d)})_{\mathbb{R}}, \) and \( A = (s)_0. \)

Take real numbers \( \xi_0, \ldots, \xi_d \) to be \( 1 = \xi_0 \gg |\xi_1| \gg \cdots \gg |\xi_d| > 0 \)

and of appropriate signs.

\[ \Phi_{A_R} : A_R^1 \rightarrow \mathbb{R}^1 \] defines a vector field \( \mathfrak{X} \) over \( A_R^1. \)

\( \mathfrak{X} \) is extended to a vector field \( \mathfrak{X} \) over \( A_R \) with finite singularities.

Denote by \( s^+(\mathfrak{X}) \) (resp. \( s^-(\mathfrak{X}) \)) the sum of positive (resp. negative) indices of singular points of \( \mathfrak{X} \), and put

\[ t_1 = \dim H^1(A_R; \mathbb{Z}/2) \] \( (i=1,2,3) \). Then we see

\[ s^+(\mathfrak{X}) \geq d + (1/3)d(d-1)(d-2), \]

\[ s^-(\mathfrak{X}) \geq (1/3)(d+1)d(d-1) + (1/3)d(d-1)(d-2). \]

Thus \( \chi(A_R) = s^+(\mathfrak{X}) - s^-(\mathfrak{X}) \geq d - (1/3)(d+1)d(d-1). \) On the other hand \( t_0 + t_1 \geq 2 + (1/3)(d-1)(d-2)(d-3). \) Hence we have

\[ P_1(A_R; \mathbb{Z}/2) = t_0 + t_1 + t_2 = 2(t_0 + t_2) - \chi(A_R) \geq d^3 - 4d^2 + 6d \ (= P_1(A; \mathbb{Z}/2)). \]

By Harnack-Thom's inequality, all equalities are hold.

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1. Preliminary: Complex Topology.

(1.0) Let $X$ be a complex manifold, $\pi: E \to X$ a holomorphic vector bundle and $s: X \to E$ a holomorphic section. Put $(s)_0 = \{ x \in X \mid s(x) = 0 \}$.

We call $s$ transverse if $s$ is transverse to the zero section $\emptyset \subset E$, that is, for any $s \in (s)_0$, $s^* T_X \otimes T_s(x)^\ell = T_s(x)^l$.

If $s$ is transverse, then $(s)_0$ is a complex submanifold of $X$.

Denote by $H$ the complex vector space $H^0(X, E)$ of totality of holomorphic sections of $E$ over $X$, and by $PH$ the projectification of $H$.

Put $Z = \{(x,[s]) \in X \times PH \mid s(x) = 0\}$ and consider the projection $\bar{\Phi}: Z \to PH$. Then $s$ is transverse if and only if $Z$ is non-singular along $\bar{\Phi}^{-1}[s]$ and $\bar{\Phi}$ is submersive over $[s]$.

In particular, for transverse sections $s, s' \in H$, $(s)_0$ and $(s')_0$ are diffeomorphic.

(1.1) Let $s \in H^0(X, E)$ be transverse. Put $Z = (s)_0$.

Then we have an exact sequence

$$ 0 \to TZ \to TX|_Z \to E|_Z \to 0, $$

of complex vector bundles. Therefore $c_t(TX|_Z) = c_t(TZ)c_t(E|_Z)$ for Chern polynomials. The Chern classes of $TZ$ are calculated by the formula $c_t(TZ) = \frac{c_t(TX|_Z)}{c_t(E|_Z)}$ (cf. [F]).
(1.2) Let $L$ be a holomorphic line bundle over a complex manifold $V$ of dimension $n$. Let $Z$ be the zero-locus of a transverse section of $L$. Then by (1.1),

$$
\chi(Z) = \langle \sum_{i+j=n+1} (-1)^j c_1(TV)(c_1(L))^j, [V] \rangle.
$$

For example, if $\dim V = 2$, then

$$
\chi(Z) = \langle c_1(TV)c_1(L) - c_1(L)^2, [V] \rangle.
$$

Furthermore, if $Z$ is connected, then

$$
\chi(Z) = 1 + (1/2)\langle c_1(L)^2 - c_1(L)c_1(TV), [V] \rangle.
$$

(1.3) Let $R$ be a non-singular curve of genus $g$. Denote by $\bar{\gamma}: V \times R \to V$ and $\gamma: V \times R \to R$ the projections. Put $L' = t^*L \otimes \gamma_R^*(r)$ over $V \times R$ for each $r$. Let $A \subset V \times R$ be the zero-locus of a transverse section of $L'$.

Then $\chi(A) = \langle \theta, [V] \rangle$, where

$$
\theta = rc_n(TV) + \sum_{i+j=n, j>0} ((j+1)r+2g-2)c_1(TV)(-c_1(L))^j,
$$

as an element of $H^{2n}(V; \mathbb{Z})$.

For example, if $\dim V = 2$, then

$$
\chi(A) = \langle r c_2(TV) - (2r+2g-2)c_1(TV)c_1(L) + (3r+2g-2)c_1(L)^2, [V] \rangle.
$$

(1.4) Example. Let $C, C'$ and $C''$ be non-singular curves...
of genus \(g, g'\) and \(g''\) respectively. Put \(X = C \times C' \times C''\), and denote projections by \(p_1, p_2\) and \(p_3\) to \(C, C'\) and \(C''\) respectively. Let \(A \subset X\) be the zero-locus of a transverse section of \(L' = p_1 \ast \mathcal{O}_C(d) \otimes p_2 \ast \mathcal{O}_{C'}(d') \otimes p_3 \ast \mathcal{O}_{C''}(d'')\). Then \(\chi(A)\) is equal to 
\[6(d-1)(d'-1)(d''-1) + (2+4g')(d-1)(d'-1) + (2+4g)(d'-1)(d''-1) + (2+4g')(d''-1)(d-1) + (2+4g'-g')(d-1) + (2+4g'g')(d-1) + 6 - 4(g+g'+g'') + 4(gg'+g''+g'').\]

(1.5) In (1.3), denote by \(\Phi: A \longrightarrow \mathbb{P}^1\) the projection to \(\mathbb{P}^1\). Put \(\delta = \text{Hom}(TA, \Phi^*\mathbb{P}^1)\). Then \(\left\langle \mathcal{O}_n(\delta), [A] \right\rangle = \left\langle \gamma, [V] \right\rangle\), where
\[
\gamma = (-1)^n r \sum_{1+j=n} (j+1)c_1(TV)(-c_1(L))^j,
\]
as an element of \(H^{2n}(V; \mathbb{Z})\).

For example, if \(\dim V = 2\), then
\[
\left\langle \mathcal{O}_2(\delta), [A] \right\rangle = r \left\langle \mathcal{O}_2(TV) - 2c_1(TV)c_1(L) + 3c_1(L)^2, [V] \right\rangle.
\]

(1.6) Let \(A\) be a non-singular hypersurface of \(\mathbb{P}^n \times \mathbb{P}^1\) of degree \((d, r)\). Then \(\chi(A) = \left\langle \mathcal{O}_n(TA), [A] \right\rangle\) is equal to
\[
(n+1)(1-d)^n r + 2\left(\frac{(1-d)^{n+1}-1}{d} + n+1\right).
\]

If \(\Phi: A \longrightarrow \mathbb{P}^1\) has only isolated critical points, then
\[
\left\langle \mathcal{O}_n(\text{Hom}(TA, \ast \mathbb{P}^1)), [A] \right\rangle\text{ is equal to } (n+1)(d-1)^n r.
\]

(1.7) Let \(A\) be a non-singular irreducible projective variety of dimension \(n\). Then \(H_i(A; \mathbb{Z})\) is torsion free for all \(i\), and \(\text{rank } H_i(A; \mathbb{Z})\) is equal to 0 (if \(n, i: \text{odd}\)), 1 (if \(n, i: \text{even}\)), \(n+1\) for \(i = n, n: \text{odd}\), \(\chi(A)\) for \(i = n, n: \text{even}\).
(1.8) If $A$ is a simply connected compact complex surface, then $P_t(A;K) = P_{-t}(A;K)$, and $P_1(A;K) = P_{-1}(A;K) = X(A)$ for any field $K$.

2. Preliminary: Real Topology.

(2.1) A **real structure** on a complex manifold $X$ is an anti-holomorphic involution $\tau: X \to X$. The pair $(X, \tau)$ is called a **real complex manifold**. Two real complex manifolds $(X, \tau), (X', \tau')$ are isomorphic if there is an isomorphism $\sigma: X \to X'$ of complex manifolds satisfying $\sigma \circ \tau = \tau' \circ \sigma$ (cf. [S]).

(2.2) Let $(X, \tau)$ be a real complex manifold. We denote by $X^\tau$ the space of fixed points of $\tau$ in $X$, and call it the **real locus** of $X$ (with respect to $\tau$). $(X, \tau)$ is a **M-manifold** if $P_1(X^\tau; \mathbb{Z}/2) = P_1(X; \mathbb{Z}/2)$ (cf. [G]). A M-manifold $(X, \tau)$ of dimension 1 (resp. 2) is called a **M-curve** (resp. **M-surface**).

(2.3) Example. The number of equivalence classes of real structures on $\mathbb{P}^n$ is one if $n$ is even and two if $n$ is odd (cf. [F], p.240).

The anti-holomorphic involution $\tau': \mathbb{P}^{2m+1} \to \mathbb{P}^{2m+1}$ defined by $\tau'[X_0:X_1:\ldots:X_{2i}:X_{2i+1}:\ldots:X_{2m}:X_{2m+1}] = [-\overline{X}_1:\overline{X}_0:\ldots:-X_{2i+1}:\overline{X}_{2i}:\ldots:-\overline{X}_{2m+1}]:\overline{X}_{2m}]$ gives a real structure not equivalent to the usual real structure defined by the complex conjugation $(\mathbb{P}^{2m+1}, \tau_{2m+1})$. We often denote by $\mathbb{P}^{2m+1}_0 = (\mathbb{P}^{2m+1}, \tau')$, $\mathbb{P}^{2m+1}_1 = (\mathbb{P}^{2m+1}, \tau_{2m+1})$. 

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Then $\mathbb{P}^{2m}$ and $\mathbb{P}^{2m+1}_1$ are $\mathbb{M}$-manifolds, but $\mathbb{P}^{2m+1}_0$ is not a $\mathbb{M}$-manifold.

(2.4) From properties of Poicaré series, we see

**Lemma.** Let $(X,T)$, $(X',T')$ be $\mathbb{M}$-manifolds. Then $(X \sqcup X'$, $T \sqcup T'$) and $(X \times X'$, $T \times T'$) are also $\mathbb{M}$-manifolds.

(2.5) **Lemma.** Let $(X,T)$ be a $\mathbb{M}$-surface with $H_1(X;\mathbb{Z}/2) = 0$ and $H_0(X;\mathbb{Z}/2) \cong \mathbb{Z}/2$. Then $\chi(X) + \chi(X_T) = 4$.

**Proof.** $P_1(X;\mathbb{Z}/2) = P_1(X;\mathbb{Z}/2) = P_1(X_T;\mathbb{Z}/2)$.

\[
P_1(X;\mathbb{Z}/2) + P_1(X_T;\mathbb{Z}/2) = 2(\dim H_0(X;\mathbb{Z}/2) + \dim H_2(X_T;\mathbb{Z}/2)) = 4.
\]

(2.6) Let $\pi: E \longrightarrow X$ be a holomorphic vector bundle over a real complex manifold $(X,T)$. A real structure of $\pi$ is a real structure $T: E \longrightarrow E$ of $E$ as a complex manifold (cf. 2.1) such that $\pi \circ T = T \circ \pi$ and the restriction $T_x: E_x \longrightarrow E_{T(x)}$ to each fiber $(x \in X)$ is conjugate linear.

We call the triple $E = (\pi;T,T)$ a real holomorphic vector bundle (cf. [A]). Notice that the restriction $\pi_T: E_T \longrightarrow X_T$ to the real locus of $\pi$ is a real vector bundle.

A holomorphic section $s \in H^0(X,E)$ of $E$ is real if $T \circ s \circ T^{-1} = s$, that is, $s \in H^0(X,E)_R$ with respect to the anti-holomorphic involution $s \longmapsto T \circ s \circ T^{-1}$.

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(2.7) **Definition.** A holomorphic section $s$ of a real holomorphic vector bundle over a real complex manifold $(X, \mathcal{T})$ is a **M-section** if $s$ is transverse, real and the zero-locus $(s)_0 \subset X$ with restricted $\mathcal{T}$ is a M-manifold.

(2.8) **Remark.** Two real holomorphic vector bundles are isomorphic as real holomorphic vector bundles if and only if they are isomorphic as holomorphic vector bundles.

On $\mathbb{P}^n$, any holomorphic line bundle has a structure of real holomorphic line bundle.

(2.9) **Poincaré-Hopf-Pugh formula (cf. [P]).**

Let $M$ be a compact $C^\infty$ manifold of dimension $n$ with boundary $\partial M$.

A tangent vector $\xi$ to $M$ at a point $x_0$ of $M$ is external if $df_{x_0}(\xi)$ is positive for some $C^\infty$ function $f$ defined in a neighborhood $U$ of $x_0$ such that $f^{-1}(0) = \partial M \cap U$, $f$ takes negative values in $(M-\partial M) \cap U$ and $df|_{\partial M \cap U}$ does not vanish (figure 1):

![Diagram](attachment:image.png)

external

Let $v : \partial M \to TM|\partial M$ be a $C^\infty$ section over $\partial M$ to the tangent bundle $TM$. 

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Assume that (a): for each \( x_0 \in \partial M \), \( v(x_0) \neq 0 \).

First put \( M_0 = M \). Next put
\[
M_1' = \{ x \in \partial M \mid v(x) \text{ is external} \},
\]
and put \( M_1 = \overline{M_1'} \), and \( \partial M_1 = M_1 - M_1' \).

Inductively, if \( M_k \) is a \( C^\infty \) manifold with boundary \( \partial M_k \) (\( k \geq 0 \)), then put
\[
M_{k+1}' = \{ x \in \partial M_k \mid (v|\partial M_k)(x) \text{ is external w.r.t. } M_k \},
\]
\( M_{k+1} = \overline{M_{k+1}'} \) and \( \partial M_{k+1} = M_{k+1} - M_{k+1}' \).

Assume that (b): \( M_k \) is a \( C^\infty \) manifold with boundary \( \partial M_k \), \( (k = 1, 2, \ldots, n-1) \).

Lemma. Let \( v \) satisfy two assumptions (a), (b) stated above. Then for any \( C^\infty \) extension \( w: M \to TM \) with isolated singularities, we have
\[
(c): \quad \text{ind } w = \sum_{i=0}^{n} (-1)^i \chi(M_i).
\]

Remark. (0) We adopt the following definition of index of a vector field: Let \( x_0 \in M \) be an isolated singular point of \( w \). Take a system of coordinates \( x_1, \ldots, x_n \) centered at \( x_0 \), and write locally
\[
w(x) = a_1(x)(\partial/\partial x_1) + \ldots + a_n(x)(\partial/\partial x_n).
\]
Define \( \text{ind}_{x_0} w = \text{deg}_0(-a) \), where \( a = (a_1, \ldots, a_n) \).
Then put \( \text{ind } w = \sum \text{ind } x_0 w \), where the sum runs over isolated singular points \( x_0 \) of \( w \).

1. If \( \emptyset M \) is empty, then (c) is the Poincaré-Hopf's formula.

2. For a \( C^\infty \) vector field \( w \) over \( M \) with only isolated singular points, there exists a non-negative \( C^\infty \) function \( f: U \to \mathbb{R} \) with the following properties:
   
   (i) \( f^{-1}(0) = \emptyset M \).
   (ii) For any sufficiently small \( \varepsilon > 0 \), \( w|_{f^{-1}(\varepsilon)} \) satisfies two assumptions (a), (b).

3. Non-linear systems of real sections.

In this section we prove Theorem 0.4.

In the situation of Theorem 0.4, put \( Z = (s_r)_0 = (s_1)_0 \)
\( (0 \leq i \leq r) \), \( s^{(r)} = \sum_{0 \leq i \leq r} \varepsilon_i s_i \lambda^i \mu^{r-1} \) and \( A^{(r)} = (s^{(r)})_0 \). Denote by \( s_1^{(r)} \) (resp. \( t_1^{(r)} \)) \((i=0,1,2)\) the number of real critical points of \( \varphi = \psi|A^{(r)}_1 \) of index 1 (resp. \( \dim H_1(A^{(r)}_1, \mathbb{Z}/2) \)).

Identify \( H^4(S; \mathbb{Z}) \) with \( \mathbb{Z} \) by the fundamental class [S].

(3.1) Proof of Theorem 0.4. By (1.2), \( g(Z) \) is equal to \( 1 + (1/2)(c_1(L)^2 - c_1(L)c_1(\text{TS})) \).

Let \( N \) be \( S_\Re \) minus the interiors of \( 2g(Z) \) empty oval.

Put \( M = \{ (x; \lambda, \mu) \in A^{(r)}_1 \mid |s^{(r-1)}(x; \lambda, \mu)| \geq \delta, x \in N \} \) for a positive number \( \delta \) with \( |\varepsilon_{r-1}| > \delta \). Then \( M \) is a \( C^\infty \)
manifold with boundary such that $\chi(M) = \chi(S_\mathbb{R}) - 2g(Z)$.

Set $w = \text{grad} \varphi_{\mathbb{R}} | M$. Then, with respect to $w$, $\chi(M_1)$ is equal to $c_1(L)^2$ (cf. 2.9) and $M_2$ is empty. Thus we see

$$\text{index } w = \chi(M) - \chi(M_1) = \chi(S_\mathbb{R}) - 2g(Z) - c_1(L)^2.$$ 

Therefore on $M$, the number of critical point of $\varphi_{\mathbb{R}}$ of index 1 is not less than $-\text{index } w = c_1(L)^2 + 2g(Z) - \chi(S_\mathbb{R})$.

Thus we have

$$s_1^{(r)} - s_1^{(r-1)} \geq 2c_1(L)^2 - c_1(L)c_1(TS) - \chi(S_\mathbb{R}) + 2,$$

$$s_0^{(r)} + s_2^{(r)} - (s_0^{(r-1)} + s_2^{(r-1)}) \geq 2g(Z)$$

$$= c_1(L)^2 - c_1(L)c_1(TS) + 2,$$

$$s_0^{(0)} = s_1^{(0)} = s_2^{(0)} = 0.$$

So we have

$$s_1^{(r)} \geq r(2c_1(L)^2 - c_1(L)c_1(TS) - \chi(S_\mathbb{R}) + 2) \quad \ldots \quad (1),$$

$$s_0^{(r)} + s_2^{(r)} \geq r(c_1(L)^2 - c_1(L)c_1(TS) + 2) \quad \ldots \quad (2).$$

By (2.5), $\chi(S) + \chi(S_\mathbb{R}) = 4$. Hence we have

$$s(\varphi_{\mathbb{R}}) = s^{(r)} + s_1^{(r)} + s_2^{(r)}$$

$$\geq r(3c_1(L)^2 - 2c_1(L)c_1(TS) + c_2(TS)) \quad \ldots \quad (3).$$

By (1.5), equalities in (1), (2) and (3) hold. Thus we have

$$\chi(A_{\mathbb{R}}) = s_0^{(r)} - s_1^{(r)} + s_2^{(r)}$$

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\begin{align*}
= r(-c_1(L)^2 - c_2(TS) + 4) \quad \ldots \quad (4).
\end{align*}

On the other hand, because of the existence of ovals, we have
\begin{align*}
t_0(r) + t_2(r) - (t_0(r-1) + t_2(r-1)) & \geq 2g(Z), \\
t_0(1) + t_2(1) & \geq 2.
\end{align*}

Thus we have
\begin{align*}
t_0(r) + t_2(r) & \geq 2g(Z)(r-1) + 2 \quad \ldots \quad (5).
\end{align*}

Therefore, by (4), (5) and (1.3), we have
\begin{align*}
P_1(A_R;Z/2) & = t_0(r) + t_1(r) + t_2(r) \\
& = 2(t_0(r) + t_2(r)) - \chi(A_R) \\
& \geq (3r-2)c_1(L)^2 - (2r-2)c_1(L)c_1(TS) + rc_2(TS) \\
& = P_1(A;Z/2) \quad \ldots \quad (6).
\end{align*}

By Harnack-Thom's inequality \( P_1(A_R;Z/2) \leq P_1(A;Z/2) \).

Hence equalities in (5) and (6) hold. This completes the proof of Theorem 0.4.

(3.2) Example. Let us consider the case \( S = \mathbb{P}^2 \). Let \( A \) be a non-singular surface of \( \mathbb{P}^2 \times \mathbb{P}^1 \) of degree \( (d,r) \). Then
\( \chi(A) = P_1(A;Z/2) = 3 + d^2 + 3(d-1)^2(r-1) \).

If \( \varphi: A \rightarrow \mathbb{P}^1 \) has only isolated critical points, then
\( s(\varphi) = \sum_{x \in A} \mu_x(\varphi) = 3(d-1)^2r \), where \( \mu_x(\varphi) \) is the Milnor number of \( \varphi \) at \( x \).
Proposition. Let \( A \subset \mathbb{P}^2 \times \mathbb{P}^1 \) be a non-singular real surface of degree \((d,r)\) such that \( \varphi: A \rightarrow \mathbb{P}^1 \) has only isolated critical points. Then we have the sharp estimate

\[
P_{\mathbb{R}}(A; \mathbb{R}/2) \leq 3 + d^2 + 3(d-1)^2(r-1),
\]

\[
(A_{\mathbb{R}}) \leq 3(d-1)^2 r.
\]

Example. Let \( \mathcal{A} = \{ \lambda F + \mu G \mid [\lambda; \mu] \in \mathbb{P}^1 \} \) be a pencil of real plane curves in \( \mathbb{P}^2 \) of degree \( d \).

\( A = (\lambda F + \mu G)_0 \subset \mathbb{P}^2 \times \mathbb{P}^1 \) is non-singular if and only if \( (F)_0 \) and \( (G)_0 \) intersect transversely in \( \mathbb{P}^2 \). If \( A \) is non-singular, then \( A \sim \mathbb{P}^2 \# \mathbb{P}^2 \# \ldots \# \mathbb{P}^2 \). In this case, if \( (F)_0 \)

and \( (G)_0 \) intersect in \( k \) points \( (0 \leq k \leq d^2, \ k \equiv d \pmod{2} ) \), then \( A_{\mathbb{R}} \sim \# \mathbb{R} \mathbb{P}^2 \). Thus \( A \) is an M-surface if and only if \( k = d^2 \).


Let \( S \) be a compact real complex surface, \( L, L' \) real holomorphic line bundles, \( s, s' \) M-sections of \( L, L' \) respectively.

Put \( C = (s)_0 \) and \( C' = (s')_0 \). Assume that \( C \) and \( C' \) are both rational and \( CC' = \langle c_1(L)c_1(L'), [S] \rangle \geq 0 \). (This assumption for \( S \) is rather restrictive (cf. [BPV], Proposition V.4.3).

Consider the following condition:
(**) For any effective divisor \( \alpha \) on \( C \) of degree \( CC' \) with \( \text{supp} \alpha \subseteq C_R \), there exists a real section \( s'' \in H^0(S,L')_R \) such that \( (s'')_0 | C = \alpha \).

Theorem 4.0. Under the condition (**), for any natural numbers \( d \) and \( e \), \( L^{\otimes d} \otimes L'^{\otimes e} \) has an M-section near \( s^{\otimes d} \otimes s'^{\otimes e} \) in \( H^0(S,L^{\otimes d} \otimes L'^{\otimes e})_R \). Furthermore, if \( CC' \) is positive, then \( L^{\otimes d} \otimes L'^{\otimes e} \) has a pair of M-sections near \( s^{\otimes d} \otimes s'^{\otimes e} \) satisfying (*) (cf. Introduction).

Corollary 4.1. If \( C^2 \geq 0 \), then under the condition (**) for \( C' = C \), for any natural number \( d \), \( L^{\otimes d} \) has an M-section near \( s^{\otimes d} \). Furthermore, if \( C^2 \) is positive, then \( L^{\otimes d} \) has a pair of M-sections near \( s^{\otimes d} \) satisfying (*).

4.2 Example. (1) \( S = \mathbb{P}^2 \), \( L = L' = \mathcal{O}_{\mathbb{P}^2}(1) \) (This corresponds to the Harnack's method).

(2) \( S = \mathbb{P}^2 \), \( L = L' = \mathcal{O}_{\mathbb{P}^2}(2) \), \( C = C' \): a real ellipse with \( C_R \neq \emptyset \) (This corresponds to the Hilbert's method).

(3) \( S = \mathbb{P}^1 \times \mathbb{P}^1 \), \( L = L' = p_1^* \mathcal{O}_{\mathbb{P}^1}(1) \otimes p_2^* \mathcal{O}_{\mathbb{P}^1}(1) \).

(4) \( S = \mathbb{P}^1 \times \mathbb{P}^1 \), \( L = p_1^* \mathcal{O}_{\mathbb{P}^1}(1) \), \( L' = p_2^* \mathcal{O}_{\mathbb{P}^1}(1) \) (This is used to show Proposition 0.2 and Corollary 0.6).
References:


(March. 24. 1986).