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Differential Equations and Grassmann Manifolds
--- from Prof. Sato's lectures ---

Kanehisa Takasaki (高崎金久)
RIMS, Kyoto University

1. Introduction

The subject of this article is an interrelation of two, seemingly distinct mathematical objects, *differential equations* and *Grassmann manifolds*. This interrelation was discovered by Prof. M. Sato in 1981 in his study on the so called *soliton equations*. He also conjectured that a similar interrelation would be found for some other differential equations in higher dimensions. It seems very natural for such a conjecture to be proposed then, because soliton equations, for several reasons, look a fairly special type of differential equations whereas the mechanism that connects them with Grassmann manifolds appears to be of universal nature. This article is concerned with this problem, i.e., how to generalise the interrelation of soliton equations and Grassmann manifolds.

As a matter of fact, it has turned out that this is a very difficult problem. Until now no definite answer has been found. Researches on this subject however are now in progress by Prof. Sato himself and also by young people in Kyoto and Tokyo. This article is intended to introduce part of their present status. inparticular, of the work by Prof. Sato. I would like to apologise here in advance that most material in this article is taken from Prof. Sato's lectures delevered at Kyoto University.

2. soliton equations

To start with, let me briefly review characteristic aspects of soliton equations. Soliton equations are also refered to as a class of *completely integrable systems* (with infinite degree of freedom). Many examples are discovered in the last ten or fifteen years (some were already known in the last century!). Probably the most familiar ones would be the Korteweg-de Vries (or KdV in short) equation. There are many other important examples but I omit their names here. I just stress the fact that they are all *nonlinear* differential equations; therefore classical techniques based on the linear superposition principle, such as the method of Green functions, can not be applied. Nevertheless it has turned out in the last fifteen years of intensive studies that soliton equations have a number of significant roperties such as:

i) Soliton equations have many *elementary* solutions (e.g.

soliton solutions, rational solutions, etc.).

ii) These solutions are written in a *closed form* using elementary functions (e.g. exponential functions, rational functions, etc.).

iii) There is a sort of *nonlinear superposition principle* of these solutions despite of the fact that the equations themselves are nonlinear ones.

iv) By superposing these elementary solutions many (if necessary, infinitely many) times one can obtain a *general solution*, etc....

These facts clearly show that solton equations are in a sense vary similar to linear differential equations. However why?

There may be several ways to answer this question, but probably the most essential point would be that a soliton equation is always accompanied with a linear system (sometimes called a scattering problem). Their relation is just the same as the situation in the integrability theorem of Frobenius, that is, a nonlinear system arises as the *integrability conditions* of a linear system.

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                 integrability conditions
linear system    ------------>            nonlinear system
                 <------------
                 existence of solutions
      :                                          :
      :                                          :
scattering problem                        soliton equation
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This is of course just the beginning of a long story; it is hard to explain in this short article why such a double structure leads to consequences as mentioned above. We shall not go further into its detail here.

Instead, I would like to show below another, more accessible example of nonlinear differential equations for which *linearity* and *nonlinearity* intertwine just as I mentioned for the case of soliton equations. This example, in fact, is deeply connected with soliton equations; we shall return to this point later.

3. A model in one dimension

Geometrically, a Grassmann manifold is the set of vector subspaces with assigned dimensions, say m, in a given vector space V :

(1) $$GM = GM(m;\ V) = \{\text{vector subspaces } U \subset V\ ;\ \dim U = m\}.$$

Let us use the sign $[U]$ if we stress that U is considered a *point* of GM rather than a vector subspace of V. Such a Grassmann manifold carries a vector bundle H called the "tautological" bundle, whose fiber at a point $[U]$ of GM is the corresponding vector space U itself:

(2) $$\begin{array}{ccccc} U & = & H_{[U]} & \subset & H \\ \downarrow & & & & \downarrow \\ [U] & \in & GM & = & GM(m;\ V) \end{array}$$

Let us here draw attention to the fact that the Grassmann manifold in itself is a nonlinear manifold whereas the vector space V and its vector subspaces are evidently linear manifolds. What we now argue below is that there are linear and nonlinear systems for which the above geometric situation is faithfully realised at the level of their solution spaces.

To be precise, they are related to a *factorisation* of an ordinary differential operator. (In fact, such an idea is by no means new, and can be found in the Galois theory of linear differential equations known as the Picard-Vessiot theory.) Let us consider an ordinary differential operator P of order N of the form

$$(3) \qquad P = (d/dx)^N + p_1(x)(d/dx)^{N-1} + \dots + p_N(x).$$

As for the coefficients, we assume that they lie in a *good* class of functions, such as that of analytic or meromorphic functions, in which the uniqueness of solutions of the initial value problem, etc. hold. The factorisation problem mentioned above is to find two ordinary differential operators such that

$$(4) \quad P = W^*W, \quad W^* = (d/dx)^n + w_1^*(x)(d/dx)^{n-1} + \dots + w_n^*(x),$$
$$W = (d/dx)^m + w_1(x)(d/dx)^m + \dots + w_m(x),$$
$$N = n + m, \quad m, n\text{: positive integers.}$$

Evidently this yields a system of nonlinear differential equations for the coefficients of W^* and W. It would be worth mentioning that if $m = n = 1$, this system reduces to the

ordinary Riccati equation of second order whose solution space is known to form a projective line $\mathbb{P}^1$. For a general case it turns out that the solution space of the nonlinear system given by (4) can be identified with a Grassmann manifold as follows:

$$(5) \qquad \{(W^*, W);\ P = W^*W\} \simeq GM(m, V),$$

where $V = \{ u;\ Pu = 0 \}$ $(\dim V = N)$. The map from the left to the right assigns to each pair (W^*, W) the solution space of the linear differential equation $Wu = 0$ which forms an m-dimensional vector subspace of V :

$$(6) \qquad (W^*, W) \ \text{--->}\ U := \{ u;\ Wu = 0 \} \subset V.$$

The inverse of this map, too, can be explicitly constructed using Wronskian determinants. To see this, we take for any m-dimensional vector subspace U of V a basis $u_0, u_1, \ldots, u_{m-1}$, and define a linear differential operator W to be:

$$(7) \qquad Wu = \frac{\det\left(\begin{array}{c|c} (d/dx)^i u_j & (d/dx)^i u \\ \hline (d/dx)^m u_j & (d/dx)^m u \end{array}\right)}{\det\left((d/dx)^i u_j \right)} \quad (0 \le i,\ j \le m - 1).$$

It is not hard to see that this gives an operator whose solution space agrees with U. To show that it indeed factors out the operator P , let us note that the ordinary division procedure of integers or polynomials in one variables can be extended to ordinary differential operators as well. In particular, the oprerator P can be represented in a unique way as follows:

$$P = QW + R, \qquad Q = (d/dx)^n + q_1(x)(d/dx)^{n-1} + \ldots + q_n(x).$$
$$R = r_0(x)(d/dx)^{m-1} + \ldots + r_{m-1}(x).$$

Since P and W both annihilate U, if follows that R also annihilates U. This situation however can not occur unless $R = 0$, because the dimensions of the solution space of R do not exceed its order ($\leq m - 1$). Thus rewriting Q as W^*, we obtain a pair (W^*, W) that factorises P as in (4).

An interesting feature of the above construction is that it also tells us what equations correspond to the tautological bundle. Each fiber $H_{[U]}$ of the tautological bundle H can be identified with the solution space of the linear differential equation $Wu = 0$. The bundle itself corresponds to the system of differential equations $P = W^*W$, $Wu = 0$, which consists of the nonlinear part $P = W^*W$ and the linear part $Wu = 0$ describing, respectively, the base space $GM(m, V)$ and the fiber $H_{[U]}$ at each point of the base space.

Thus it turns out that a Grassmann manifold is not merely a geometric object, but also admits an analytical (or "algebro-analytical") realisation as the solution space of some differential equations.

$$\begin{array}{ccc} H & \langle\text{-----}\rangle & P = W^*W,\ Wu = 0 \\ \downarrow & & \downarrow \\ GM(m, V) & \langle\text{-----}\rangle & P = W^*W \\ \langle\text{GEOMETRY}\rangle & & \langle\text{DIFFERENTIAL EQUATIONS}\rangle \end{array}$$

4. Dynamical motion on a Grassmann manifold

What we have viewed above is a typical example of the subject of this article, interrlations of differential equations and Grassmann manifolds. This example, as already mentioned, in itself can not be identified with a soliton equation, but they are in fact in a close relationship. The essence is the fact that *a soliton equation corresponds to dynamical motion on a Grassmann manifold*; this was the main conslusion of Prof. Sato's discovery in 1981. We now turn to this topic and illustrate his geometric interpretation of soliton equations in the finite dimensional case discussed in §3. (In fact, a finite dimensional Grassmann manifold can parametrise only a special class of solutions of a soliton equation such as rational and soliton equations, which are "elementary" in the sense mentioned in §2. More general solutions, as stressed by Prof. Sato, lie in their limit to an infinite dimensional Grassmann manifold which he called the "universal Grassmann manifold".)

In order to obtain a *commuting* set of dynamical flows on $GM(m, V)$, we now start from the situation where P is an ordinary differential operator with *constant* coefficients, i.e. $p_1, \ldots, p_N$ are constants. This means that P commutes with all the powers $(d/dx)^k$, $k = 1, 2, \ldots$,

$$(8) \qquad [P, (d/dx)^k] = 0,$$

and therefore the $(d/dx)^k$'s can act on the solution space, V,

of the linear equation $Pu = 0$. (It would be worth noting that they give infinitesimal generators of *inner symmetries* of this linear equation; this point of view seems to suggest some generalisation of the present construction.) Let us write the linear endomorphism induced by (d/dx) on V as Λ :

$$(9) \qquad \Lambda = d/dx : V \dashrightarrow V.$$

Accordingly the powers Λ^k, $k = 1, 2, \ldots,$ agree with the linear endomorphisms induced by $(d/dx)^k$. Now we *exponentiate* the above infinitesimal inner symmetries of the linear equation $Pu = 0$ to define the following commuting flows on V :

$$(10) \qquad \exp \sum_{k=1}^{\infty} t_k \Lambda^k : V \dashrightarrow V,$$

where $t_1, t_2, \ldots,$ are time variables. This gives rise to a set of commuting dynamical flows on $GM(m, V)$ parametrised by the multi-time variables $t = (t_1, t_2, \ldots)$:

$$(11) \qquad \begin{aligned} \exp \sum t_k \Lambda^k : GM(m, V) &\dashrightarrow GM(m, V) \\ [U] &\dashrightarrow [\exp(\textstyle\sum t_k \Lambda^k) U] \end{aligned}$$

The above dynamical flows on $GM(m, V)$ can be transformed into time evolutions of ordinary differential operators $W = W(t,x,d/dx)$ and $W^* = W^*(t,x,d/dx)$ that arise as factors of P, $P = W^* W$. What equations will they satisfy then?

The answer is as follows: They satisfy the equations

$$(12) \qquad \frac{\partial W}{\partial t_k} = B_k W - W\left(\frac{d}{dx}\right)^k, \quad \frac{\partial W^*}{\partial t_k} = \left(\frac{d}{dx}\right)^k W^* - W^* B_k,$$

where B_k, $k = 1, 2, \ldots,$ are ordinary differential operators

of order k, which are connected with W and W^* by simple formulae though I omit the detail here. I just mention that the coefficients of the B_k's become polynomials of derivatives of the coefficients of the W and W^*. Therefore, in particular, eqs. (12) may be thought of as a system of nonlinear differential equations for the coefficients of the W and W^*.

This nonlinear system, in fact, agrees with a special case of the soliton equations studied by Prof. Sato in 1981. To be more precise, it describes special solutions, such as rational solutions and soliton solutions, of a soliton equations known under the name of the Kadomtsev-Petviashvili (or KP, in short) equation. For example, the solutions thus obtained correspond to soliton solutions if V consists of linear combinations of exponential functions $\exp(\lambda_j x)$ where λ_j are constants, and to rational solutions if V is formed by polynomials in x. For the former case, for example, linear operator (10) changes the exponential functions $\exp(\lambda_j x)$ into $\exp(\lambda_j x + \sum \lambda_j^k t_k)$, which indeed takes the familiar form of exponential functions that occur in a variety of explicit formulae of soliton solutions of the KP equation. For the latter case, on the other hand, the action of the linear operator becomes more complicated and produces linear combinations of the so called "Schur polynomials" that played central roles in the work of Prof. Sato.

The present framework can also explain a mechanism under which solutions of other soliton equations are derived as a subset of those of the KP equation. For example, under the

condition that the U is invariant under the action of Λ^2, i.e.

(13) $$\Lambda^2 U = (d/dx)^2 U \subset U,$$

the above solutions correspond to those of the KdV equation; if the above condition is replated by

(14) $$\Lambda^3 U = (d/dx)^3 U \subset U,$$

then we obtain solutions of another soliton equation, known under the name of Boussinesq equation. It has been known since the 70's that the KdV and Boussinesq equations may be thought of as special cases ("reductions") of the KP equation; in other words, the solution spaces of the former form subsets of that of the latter. What I mentioned above shows how this fact can be understood in a more gemetric framework; eqs. (13) and (14) indeed define the subsets (subvarieties) of the Grassmann manifold that correspond to solutions of the KdV and Boussinesq equations respectively.

Of course, as mentioned above, solutions thus obtained are of considerably limited type. More general and complicated solutions does not lie in finite dimensional Grassmann manifolds as taken above, but in an appropriate infinite dimensional limit. The "universal" Grassmann manifold is a good candidate for such a limit, and it was indeed proved by Prof. Sato that basically the same machinary as discussed above can work in the framework of the universal Grassmann manifold, leading to a satisfactory way of understanding of more general solutions.

5. A model of higher dimensinal generalisation

Now I would like to turn to the problem of higher dimensional generalisation. This problem, as mentioned in the beginning of this article, is at present very hard to attack and no definite answer is known until now, though there are several speculations. Below I show one of them also due to Prof. Sato.

A central idea is to replace the linear *ordinary* differential equations used in the previous setting by *partial* ones. Of course, in order to put the situation under good control, these equations are required to be such that the solution space be *finite dimensional*. Therefore, in particular, we have to take a *system* of linear equations, not a single equation like $Pu = 0$ or $Wu = 0$ as used in the previous setting. A system of linear differential equations with the above property, i.e. the finite dimensionality of its solution space, is called a "holonomic system" in the terminology of "algebraic analysys". (A precise definition of this notion requires a variety of cohomological objects, so I omit it here.) For example, an integrable Pfaffian system is a holomoic system. Thus, noting that the linear equations $Pu = 0$ and $Wu = 0$ are a pair of holonomic systems (in one dimension) under the "incidence" relation $V \supset U$, a higher dimensional counterpart of that setting would be, naturally, a pair of holonomic systems under a similar "incidence" relation of their solution spaces:

one dimensional case	higher dimensional case
V: $Pu = 0$	$P_j u = 0$ (holonomic system)
U: $Wu = 0$	$W_k u = 0$ (holonomic system)
with incudence relation	with incidence relation
$U \subset V$	$U \subset V$

Again, under some appropriate situation, it can be shown that holonomic systems $W_k u = 0$ with incidence relation $U \subset V$ and with $\dim U = m$ are parametrised by (a subset of) the Grassmann manifold $GM(m, V)$.

I now illustrate such an example in, say, two dimensions with indipendent variables (x, y). Take a system of N functions $u_0 = u_0(x, y), \ldots, u_{N-1}(x, y)$ for which the Wronskian with respect to the first variable does not vanish:

$$\det((\partial/\partial x)^i u_j) \neq 0 \quad (i, j = 0, \ldots, N-1). \tag{15}$$

It is then not hard to see that there are two linear differential operators P_0, P_1 of the form

$$\begin{aligned} P_0 &= (\partial/\partial x)^N + a_1(x,y)(\partial/\partial x)^{N-1} + \ldots + a_N(x,y), \\ P_1 &= (\partial/\partial y) + b_1(x,y)(\partial/\partial x)^{N-1} + \ldots + b_N(x,y) \end{aligned} \tag{16}$$

for which the u_j's give a basis of the solution space of the holonomic system

$$P_0 u = 0, \quad P_1 u = 0. \tag{17}$$

This can be checked with almost the same argument as employed

for the one dimensional case. (In particular, we can obtain explicit formulae for P_0, P_1 using Wronskian determinants as in (7).) Thus the vector space V spanned by the u_j's is now identified with the solution space of holonomic system (17). Just the same argument, applied to an m-dimensional subspace U of V, leads to the consequence that there are two linear differential operators W_0, W_1 of the form

$$(18)\qquad \begin{aligned} W_0 &= (\partial/\partial x)^m + w_1(x,y)(\partial/\partial x)^{m-1} + \dots + w_m(x,y), \\ W_1 &= (\partial/\partial y) + v_1(x,y)(\partial/\partial x)^{m-1} + \dots + v_m(x,y) \end{aligned}$$

for which the vector space U agrees with the solution space of the holonomic system

$$(19)\qquad W_0 u = 0, \quad W_1 u = 0.$$

The incidence relation $U \subset V$ between the solution spaces can also be characterised, at the level of differential operators, by the existence of linear differential operators $A_{00}, \dots, A_{11}$ for which the following equations are satisfied:

$$(20)\qquad P_0 = A_{00}W_0 + W_{01}W_1, \quad P_1 = A_{10}W_0 + A_{11}W_1.$$

The last equations can be thought of as a counterpart of the factorisation equation (4) in the one dimensional case. Thus, viewed as differential equations for the coefficients of the operators included, eqs. (20) give a system of nonlinear differential equaions whose solutions are parametrised by the Grassmann manifold $GM(m, V)$.

The above argument can be reformulated in a more intrinsic

way using the notion of modules over a ring of linear differential opearators. The use of such rings and modules in a problem of analysis, which can date back to the end of the 50's when the theory of hyperfunctions made its first step, is one of the basic ideas of "algebraic analysis" initiated by Prof. Sato. Let $\mathcal{D}$ denote the ring of linear differential operators in two variables (x, y), $\mathcal{D} = \{ \sum a_{ij}(x,y)\partial^{i+j}/\partial x^i \partial y^j ;$ $a_{ij}(x,y) \in \mathcal{O} \}$, where $\mathcal{O}$ is a differential algebra which I do not specify for simplicity (in fact, the choice of $\mathcal{O}$ includes some delicate problems though I do not discuss them here). Then the previous holonomic systems corresponds to left $\mathcal{D}$ - modules as follows:

$$(21) \qquad \begin{array}{lll} P_j u = 0 & \text{<----->} & M := \mathcal{D} / I, \quad I := \sum \mathcal{D} P_j, \\ W_k u = 0 & \text{<----->} & N := \mathcal{D} / J, \quad J := \sum \mathcal{D} W_k. \end{array}$$

Note that I and J are left ideals of $\mathcal{D}$, generated by the P_k's and W_j's respectively. Then the relation as shown in (20) can be more concisely restated as:

$$(22) \qquad I \subset J,$$

or equivallently, it induces a surjective homomorphism $M \dashrightarrow N$ of $\mathcal{D}$-modules. An advantage of such a reformulation is that it provides an intrinsic way of understanding of differential equations in a form paralell to the formalism adopted in algebraic geometry.

Nonlinear equations associated with commuting dynamical flows can also be derived almost the same way as in the one

dimensional case.

It should be stressed that in applications, if they exist, the above choise of the holonomic systems might be modified to a large extend. For example, the form of the first holonomic system $P_j u = 0$ taken above seems to be fairly special; probably more general forms would be required in applications; if the first holonomic system is chosen as such, then, accordingly, the second holonomic system would take a more general form as well. Reseraches in this direction are now in progress though I omit the detail.

6. Outlook

The above generalisation is by no means a final one. First of all, we could just obtained examples where the Grassmann manifolds concerned are finite dimensional one. The final form of the theory, if it exists, should be such that an infinite dimensional Grassmann manifold gives a parametrisation of the solution space of a system of differential equations, as that was indeed the case for soliton equations. If there is such a nonlinear system, the example constructed above would be derived as a special reduction, or it would be rather better to say that such a final formulation could be attained in an infinite dimensional limit of these examples. At present however no one knows how to take such a limit.

It would be worth noting that such an approach is not a

unique possibility to access the present problem. Another approach would be to study good examples of nonlinear equations known in, for example, physics. In this respect the self-duality equations of both Yang-Mills and gravitational fields are of particular interest, because for several reasons they are recognised as completely integrable systems in higher dimensions. In recent works I could indeed found a connection between these equations and some sorts of infinite dimensional Grassmann manifolds. The frame work adopted there is somewhat different from the present one, so clarfying their relationship would be a very important problem. A speculative argument, though I omit it here, suggests that a class of self-dual solutions of Yang-Mills equation, probably including instanton solutions, can be derived from the finite dimensional model discussed in §5. This might provide some hint to the above problem.

The present status of reserach is anyway far from the goal. Probably some crucial idea or a new point of view is still lacked there. Prof. Sato's interest, as far as I could guess from his recent lectures, now seems to aim at very abstract and general problems. One of them, which seems to form central part of his ideas, is to find a categry of noncommutative rings and their modules in which the notion of differential equations (both linear and nonlinear; not limited to "completely integrable" ones) is reformulated in an intrinsic way, just as the notion of algebraic equations (and algebraic varieties as their solutions) is given a firm foundation on the basis of the theory of commutative algebras. A key to the problem of

"complete integrability" in higher dimensions, too, might be obtained in such a general point of view. As a matter of fact, at present, we do not even have a rigorous and general mathematical definition of "complete integrability" for nonlinear partial differential equations. It seems that an appropriate definition of this vague notion can not be found without a deeper understanding of general aspects of differential equations. What we really need would be therefore a *general theory* of differential equations, which has been Prof. Sato's *Jugendtraum*.

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