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Solvable lattice models and the elliptic theta functions

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1. In the theory of integrable systems important roles are played by theta functions. In this note we describe one such connection between solvable lattice models in statistical mechanics and the elliptic theta and modular functions.

In 2 some basic results on the 2 dimensional Ising model are reviewed [1]. In particular, the magnetization is expressed in terms of the elliptic moduli. In 3 the star-triangle relation and a class of elliptic solutions [2] are given. The simplest case of this class is the Ising model. In 4 as the generalization of the magnetization result the local spin probabilities of this model are shown to be expressed in terms of modular functions [3].

2. Consider a two dimensional square lattice Λ. To each site i of Λ we associate a spin variable σ_i which takes the values ± 1. To each bond, say, between two sites i and j, we associate the Boltzmann weight

$$w_{ij}(\sigma_i,\sigma_j) = e^{K_1\sigma_i\sigma_j} \quad \text{if } (i,j) \text{ is a horizontal pair,}$$

$$= e^{K_2\sigma_i\sigma_j} \quad \text{if } (i,j) \text{ is a vertical pair,} \tag{1}$$

where $K_1, K_2 > 0$ are constants.

The magnetization M_0 is the expectation value of the center spin σ_0 with respect to the probability determined by these Boltzmann weights:

$$M_0 = \sum_{\sigma} \prod_{(i,j)} \sigma_0 w_{ij}(\sigma_i,\sigma_j)/Z, \tag{2}$$

$$Z = \sum_{\sigma} \prod_{(i,j)} w_{ij}(\sigma_i,\sigma_j).$$

The Boltzmann weights (1) exhibits the $\mathbb{Z}_2$ symmetry: $w_{ij}(\sigma_i,\sigma_j) = w_{ij}(-\sigma_i,-\sigma_j)$. Therefore, the formal computation of (2) gives rise to the result that $M_0=0$. But, if one first compute the magnetization on a finite lattice fixing the boundary spins to 1 and then take the thermodynamic limit where the lattice size $\uparrow\infty$, one may get a non zero result. In fact, Young [4] obtained

$$M_0 = (1-k^2)^{1/8} \quad \text{if } k < 1,$$

$$= 0 \quad \text{if } k > 1, \tag{3}$$

where

$$k = 1/(\sinh 2K_1 \sinh 2K_2). \tag{4}$$

Note that (4) determines an elliptic curve in the phase space of (z_1,z_2) where $z_i = e^{2K_i}$ $(i=1,2)$. To put it in a different way, in the non-zero region of the magnetization the M_0-constant curves constitute an elliptic fibration. Baxter [5] revealed the mechanism of such relation between solvable lattice and elliptic fibrations. In 3 we recall Baxter's idea and present a class of solvable models obtained in [2].

3. Consider a generalization of the Ising model such that the spin variable σ_i now takes the values $0,1,\ldots,N-1$. Let us denote the Boltzmann weights by $w_H(\sigma_i,\sigma_j)$ if (i,j) is a horizontal pair and by $w_V(\sigma_i,\sigma_j)$ if (i,j) is a vertical pair. A sufficient condition for solvability is known as the star-triangle relation: We need a family of weights $w_H(a,b|u)$ and $w_V(a,b|u)$ $(a,b=0,1,\ldots,N-1)$ parametrized by a complex parameter u satisfying

$$\sum_g w_H(a,g|u)w_V(c,g|u+v)w_H(g,b|v)$$

$$= \chi(u)w_V(c,a|v)w_H(a,b|u+v)w_V(c,b|v), \tag{5}$$

where $\chi(u)$ is a normalization factor independent of a,b,c. Note the fact that (5) is a kind of addition formula, which suggests solutions to be given in terms of theta functions.

In [6] Fateev and Zamolodchikov found a solution of (5) for arbitrary N, which satisfies the $\mathbb{Z}_N$ symmetry

$$w_{ij}(a+1,\ b+1) = w_{ij}(a,b), \tag{6}$$

where $a,b \in \mathbb{Z}_N$. Their solution for $N=2$ coincides with the Ising model at the critical temperature, i.e. $k=1$, where the elliptic curve (4) degenerates to a rational one.

Thus it is tempting to seek for a family of solutions to (5) parametrized by elliptic theta functions and extending to the Fateev-Zamolodchikov solution. In [2] Kashiwara and Miwa presented such solutions by discarding the $\mathbb{Z}_N$ symmetry (6).

Let K be the complete elliptic integral and let $H(u)$ and $\Theta(u)$ be the Jacobian theta functions. We denote by n the integer part of $N/2$. Then define for $a=0,\ldots,n$

$$h_a(u) = \prod_{\ell=0}^{a-1} H((2\ell+u)K/N) \prod_{\ell=a+1}^{n} H((2\ell-u)K/N),$$

$$t_a(u) = \prod_{\ell=0}^{a-1} \Theta((2\ell+u)K/N) \prod_{\ell=a+1}^{n} \Theta((2\ell-u)K/N),$$

$$R_a = \sqrt{\Theta(2sK/N)/\Theta((4a+2s)K/N)},$$

where s is an integer. The Boltzmann weights given by Kashiwara and Miwa are

$$w_H(a,b|u,r) = \frac{r_a r_b}{R_a R_b} W_{ab}(u),$$

$$w_V(a,b|u,r) = \frac{1}{r_a r_b} W_{ab}(1-u),$$

$$W_{ab}(u) = h_{a-b}(u) t_{a+b+s}(u). \tag{7}$$

Here r_a $(a=0,1,\ldots,N-1)$ are multiplicative parameters which ensures the crossing symmetry

$$w_V(a,b|u,r) = w_H(a,b|1-u,R/r).$$

As mentioned already, the solution (7) breaks the $\mathbb{Z}_N$ symmetry, but still enjoys the $\mathbb{Z}_2$ symmetry

$$w_{ij}(-a-s,-b-s) = w_{ij}(a,b). \tag{8}$$

4. The corner transfer matrix was invented by Baxter [5] for the computation of the magnetization. The method is also applicable for general N-state cases in the computation of the probability that the center spin σ_0 takes the value a when the boundary spins are fixed to b;

$$P(a|b) = \sum_{\substack{\sigma_0=a \\ \sigma_{\text{boundary}}=b}} \prod_{(i,j)} w_{ij}(\sigma_i,\sigma_j)/Z.$$

To be precise, we should modify the model by performing the duality transformation with respect to the $\mathbb{Z}_2$ symmetry (8) so that a,b take the values 0,1,...,n-s. (See [3] for details.)

The result is remarkable in the sense it is related to the string functions of the level N $A_1^{(1)}$ modules: We set

$$\theta^{(\pm)}_{\ell,N}(\tau) = \sum_{\nu\in\mathbb{Z}} (\mp)^{\nu} \exp\left(\frac{N\pi i\tau}{2}\left(\nu-\frac{1}{2}+\frac{\ell}{N}\right)^2\right),$$

where $\tau = 2iK/NK'$. Here we consider only the case N=2n, s=1. (See [3] for the complete results.) Then we have

$$P(a|b) = \frac{\theta^{(-)}_{2a+1,N}(\tau)c^N_{2a+1,2b+1}(\tau)}{M^N_b(\tau)}, \tag{9}$$

$$M^N_b(\tau) = \theta^{(+)}_{2b+2,N+2}(\tau)/\theta^{(+)}_{1,2}(\tau).$$

In (9), $c^N_{2a+1,2b+1}(\tau)$ is the main part, which coincides with the string function as explained below.

Consider a binary quadratic form $B(x,y)=2(N+2)x^2-2Ny^2$ on $L = \mathbb{Z} \oplus \mathbb{Z}$. We denote by L^* the dual lattice to L; $L^* = \frac{\mathbb{Z}}{2(N+2)} \oplus \frac{\mathbb{Z}}{2N}$. We set $G_0 = \{g^n | n \in \mathbb{Z}\}$ where $g = \begin{pmatrix} N+1 & N \\ N+2 & N+1 \end{pmatrix}^2$. Note that G_0 is contained in the isotropy subgroup for B and acts trivially on L^* mod L. For $\mu \in L^*$ we define

$$\Theta^B_\mu(\tau) = \sum_{\substack{(x,y)\in G_0\backslash(L+\mu) \\ B(x,y)>0}} \operatorname{sgn} x \; e^{\pi i \tau B(x,y)}.$$

This is a special case of Hecke's indefinite modular forms, and $c^N_{j,k}(\tau)$ is related to it by

$$\eta(\tau)^3 c^N_{j,k}(\tau) = \Theta^B_\mu(\tau)$$

where $\mu = \left(\frac{k+1}{2(N+2)}, \frac{j}{2N}\right)$ and $\eta(\tau) = q^{1/24} \prod_{j=1}^{\infty} (1-q^j)$.

In the study of the character formulas for the level N $A^{(1)}_1$ modules Kac and Peterson found $c^N_{j,k}(\tau)$ appears as the string function [7]. Thus the Kashiwara-Miwa model seems to be

closely related to the level N $A_1^{(1)}$ modules. This may not be surprising because the result of Fateev-Zamolodchikov [8], which deals with the critical and continuum limit of the Kashiwara-Miwa model, is directly related to the level N $A_1^{(1)}$ module. Still we have not unveiled a structural background for the formula (9).

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