Title

Topologically Extremal Real Surfaces in $\mathbb{P}^2 \times \mathbb{P}^1$ and $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$

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Topologically Extremal Real Surfaces in 

\( \mathbb{P}^2 \times \mathbb{P}^1 \) and \( \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \).

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From a general viewpoint we illustrate a method of construction of surfaces in \( \mathbb{P}^2 \times \mathbb{P}^1 \) and \( \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \) defined over \( \mathbb{R} \) having topologically extremal properties. Precisely we show that for each \( d, e \) and \( r \) there exists an \( M \)-surface \( A \) in \( \mathbb{P}^2 \times \mathbb{P}^1 \) (resp. \( \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \)) of degree \( (d,r) \) (resp. \( (d,e,r) \)) such that the projection \( A \rightarrow \mathbb{P}^1 \) has the maximal number of real critical points. The construction of \( M \)-surfaces in \( \mathbb{P}^3 \) by O.Ya.Viro is also made more clear.

0. Introduction.

Harnack [H] pointed out that the number of components in the real locus of a curve in \( \mathbb{P}^2 \) of degree \( d \) defined over \( \mathbb{R} \) does not exceed \( 1+(1/2)(d-1)(d-2) \) and, for each \( d \), there exists a non-singular curve in \( \mathbb{P}^2 \) of degree \( d \) defined over \( \mathbb{R} \), the real locus of which has exactly \( 1+(1/2)(d-1)(d-2) \) components.

Hilbert in his 16th problem proposed to investigate
topological restrictions for hypersurfaces in $\mathbb{P}^n$ of fixed degree defined over $\mathbb{R}$.

One may regard an real algebraic function as an one-parameter family of hypersurfaces defined over $\mathbb{R}$, and it is natural to investigate the topological restrictions for hypersurfaces in $\mathbb{P}^n \times \mathbb{P}^1$ of fixed degree defined over $\mathbb{R}$.

Let $A \subset \mathbb{P}^n \times \mathbb{P}^1$ be a real hypersurface of degree $(d, r)$, that is, the zero-locus of a polynomial $\sum_{0 \leq i \leq r} F_i (x_0, \ldots, x_n) \lambda^{r-i} \mu^i$, where $F_i (0 \leq i \leq r)$ is a real homogeneous polynomial of degree $d$.

Consider the projection $\varphi : A \to \mathbb{P}^1$. Our main object is the topology of real locus $A_R$ of $A$ and singularities of the restriction $\varphi_R : A_R \to \mathbb{R} \mathbb{P}^1$ of $\varphi$ to $A_R$.

We denote by $P_t(X, K)$ the Poincaré series of a space $X$ over a field $K$ with indeterminate $t$, and by $s(f)$ the number of critical points of a function $f : X \to R$ from a $n$-dimensional manifold to an one-dimensional manifold.

If $A \subset \mathbb{P}^n \times \mathbb{P}^1$ is non-singular, then the diffeomorphism type of $A$ is determined by $(d, r)$. For example,

$$P(A, K) = \begin{cases} \chi(A) & \text{(n:even)}, \\ 2(n+1) - \chi(A) & \text{(n:odd)} \end{cases}$$

for any $K$,

$$\chi(A) = (n+1)(1-d)^n r + 2 \frac{(1-d)^{n+1} - 1}{d} + n+1), \text{ (cf. 1.6).}$$

We call $A$ generic if $A$ is non-singular and $\varphi : A \to \mathbb{P}^1$ has only non-degenerate critical points.
If $A$ is generic, then $s(\varphi) = (n+1)(d-1)^n r$ (cf. 1.6).

By Harnack-Thom's inequality ([G]), we have an uniform estimate:

$$
\begin{align*}
\left\{ \begin{array}{l}
P_1(A;\mathbb{Z}/2) \leq P_1(A;\mathbb{Z}/2), \\
s(\varphi_R) \leq s(\varphi).
\end{array} \right.
\end{align*}
$$

In this note from a general viewpoint we show the following

**Theorem 0.1.** For $n = 1, 2$ and for each $(d, r)$, the estimate (0.0) is sharp, that is, there exists a generic real hypersurface of $\mathbb{P}^n \times \mathbb{P}^1$ of degree $(d, r)$ attaining both equalities in (0.0).

Notice that in the case $r = 1$ Theorem 0.1 is proved in [I]. A finer result is obtained in the case $n = 1$. For $A \subset \mathbb{P}^1 \times \mathbb{P}^1$, we denote by $\pi: A \rightarrow \mathbb{P}^1$ the projection to the first component.

**Proposition 0.2.** For non-singular real curves $A \subset \mathbb{P}^1 \times \mathbb{P}^1$ of degree $(d, e)$ such that both $\varphi$, $\pi$ have only non-degenerate critical points, there exists the sharp estimate:

$$
\begin{align*}
P_1(A;\mathbb{Z}/2) &\leq 2 + 2(d-1)(e-1), \\
s(\varphi_R) &\leq 2(d-1)e, \\
s(\pi_R) &\leq 2d(e-1).
\end{align*}
$$

Now let us formulate a general theorem which implies Theorem 0.1.
Let $S$ be a real complex surface (cf. 2.1), $C \subset S$ be a real curve possibly with singularities. A non-singular component $E$ of $C_\mathbb{R} \subset S_\mathbb{R}$ is an oval (resp. an empty oval) if there exists an embedding $i: D^2 \to S_\mathbb{R}$ such that $i(\partial D^2) = E$ (and that $i(\text{int } D^2) \cap C_\mathbb{R}$ is empty).

Let $S$ be compact, $L$ a real holomorphic line bundle (cf. 2.6), $s_0, s_1$ $M$-sections of $L$ (cf. 2.7).

Consider the following condition (*):

(*i) The zero-loci $(s_0)_0$ and $(s_1)_0$ are both connected and of genus $g$.

(*ii) $(s_0)_0$ and $(s_1)_0$ intersect in $\langle c_1(L)^2, [S] \rangle$ points in $S_\mathbb{R}$.

(*iii) The real locus of $(s_0 s_1)_0 = (s_0)_0 \cup (s_1)_0$ has $2g$ empty ovals.

We denote by $\mathbb{P}_1^1$ the real complex curve $(\mathbb{P}_1^1, \tau_1)$, where $\tau_1$ is the complex conjugation (cf. 2.3). Fix a pair of $M$-sections $\lambda, \mu$ of $\mathcal{O}_1^1(1)$ such that $(\lambda)_0 \neq (\mu)_0$.

Denote by $\Psi: S \times \mathbb{P}_1^1 \to \mathbb{P}_1^1$, $\iota: S \times \mathbb{P}_1^1 \to S$ the projections. For a transverse section $s$ of $3^*L \otimes \varphi^*\mathcal{O}_1^1(r)$ (cf. 1.3), denote by $\Psi: (s)_0 \to \mathbb{P}_1^1$, $\iota: (s)_0 \to S$ the restrictions of projections. Then, associated to $s$, there is a natural section of $\text{Hom}(T(s)_0, \mathfrak{H}^*_M \mathbb{P}_1^1)$ defined by the tangent map of $\varphi$. 

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Theorem 0.4. Let $S$ be an $M$-surface with connected real part $S_{\mathbb{R}}$, $L$ be a real holomorphic line bundle with a pair $s_0, s_1$ of $M$-sections of $L$ satisfying the condition (*).

Then, for any $r$, there exists an $M$-section $s$ of $\mathcal{O}(s_0 \otimes 1)_{\mathbb{P}^1}$ over $S \times \mathbb{P}^1$ near $s_0 \otimes 1$, which associates an $M$-section of $\text{Hom}(T(s)_0, \mathcal{O}(s_0 \otimes \mathbb{P}^1))$ defined by the projection $\varphi: (s)_0 \rightarrow \mathbb{P}^1$.

Explicitly, $s$ can be taken in a form

$$\sum_{0 \leq i \leq r} \epsilon_i s_i t_i^{r-i},$$

where $s_i = s_0$ ($i$: even), $s_i = s_1$ ($i$: odd) and $\epsilon_0, \epsilon_1, \ldots, \epsilon_r$ are real numbers with $1 = \epsilon_0 \gg |\epsilon_1| \gg \ldots \gg |\epsilon_r| > 0$.

Remark 0.5. A sufficient condition for the existence of a pair of $M$-sections satisfying (*) is given in section 4. Theorem 0.4 with this sufficient condition implies immediately Theorem 0.1 in the case $n = 2$.

Putting $S = \mathbb{P}^1 \times \mathbb{P}^1$ ($= \mathbb{P}^1 \times \mathbb{P}^1$) and $L = \mathcal{O}(d) \otimes \mathcal{O}(r)$ over $S$, we have

Corollary 0.6. For non-singular real surface $A \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ of degree $(d,e,r)$ such that $\varphi: A \rightarrow \mathbb{P}^1$ has only non-degenerate critical points, there exists the sharp estimate:

$$\begin{cases} P_1(A_R; 2/2) \leq 6de-4de-4er-4rd+4d+4e+4r, \\
 s(\varphi) \leq (6de-4d-4e+4)r. \end{cases}$$
From Theorem 0.4, it naturally arises the following general problem:

Problem 0.7. Let $E$ be a real holomorphic vector bundle over a real complex manifold. Give a criterion for the existence or the non-existence of $M$-sections of $E$.

Lastly we intend to clarify the construction of $M$-surfaces in $\mathbb{P}^3$ by Viro [V].

**Theorem 0.8.** (Viro) For non-singular real surfaces $A$ in $\mathbb{P}^3$ of degree $d$, there exists the sharp estimate:

$$p_1(A_R;3/2) \leq d^3 - 4d^2 + 6d.$$ 

Let $X_0, X_1, X_2, X_3$ be homogeneous coordinates of $\mathbb{P}^3$. Put $\mathbb{P}^2 = \{X_3 = 0\}$, $\mathbb{P}^1 = \{X_3 = X_2 = 0\}$ and $\mathcal{U} = \{X_0 = X_1 = 0\}$.

Let $\varphi : \mathbb{P}^3 - \mathcal{U} \rightarrow \mathbb{P}^1$ be a projection. Fix a tubular neighborhood $U$ of $\mathcal{U}$ in $\mathbb{P}^3$ such that $U \cap \mathbb{P}^1$ is empty.

Observe that for each $d$ there exist $M$-sections $s_0, \ldots, s_d$ of $\mathcal{O}_{\mathbb{P}^2}(0), \ldots, \mathcal{O}_{\mathbb{P}^d}(d)$ near $X_2^0, \ldots, X_2^d$ respectively such that $(s_1)_0$ and $(s_{i+1})_0$ intersect in $i(i+1)$ points in $\mathbb{RP}^2$, the real locus of $(s_1s_{i+1})_0$ has $(1/2)(i-1)(i-2) + (1/2)i(i-1)$ empty ovals ($0 \leq i \leq d-1$) and $\varphi | (s_1)_0$ has $(i-1)i$ real critical points ($0 \leq i \leq d$). Naturally each $s_1$ is extended to a section $\tilde{s}_1$ of $\mathcal{O}_{\mathbb{P}^3}(1)$ ($0 \leq i \leq d$).
Put \( s = \sum_{0 \leq i \leq d} \xi_1 x_2^{d-i} s_1 \in H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(d))_{\mathbb{R}}, \) and \( A = (s)_0. \)

Take real numbers \( \xi_0, \ldots , \xi_d \) to be \( 1 = \xi_0 \gg |\xi_1| \gg \ldots \gg |\xi_d| > 0 \)
and of appropriate signs.

\( q_{\mathbb{R}^t} : A_{\mathbb{R}} - U \rightarrow \mathbb{R}^1 \)
defines a vector field \( \mathfrak{F}' \) over \( A_{\mathbb{R}} - U. \)
\( \mathfrak{F}' \) is extended to a vector field \( \mathfrak{F} \) over \( A_{\mathbb{R}} \) with finite
singularities.

Denote by \( s^+(\mathfrak{F}) \) (resp. \( s^-(\mathfrak{F}) \)) the sum of positive (resp.
negative) indices of singular points of \( \mathfrak{F} \), and put
\( t_1 = \dim H_1(A_{\mathbb{R}}; \mathbb{Z}/2) \) (i=1,2,3). Then we see
\[
\begin{align*}
  s^+(\mathfrak{F}) & \geq d + (1/3)d(d-1)(d-2), \\
  s^-(\mathfrak{F}) & \geq (1/3)(d+1)d(d-1) + (1/3)d(d-1)(d-2).
\end{align*}
\]
Thus \( \chi(A_{\mathbb{R}}) = s^+(\mathfrak{F}) - s^-(\mathfrak{F}) \geq d - (1/3)(d+1)d(d-1). \) On the
other hand \( t_0 + t_1 \geq 2 + (1/3)(d-1)(d-2)(d-3). \) Hence we have
\[
\begin{align*}
P_1(A_{\mathbb{R}}; \mathbb{Z}/2) &= t_0 + t_1 + t_2 \\
&= 2(t_0 + t_2) - \chi(A_{\mathbb{R}}) \\
&\geq d^3 - 4d^2 + 6d \quad (= P_1(A; \mathbb{Z}/2)).
\end{align*}
\]

By Harnack-Thom's inequality, all equalities are hold.

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1. Preliminary: Complex Topology.

(1.0) Let $X$ be a complex manifold, $\pi: E \rightarrow X$ a holomorphic vector bundle and $s: X \rightarrow E$ a holomorphic section. Put $(s)_0 = \{x \in X \mid s(x) = 0\}$.

We call $s$ **transverse** if $s$ is transverse to the zero section $\mathcal{Z} \subset E$, that is, for any $s \in (s)_0$, $s^* T_X \otimes T_s(x)^⊥ = T_s(x)^⊥$.

If $s$ is transverse, then $(s)_0$ is a complex submanifold of $X$.

Denote by $H$ the complex vector space $H^0(X, E)$ of totality of holomorphic sections of $E$ over $X$, and by $PH$ the projectification of $H$.

Put $Z = \{(x, [s]) \in X \times PH \mid s(x) = 0\}$ and consider the projection $\tilde{\Phi}: Z \rightarrow PH$. Then $s$ is transverse if and only if $Z$ is non-singular along $\tilde{\Phi}^{-1}[s]$ and $\tilde{\Phi}$ is submersive over $[s]$.

In particular, for transverse sections $s, s' \in H$, $(s)_0$ and $(s')_0$ are diffeomorphic.

(1.1) Let $s \in H^0(X, E)$ be transverse. Put $Z = (s)_0$. Then we have an exact sequence

$$0 \rightarrow TZ \rightarrow TX|_Z \rightarrow E|_Z \rightarrow 0,$$

of complex vector bundles. Therefore $c_t(TX|_Z) = c_t(TZ)c_t(E|_Z)$ for Chern polynomials. The Chern classes of $TZ$ are calculated by the formula $c_t(TZ) = \frac{c_t(TX|_Z)}{c_t(E|_Z)}$ (cf. [F]).
(1.2) Let \( L \) be a holomorphic line bundle over a complex manifold \( V \) of dimension \( n \). Let \( Z \) be the zero-locus of a transverse section of \( L \). Then by (1.1),
\[
\chi(Z) = \left\langle -1 \sum_{i+j = n+1} (-1)^i c_1(TV)(c_1(L))^j, [V] \right\rangle.
\]
For example, if \( \dim V = 2 \), then
\[
\chi(Z) = \left\langle c_1(TV)c_1(L) - c_1(L)^2, [V] \right\rangle.
\]
Furthermore, if \( Z \) is connected, then
\[
\chi(Z) = 1 + (1/2)\left\langle c_1(L)^2 - c_1(L)c_1(TV), [V] \right\rangle.
\]

(1.3) Let \( R \) be a non-singular curve of genus \( g \). Denote by \( \psi: V \times R \to V \) and \( \rho: V \times R \to R \) the projections. Put \( L' = \psi^* L \odot \rho^* (r) \) over \( V \times R \) for each \( r \). Let \( A \subseteq V \times R \) be the zero-locus of a transverse section of \( L' \).

Then \( \chi(A) = \langle \rho, [V] \rangle \), where
\[
\rho = r c_n(TV) + \sum_{i+j = n, j > 0} ((j+1)r+2g-2)c_1(TV)(-c_1(L))^j,
\]
as an element of \( H^{2n}(V; \mathbb{Z}) \).

For example, if \( \dim V = 2 \), then
\[
\chi(A) = \left\langle rc_2(TV) - (2r+2g-2)c_1(TV)c_1(L) + (3r+2g-2)c_1(L)^2, [V] \right\rangle.
\]

(1.4) Example. Let \( C, C' \) and \( C'' \) be non-singular curves
of genus $g, g'$ and $g''$ respectively. Put $X = C \times C' \times C''$, and denote projections by $p_1, p_2$ and $p_3$ to $C, C'$ and $C''$ respectively. Let $A \subset X$ be the zero-locus of a transverse section of $L' = p_1^* O_C(d) \otimes p_2^* O_{C'}(d') \otimes p_3^* O_{C''}(d'')$. Then $\chi(A)$ is equal to $6(d-l)(d'-l)(d''-l) + (2+4g'')(d-l)(d'-l) + (2+4g)(d'-l)(d''-l) + (2+4g')(d''-l)(d-l) + (2+4g'g')(d-l) + (2+4g''g)(d'-l) + (2+4g'g')(d''-l) + 6 - 4(g+g'+g'') + 4(gg'+g''+g'g).

(1.5) In (1.3), denote by $\varphi: A \longrightarrow \mathbb{P}$ the projection to $\mathbb{P}$. Put $\xi = \text{Hom}(TA, \varphi^* \mathbb{P})$. Then $\langle c_n(\xi), [A] \rangle = \langle \gamma, [V] \rangle$, where

$$\gamma = (-1)^{n} r \sum_{j=1}^{r} (j+1)c_1(TV)(-c_1(L))^j,$$

as an element of $H^{2n}(V; \mathbb{Z})$.

For example, if $\text{dim } V = 2$, then

$$\langle c_2(\xi), [A] \rangle = r \langle c_2(TV) - 2c_1(TV)c_1(L) + 3c_1(L)^2, [V] \rangle.$$

(1.6) Let $A$ be a non-singular hypersurface of $\mathbb{P}^n \times \mathbb{P}^1$ of degree $(d,r)$. Then $\chi(A) = \langle c_n(TA), [A] \rangle$ is equal to

$$(n+1)(1-d)^n r + 2(\frac{(1-d)^{n+1}}{d} + n+1).$$

If $\varphi: A \longrightarrow \mathbb{P}^1$ has only isolated critical points, then

$s(\varphi) = \langle c_n(\text{Hom}(TA, \varphi^* \mathbb{P}^1)), [A] \rangle$ is equal to $(n+1)(d-l)^n r$.

(1.7) Let $A$ be a non-singular irreducible projective variety of dimension $n$. Then $H_i(A; \mathbb{Z})$ is torsion free for all $i$, and $\text{rank } H_1(A; \mathbb{Z})$ is equal to $0$ ($i \neq n$, $i$: odd), $1$ ($i \neq n$, $i$: even), $n+1$ $\chi(A)$ ($i=n$, $n$: odd), $\chi(A)-n$ ($i=n$, $n$: even).
(1.8) If $A$ is a simply connected compact complex surface, then $P_t(A;K) = P_{-t}(A;K)$, and $P_1(A;K) = P_{-1}(A;K) = \chi(A)$ for any field $K$.

2. Preliminary: Real Topology.

(2.1) A real structure on a complex manifold $X$ is an anti-holomorphic involution $\tau: X \rightarrow X$. The pair $(X,\tau)$ is called a real complex manifold. Two real complex manifolds $(X,\tau), (X',\tau')$ are isomorphic if there is an isomorphism $\sigma: X \rightarrow X'$ of complex manifolds satisfying $\sigma \circ \tau = \tau' \circ \sigma$ (cf. [S]).

(2.2) Let $(X,\tau)$ be a real complex manifold. We denote by $X_{\mathbb{R}}$ the space $X^\tau$ of fixed points of $\tau$ in $X$, and call it the real locus of $X$ (with respect to $\tau$).

$(X,\tau)$ is a M-manifold if $P_1(X_{\mathbb{R}};\mathbb{Z}/2) = P_1(X;\mathbb{Z}/2)$ (cf. [G]). A M-manifold $(X,\tau)$ of dimension 1 (resp. 2) is called a M-curve (resp. M-surface).

(2.3) Example. The number of equivalence classes of real structures on $\mathbb{P}^n$ is one if $n$ is even and two if $n$ is odd (cf. [F], p.240).

The anti-holomorphic involution $\tau': \mathbb{P}^{2m+1} \rightarrow \mathbb{P}^{2m+1}$ defined by $\tau'[X_0:X_1:\cdots:X_{2i}:X_{2i+1}:\cdots:X_{2m}:X_{2m+1}] = [-\bar{X}_1: \bar{X}_0: \cdots :-X_{2i+1}: \bar{X}_{2i}: \cdots :-X_{2m+1}: \bar{X}_{2m}]$ gives a real structure not equivalent to the usual real structure defined by the complex conjugation $(\mathbb{P}^{2m+1}, \tau_{2m+1})$. We often denote by $\mathbb{P}^{2m+1}_0 = (\mathbb{P}^{2m+1}, \tau')$, $\mathbb{P}^{2m+1}_1 = (\mathbb{P}^{2m+1}, \tau_{2m+1})$.  

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Then $\mathbb{P}^{2m}$ and $\mathbb{P}^{2m+1}_0$ are M-manifolds, but $\mathbb{P}^{2m+1}_0$ is not a M-manifold.

(2.4) From properties of Poicaré series, we see

**Lemma.** Let $(X, \tau)$, $(X', \tau')$ be M-manifolds. Then $(X \perp X', \tau \perp \tau')$ and $(X \times X', \tau \times \tau')$ are also M-manifolds.

(2.5) **Lemma.** Let $(X, \tau)$ be a M-surface with $H_1(X; \mathbb{Z}/2) = 0$ and $H_0(X_R; \mathbb{Z}/2) \cong \mathbb{Z}/2$. Then $\chi(X) + \chi(X_R) = 4$.

**Proof.** $P_{-1}(X; \mathbb{Z}/2) = P_{1}(X; \mathbb{Z}/2) = P_{1}(X_R; \mathbb{Z}/2)$.

$P_{1}(X_R; \mathbb{Z}/2) + P_{-1}(X_R; \mathbb{Z}/2) = 2(\dim H_0(X_R; \mathbb{Z}/2) + \dim H_2(X_R; \mathbb{Z}/2)) = 4$.

(2.6) Let $\pi: E \longrightarrow X$ be a holomorphic vector bundle over a real complex manifold $(X, \tau)$. A real structure of $\pi$ is a real structure $T: E \longrightarrow E$ of $E$ as a complex manifold (cf. 2.1) such that $\pi \circ T = \tau \circ \pi$ and the restriction $T_x: E_x \rightarrow E_{\tau(x)}$ to each fiber $(x \in X)$ is conjugate linear.

We call the triple $E = (\pi; T, \tau)$ a real holomorphic vector bundle (cf. [A]). Notice that the restriction $\pi_R: E_R \rightarrow X_R$ to the real locus of $\pi$ is a real vector bundle.

A holomorphic section $s \in H^0(X, E)$ of $E$ is real if $T_s \circ \tau^{-1} = s$, that is, $s \in H^0(X, E)_R$ with respect to the anti-holomorphic involution $s \mapsto T_s \circ \tau^{-1}$.
(2.7) **Definition.** A holomorphic section \( s \) of a real holomorphic vector bundle over a real complex manifold \((X, \mathcal{T})\) is a **M-section** if \( s \) is transverse, real and the zero-locus \((s)_0 \subset X\) with restricted \( \mathcal{T} \) is a M-manifold.

(2.8) **Remark.** Two real holomorphic vector bundles are isomorphic as real holomorphic vector bundles if and only if they are isomorphic as holomorphic vector bundles.

On \( \mathbb{P}^n \), any holomorphic line bundle has a structure of real holomorphic line bundle.

(2.9) **Poincaré-Hopf-Pugh formula (cf. [P]).**

Let \( M \) be a compact \( C^\infty \) manifold of dimension \( n \) with boundary \( \partial M \).

A tangent vector \( \xi \) to \( M \) at a point \( x_0 \) of \( M \) is external if \( df_{x_0}(\xi) \) is positive for some \( C^\infty \) function \( f \) defined in a neighborhood \( U \) of \( x_0 \) such that \( f^{-1}(0) = \partial M \cap U \), \( f \) takes negative values in \( (M-\partial M) \cap U \) and \( df|_{\partial M \cap U} \) does not vanish (figure 1):

\[ \begin{array}{c}
\xi \\
\downarrow \\
_{x_0} \\
M
\end{array} \]

![Diagram of tangent vector with external condition](image)

Let \( v: \partial M \to TM|\partial M \) be a \( C^\infty \) section over \( \partial M \) to the tangent bundle \( TM \).
Assume that (a): for each \( x_0 \in \partial M \), \( v(x_0) \neq 0 \).

First put \( M_0 = M \). Next put

\[
M_1' = \{ x \in \partial M \mid v(x) \text{ is external} \},
\]

and put \( M_1 = \overline{M_1} \), and \( \partial M_1 = M_1 - M_1' \).

Inductively, if \( M_k \) is a \( C^\infty \) manifold with boundary \( \partial M_k \) (\( k \geq 0 \)), then put

\[
M_{k+1}' = \{ x \in \partial M_k \mid (v|\partial M_k)(x) \text{ is external w.r.t. } M_k \},
\]

\( M_{k+1} = \overline{M_{k+1}} \) and \( \partial M_{k+1} = M_{k+1} - M_{k+1}' \).

Assume that (b): \( M_k \) is a \( C^\infty \) manifold with boundary \( \partial M_k \), (\( k = 1, 2, \ldots, n-1 \)).

Lemma. Let \( v \) satisfy two assumptions (a), (b) stated above. Then for any \( C^\infty \) extension \( w: M \to TM \) with isolated singularities, we have

\[
(c): \quad \text{ind } w = \sum_{i=0}^{n} (-1)^i \chi(M_i).
\]

Remark. (0) We adopt the following definition of index of a vector field: Let \( x_0 \in M \) be an isolated singular point of \( w \). Take a system of coordinates \( x_1, \ldots, x_n \) centered at \( x_0 \), and write locally

\[
w(x) = a_1(x)(\partial/\partial x_1) + \ldots + a_n(x)(\partial/\partial x_n).
\]

Define \( \text{ind}_{x_0} w = \deg_0(-a) \), where \( a = (a_1, \ldots, a_n) \).
Then put $\text{ind } w = \sum \text{ind } x_0 w$, where the sum runs over isolated singular points $x_0$ of $w$.

(1) If $\emptyset M$ is empty, then (c) is the Poincaré-Hopf's formula.

(2) For a $C^\infty$ vector field $w$ over $M$ with only isolated singular points, there exists a non-negative $C^\infty$ function $f: U \to \mathbb{R}$ with the following properties:

(i) $f^{-1}(0) = \emptyset M$. (ii) For any sufficiently small $\varepsilon > 0$, $w|f^{-1}(\varepsilon)$ satisfies two assumptions (a), (b).

3. Non-linear systems of real sections.

In this section we prove Theorem 0.4.

In the situation of Theorem 0.4, put $Z = (s_r)_0 \sim (s_1)_0$

$(0 \leq i < r)$, $s^{(r)} = \sum_{0 \leq i < r} \varepsilon_i s_{i-1} x^{r-i-1}$ and $A^{(r)} = (s^{(r)})_0$. Denote

by $s_i^{(r)}$ (resp. $t_i^{(r)}$) $(i = 0, 1, 2)$ the number of real critical points of $\psi = \psi|A^{(r)}$ of index 1 (resp. dim $H_1(A^{(r)}; \mathbb{Z}/2)$).

Identify $H^1(S; \mathbb{Z})$ with $\mathbb{Z}$ by the fundamental class $[S]$.

(3.1) Proof of Theorem 0.4. By (1.2), $g(Z)$ is equal to

$1 + (1/2)(c_1(L)^2 - c_1(L)c_1(TS))$.

Let $N$ be $S_{ \mathbb{R} }$ minus the interiors of $2g(Z)$ empty ovals. Put $M = \{(x; \lambda, \mu) \in A^{(r)}_{ \mathbb{R} } | |s^{(r-1)}(x; \lambda, \mu)| \geq \delta, x \in N\}$ for a positive number $\delta$ with $|\varepsilon^{(r-1)}_x| > \delta > |\varepsilon^{(r)}_x| > 0$. Then $M$ is a $C^\infty$
manifold with boundary such that $\chi(M) = \chi(S_R) - 2g(Z)$.

Set $w = \text{grad} \varphi_R | M$. Then, with respect to $w$, $\chi(M_1)$ is
equal to $c_1(L)^2$ (cf. 2.9) and $M_2$ is empty. Thus we see

$$\text{index } w = \chi(M) - \chi(M_1) = \chi(S_R) - 2g(Z) - c_1(L)^2.$$ 

Therefore on $M$, the number of critical point of $\varphi_R$ of
index 1 is not less than $-\text{index } w = c_1(L)^2 + 2g(Z) - \chi(S_R)$.

Thus we have

$$s_1(r) - s_1(r-1) \geq 2c_1(L)^2 - c_1(L)c_1(TS) - \chi(S_R) + 2,$$

$$s_0(r) + s_2(r) - (s_0(r-1) + s_2(r-1)) \geq 2g(Z)$$

$$= c_1(L)^2 - c_1(L)c_1(TS) + 2,$$

$$s_0(0) = s_1(0) = s_2(0) = 0.$$ 

So we have

$$s_1(r) \geq r(2c_1(L)^2 - c_1(L)c_1(TS) - \chi(S_R) + 2) \quad \ldots \quad (1),$$

$$s_0(r) + s_2(r) \geq r(c_1(L)^2 - c_1(L)c_1(TS) + 2) \quad \ldots \quad (2).$$

By (2.5), $\chi(S) + \chi(S_R) = 4$. Hence we have

$$s(\varphi_R) = s(r) + s_1(r) + s_2(r)$$

$$\geq r(3c_1(L)^2 - 2c_1(L)c_1(TS) + c_2(TS)) \quad \ldots \quad (3).$$

By (1.5), equalities in (1), (2) and (3) hold. Thus
we have

$$\chi(A_R) = s_0(r) - s_1(r) + s_2(r)$$
\[ = r(-c_1(L)^2 - c_2(TS) + 4) \quad \ldots \ (4). \]

On the other hand, because of the existence of ovals, we have

\[
t_0^{(r)}(r) + t_2^{(r)} - (t_0^{(r-1)} + t_2^{(r-1)}) \geq 2g(Z),
\]

\[
t_0^{(1)} + t_2^{(1)} \geq 2.
\]

Thus we have

\[
t_0^{(r)} + t_2^{(r)} \geq 2g(Z)(r-1) + 2 \quad \ldots \ (5).
\]

Therefore, by (4), (5) and (1.3), we have

\[
P_1(A_R; \mathbb{Z}/2) = t_0^{(r)} + t_1^{(r)} + t_2^{(r)}
\]

\[
= 2(t_0^{(r)} + t_2^{(r)}) - \chi(A_R)
\]

\[
\geq (3r-2)c_1(L)^2 - (2r-2)c_1(L)c_1(TS) + rc_2(TS)
\]

\[
= P_1(A; \mathbb{Z}/2) \quad \ldots \ (6).
\]

By Harnack-Thom's inequality \( P_1(A_R; \mathbb{Z}/2) \leq P_1(A; \mathbb{Z}/2) \).

Hence equalities in (5) and (6) hold. This completes the proof of Theorem 0.4.

(3.2) Example. Let us consider the case \( S = \mathbb{P}^2 \). Let \( A \) be a non-singular surface of \( \mathbb{P}^2 \times \mathbb{P}^1 \) of degree \( (d,r) \). Then

\[ \chi(A) = P_1(A; \mathbb{Z}/2) = 3 + d^2 + 3(d-1)^2(r-1). \]

If \( \varphi: A \rightarrow \mathbb{P}^1 \) has only isolated critical points, then

\[ s(\varphi) = \sum_{x \in A} \mu_x(\varphi) = 3(d-1)^2r, \] where \( \mu_x(\varphi) \) is the Milnor number of \( \varphi \) at \( x \).
Proposition. Let \( A \subset \mathbb{P}^2 \times \mathbb{P}^1 \) be a non-singular real surface of degree \( (d,r) \) such that \( \varphi: A \rightarrow \mathbb{P}^1 \) has only isolated critical points. Then we have the sharp estimate
\[
P_1(A_R; \mathbb{Z}/2) \leq 3 + d^2 + 3(d-1)^2(r-1),
\]
\[
(A_R) \leq 3(d-1)^2r.
\]

Example. Let \( \mathcal{A} = \{ \lambda F + \mu G \mid [\lambda; \mu] \in \mathbb{P}^1 \} \) be a pencil of real plane curves in \( \mathbb{P}^2 \) of degree \( d \).

\( A = (\lambda F + \mu G)_0 \subset \mathbb{P}^2 \times \mathbb{P}^1 \) is non-singular if and only if \( (F)_0 \) and \( (G)_0 \) intersect transversely in \( \mathbb{P}^2 \). If \( A \) is non-singular, then \( A \sim \mathbb{P}^2 \# \mathbb{P}^2 \# \cdots \# \mathbb{P}^2 \). In this case, if \( (F)_0 \) and \( (G)_0 \) intersect in \( k \) points \( (0 \leq k \leq d^2, k \equiv d \text{ (mod. 2)} ) \), then \( A_R \sim \#_{k+1} \mathbb{R} \mathbb{P}^2 \). Thus \( A \) is an \( M \)-surface if and only if \( k = d^2 \).


Let \( S \) be a compact real complex surface, \( L, L' \) real holomorphic line bundles, \( s, s' \) \( M \)-sections of \( L, L' \) respectively.

Put \( C = (s)_0 \) and \( C' = (s')_0 \). Assume that \( C \) and \( C' \) are both rational and \( CC' = \langle c_1(L)c_1(L'), [S] \rangle \geq 0 \). (This assumption for \( S \) is rather restrictive (cf. [BPV], Proposition V.4.3).)

Consider the following condition:
(**) For any effective divisor $\alpha$ on $C$ of degree $CC'$ with $\text{supp} \alpha \subseteq C_R$, there exists a real section $s'' \in H^0(S,L_1)'_R$ such that $(s'')_0|C = \alpha$.

Theorem 4.0. Under the condition (**), for any natural numbers $d$ and $e$, $L^{\otimes d} \otimes L^{\otimes e}$ has an M-section near $s^{\otimes d} \otimes s^{\otimes e}$ in $H^0(S,L^{\otimes d} \otimes L^{\otimes e})'_R$. Furthermore, if $CC'$ is positive, then $L^{\otimes d} \otimes L^{\otimes e}$ has a pair of M-sections near $s^{\otimes d} \otimes s^{\otimes e}$ satisfying (*) (cf. Introduction).

Corollary 4.1. If $C^2 > 0$, then under the condition (**), for $C' = C$, for any natural number $d$, $L^{\otimes d}$ has an M-section near $s^{\otimes d}$. Furthermore, if $C^2$ is positive, then $L^{\otimes d}$ has a pair of M-sections near $s^{\otimes d}$ satisfying (*).

(4.2) Example. (1) $S = \mathbb{P}^2$, $L = L' = \mathcal{O}_{\mathbb{P}^2}(1)$ (This corresponds to the Harnack's method).

(2) $S = \mathbb{P}^2$, $L = L' = \mathcal{O}_{\mathbb{P}^2}(2)$, $C = C'$: a real ellipse with $C_R \neq \emptyset$ (This corresponds to the Hilbert's method).

(3) $S = \mathbb{P}^1 \times \mathbb{P}^1$, $L = L' = p_1^* \mathcal{O}_{\mathbb{P}^1}(1) \otimes p_2^* \mathcal{O}_{\mathbb{P}^1}(1)$.

(4) $S = \mathbb{P}^1 \times \mathbb{P}^1$, $L = p_1^* \mathcal{O}_{\mathbb{P}^1}(1)$, $L' = p_2^* \mathcal{O}_{\mathbb{P}^1}(1)$ (This is used to show Proposition 0.2 and Corollary 0.6).
References:


(March.24.1986).