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On higher-order terms in Asymptotic expansions for irreducible characters of semisimple Lie groups

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Introduction.

1. Let G be a real connected semisimple Lie group, $\mathfrak{g}$ its Lie algebra. To make Harish-Chandra's theory on invariant eigen-distributions applicable, we assume following two conditions to hold:

i) the center of G is finite,

ii) G is acceptable in the sense of Harish-Chandra.

These conditions are satisfied by the examples considered in what follows, namely, the groups $G=SL(2,\mathbb{R})$ and $SL(2,\mathbb{C})$. More generally, if G is a real connected semisimple Lie group with finite center, then the condition ii) above is satisfied by a suitable finite-fold covering group of G.

Let π be a continuous representation of G on a Hilbert space. We assume π to be quasi-simple in the sense of Harish-Chandra. Then the character Θ_π of π is defined, and is a distribution on G:

$$\Theta_\pi : \phi \longmapsto \mathrm{trace}\int_G \pi(g)\phi(g)dg, \quad \phi \in \mathcal{D}(G).$$

Θ_π is an invariant eigendistribution, in the sense that

① it is invariant under the inner automophisms of G,

② it is a simultaneous eigenfunction (in distribution sense) of all differential operators on G which are invariant under right- and left-translations of G.

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The detailed study of characters gives us much information about the original representation π; for example, we have

<u>Proposition.</u> Let π_1 and π_2 be two irreducible unitary representations of G. Then they are unitarily equivalent if and only if their characters coincide:

$$\Theta_{\pi_1} = \Theta_{\pi_2} .$$

Thus the characters have been the subject of many deep studies, due originally to Harish-Chandra.

2. Let exp denote the exponential mapping, which is a local diffeomorphism from $\mathfrak{g}$ to G:

$$\exp : \mathfrak{g} \longrightarrow G.$$

This mapping is not in general injective or surjective, but if we restrict it to a sufficiently small neighborhood Ω of 0 in $\mathfrak{g}$, it is a diffeomorphism:

$$\exp : \Omega \xrightarrow[\text{diffeom.}]{} \exp(\Omega), \qquad 0 \in \Omega \subseteq \mathfrak{g} .$$

Now we pull back the distribution Θ_π and obtain a distribution θ_π on Ω in the following manner. For $\phi \in \mathcal{D}(\Omega)$, we define

$$\langle \theta_\pi , \phi \rangle \equiv \langle \Theta_\pi , \tilde{\phi} \rangle ,$$

where $\tilde{\phi} \in \mathcal{D}(\exp(\Omega))$ is defined by

$$\tilde{\phi}(\exp X) \equiv \phi(X) / \xi(X) ,$$

with ξ an auxiliary function on $\mathfrak{g}$ defined by

$$\xi(X)^2 \equiv (\text{Jacobian of } \exp \text{ at } X) , \qquad \xi(0) = 1 .$$

Then the so-defined $\Theta_\pi \in \mathcal{D}'(\Omega)$ is an invariant eigen-distribution, in the sense that

① it is invariant under the adjoint action Ad of G,

② it is a simultaneous eigenfunction (in distribution sense) of all constant-coefficient differential operators on $\mathfrak{g}$ which are invariant under the adjoint action.

3. As mentioned above, the detailed study of Θ_π and Θ_π was initiated by Harish-Chandra. He proved in particular that Θ_π (resp. Θ_π) is identifiable to a locally summable function which is actually real-analytic on G (resp. on $\Omega \subseteq \mathfrak{g}$) except for a certain set of lower dimension.

4. The study of Θ_π and Θ_π is being continued by many authors. Among other people, Barbasch and Vogan defined in [1] an asymptotic expansion near $0 \in \mathfrak{g}$ for such Θ_π, in the following form:

$$\Theta_\pi(f_t) \sim \sum_{i=i_0}^{\infty} t^i D_i(f) \qquad \text{as } t \downarrow 0,$$

where D_i's are suitable tempered distributions on $\mathfrak{g}$, and when f is in $\mathcal{D}(\Omega)$, $f_t \in \mathcal{D}(\mathfrak{g})$ is defined by

$$f_t(X) \equiv t^{-n} \cdot f(X/t), \quad X \in \mathfrak{g}, \quad n = \dim G.$$

Here we have extended f from Ω to the whole of $\mathfrak{g}$ by putting $f \equiv 0$ outside Ω. We review the details of the definition in Chapter 1.

At this point, an interesting object is the set $\mathrm{cl}(\bigcup_i \operatorname{supp} \hat{D}_i) \subseteq \mathfrak{g}^*$, where $\hat{D}_i$ is the Fourier transform of D_i,

and "supp" stands for the support, and "cl" means the closure. Barbasch and Vogan call this set the asymptotic support of Θ_π, and write $AS(\Theta_\pi) \equiv cl(\bigcup_i \text{supp}\, \hat{D}_i)$. It can be shown that $\bigcup_i \text{supp}\, \hat{D}_i$ is a closed subset of $\mathfrak{g}^*$, so that the symbol "cl" is actually unnecessary.

$AS(\Theta_\pi)$ can be considered as something indicating in which direction Θ_π is singular (in the tangent space at 0). We note that similar objects have been the subjects of Kashiwara-Vergne [9] and Howe [8].

Among the distributional coefficients D_i in the expansion, the lowest-order non-vanishing term D_{i_0} turned out to be very important. To be particular, we have by [1] and [2]:

① i_0 is necessarily non-positive. More precisely,

$$-i_0 = \text{G.K-dim}(U(\mathfrak{g}_c)/I_\pi),$$

where G.K-dim means the Gel'fand-Kirillov dimension of a $\mathbb{C}$-algebra, and $U(\mathfrak{g}_c)$ is the universal enveloping algebra of the complexification $\mathfrak{g}_c$ of $\mathfrak{g}$, and I_π denotes the annihilator ideal of π.

② When G is a complex group, D_{i_0} is the Fourier transform of a certain important distribution on $\mathfrak{g}^*$, called a nilpotent orbital integral (see Chapter 1).

③ In case where $AS(\Theta_\pi)$ is an irreducible variety, we have $AS(\Theta_\pi) = \text{supp}\, \hat{D}_{i_0}$, and $AS(\Theta_\pi)$ and/or $\text{supp}\, \hat{D}_{i_0}$ serve as a tool to get a concrete description for the theory of classification of primitive ideals for $\mathfrak{g}$, which has been in an abstract sense already completed by Duflo, Joseph and Vogan.

For ② above, we remark that the results in [2], [3] were later generalized by Hotta-Kashiwara [7].

5. A natural question to ask is, therefore, the one concerning the higher-order terms in the expansion. Motivated thus, we examine in this paper the explicit forms of the higher-order terms D_i and their Fourier transforms, for irreducible representations of certain groups. Our hope is to get from them some information about the original representation.

As the simplest examples, we take $G=SL(2,\mathbb{R})$ and $SL(2,\mathbb{C})$. Our result is complete in that it treats all irreducible representations (unitary or not) of these groups. It turns out that the higher-order terms are characterized by the appearance of certain invariant differential operators.

6. Let us now explain the content of each chapter.

In Chapter 1, we review the definition and elementary properties of the asymptotic expansion, given in [1].

In Chapter 2, we treat the case of $G=SL(2,\mathbb{R})$. We list all D_i and their Fourier transforms, for all irreducible representations of this group. We find especially that for a "principal series" representation π, $AS(\Theta_\pi)$ is not supported on the closure of a single orbit.

Chapter 3 treats $G=SL(2,\mathbb{C})$. Again we give a complete list, for all irreducible representations of this group. Here the following two differential operators come in:

$$\Box_c \equiv (\partial/\partial x)^2 + (\partial/\partial y)^2 - (\partial/\partial z)^2 ,$$

$$\bar{\Box}_c \equiv (\partial/\partial\bar{x})^2 + (\partial/\partial\bar{y})^2 - (\partial/\partial\bar{z})^2,$$

where x, y and z are complex numbers.

In Chapter 4 we make some observation on the connection between the asymptotic expansion and character formulae of Kirillov type. There we encounter a problem of justifying a Taylor series expansion of the following kind:

$$\delta(P(x,y,z)+c) = \\ = \delta(P(x,y,z)) + c\cdot\delta'(P(x,y,z)) + \\ + \tfrac{1}{2}\cdot c^2\cdot\delta''(P(x,y,z)) + \cdots\cdots .$$

Here δ denotes the Dirac delta function in one variable, and the hypersurface defined by P=0 has singularities. The answer to this problem and its generalization to groups of higher ranks would require some rather deep knowledge about the G-orbit structure of the Lie algebra.

<u>Chapter 1.</u> Definition and elementary properties of asymptotic expansion.

In this chapter we recall some basic definitions and propositions that we will use later. We begin by reviewing the definition of a nilpotent orbital integral.

1-1. Nilpotent orbital integrals.

Let $\mathfrak{g}$ be a real semisimple Lie algebra, G the adjoint group of $\mathfrak{g}$. An element $X \in \mathfrak{g}$ is called nilpotent if ad(X) is a nilpotent linear transformation in $\mathfrak{g}$. We identify $\mathfrak{g}^* \equiv \mathrm{Hom}_{\mathbb{R}}(\mathfrak{g}, \mathbb{R})$ with $\mathfrak{g}$ via Killing form. Thus we say $Y \in \mathfrak{g}^*$ is nilpotent if Y is, as an element of $\mathfrak{g}$, nilpotent. G acts on $\mathfrak{g}^*$ by coadjoint action. We denote by $\mathcal{N}$ the set of all nilpotent elements in $\mathfrak{g}^*$. If X is in $\mathcal{N}$, the G-orbit Coad(G)$\cdot X$ is called a nilpotent orbit.

The G-homogeneous space $\mathcal{O}$ =Coad(G)$\cdot X$ has a G-invariant measure. Call it μ. We have the following:

Proposition 1-1 (Ranga-Rao, [10]) For $\phi \in \mathcal{D}(\mathfrak{g}^*)$, let $\phi|_{\mathcal{O}}$ be its restriction to $\mathcal{O}$ = Coad(G)$\cdot X$. Then

$$\mu_{\mathcal{O}} : \phi \longmapsto \int \phi|_{\mathcal{O}} \cdot \mu$$

is a distribution on $\mathfrak{g}^*$: $\mu_{\mathcal{O}} \in \mathcal{D}'(\mathfrak{g}^*)$.

We call $\mu_{\mathcal{O}}$ the nilpotent orbital integral corresponding to the orbit $\mathcal{O}$.

1-2. Definition of asymptotic expansion.

As in Introduction, let

$$f_t(x) \equiv t^{-n} \cdot f(x/t) \quad , \quad n = \dim \mathfrak{g} ,$$

for $f \in \mathcal{D}(\mathfrak{g})$. Let $\theta \in \mathcal{D}'(\Omega)$, where Ω is a fixed neighborhood of 0 in $\mathfrak{g}$.

Definition. We write

$$\theta(f_t) \sim \sum_{i=i_0}^{\infty} t^i D_i(f) \quad \text{as } t \downarrow 0,$$

with $\{D_i\}$ a family of distributions on $\mathfrak{g}$, if the following condition (A) holds:

Condition (A): For any positive integer N, and a compact subset $K \subseteq \mathfrak{g}$, there exist a constant $C_{N,K} > 0$, a positive integer $k_{N,K}$, and a constant $\varepsilon_{N,K} > 0$, such that if $\operatorname{supp} f \subseteq K$ and $0 < t \leqq \varepsilon_{N,K}$,

then

$$\operatorname{supp} f_t \subseteq \Omega$$

and

$$\left| \theta(f_t) - \sum_{i=i_0}^{N} t^i D_i(f) \right| < C_{N,K} \cdot \sup_{|\alpha| \leqq k_{N,K}} |D^\alpha f| \cdot t^{N+1}.$$

Here we used a multi-index α : $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$, $\alpha_j \geqq 0$, $|\alpha| = \sum \alpha_j$, and $D^\alpha \equiv (\partial/\partial x_1)^{\alpha_1} \cdots\cdots (\partial/\partial x_n)^{\alpha_n}$.

1-3. Basic results for asymptotic expansion [1].

The fundamental properties of the asymptotic expansion are collected in the following proposition.

Proposition 1-2. ① If a distribution θ admits an asymptotic expansion, the distributional coefficients D_i's are unique.

② D_i is homogeneous of degree i:

$$D_i(f_t) = t^i D_i(f),$$

and is tempered (since any homogeneous distribution is necessarily tempered).

③ Any invariant eigendistribution defined in a neighborhood of 0 in $\mathfrak{g}$ admits an asymptotic expansion.

④ If θ is an invariant eigendistribution defined in a neighborhood of 0 in $\mathfrak{g}$, then D_i is G-invariant, and

$$\operatorname{supp} \hat{D}_i \subseteq \mathcal{N} \equiv \{\text{nilpotent elements in } \mathfrak{g}^* \simeq \mathfrak{g}\}.$$

<u>1-4. An explicit description for the distributional coefficients</u> D_i .

Now let θ be an invariant eigendistribution in a neighborhood of 0 in $\mathfrak{g}$. Then by ③ above, θ admits an asymptotic totic expansion:

$$\theta(f_t) \sim \sum_{i=i_0}^{\infty} t^i D_i(f) \quad \text{as} \quad t \downarrow 0 .$$

According to [1] , p.35, an explicit description for D_i is available, using Harish-Chandra's general formula for an invariant eigendistribution restricted to a Cartan subalgebra. To explain this, we introduce some notation.

Let G be a real connected acceptable semisimple Lie group with finite center. Let $\mathfrak{g}$ = Lie(G) be its Lie algebra, $\mathfrak{h}$ a Cartan subalgebra, and $\mathfrak{h}^+$ a connected component of $\mathfrak{h}' = \mathfrak{h} \cap \mathfrak{g}'$, the set of regularelements in $\mathfrak{h}$. ($\mathfrak{g}'$ = the set of regular elements in $\mathfrak{g}$.)

<u>Definition 1-3.</u> Let h be a function on $\mathfrak{h}^+$, taking the same value on any two points on a single G-orbit. We define h_G , a function on $\mathfrak{g}$, by

$$h_G(X) \equiv \begin{cases} h(Y), & \text{if } X = \mathrm{Ad}(g)\cdot Y \text{ for some } g \in G, \\ 0, & \text{otherwise.} \end{cases}$$

Thus h_G is a G-invariant function which vanishes for $X \notin Ad(G)\cdot \mathfrak{h}^+$.

Now let $\mathfrak{h}_1, \mathfrak{h}_2, \cdots\cdots, \mathfrak{h}_s$ be a complete set of representatives of conjugacy classes of Cartan subalgebras in $\mathfrak{g}$. Also let $\mathfrak{h}_\ell^1, \mathfrak{h}_\ell^2, \cdots\cdots, \mathfrak{h}_\ell^{k_\ell}$ be the connected components of $\mathfrak{h}_\ell' = \mathfrak{h}_\ell \cap \mathfrak{g}'$.

We recall that, on a fixed $\mathfrak{h}_\ell^j$, θ has the form

$$\theta(X) = J_\ell^j(X) \,/\, \pi_\ell(X) \quad ,$$

where $\pi_\ell(X) = \prod_{\alpha>0} \alpha(X)$, the product of all positive roots for $\mathfrak{h}_\ell$, for a fixed order on $\mathfrak{h}_\ell^*$, and J_ℓ^j is a real-analytic function in X. We write the Taylor series expansion of J_ℓ^j as

$$J_\ell^j(X) = \sum_{i=0}^{\infty} (J_\ell^j)_i(X),$$

where $(J_\ell^j)_i$ denotes the homogeneous part of degree i.

<u>Proposition 1-4 ([1], Theorem 3-2).</u>

Using the notation of Definition 1-3, we have

① for $i \geqq 0$, the function

$$\sum_{\ell, j} [(J_\ell^j)_i \,/\, \pi_\ell]_G$$

is locally summable.

② the distributional coefficients D_i are given by

$$\theta(f_t) \sim \sum_{i=i_0}^{\infty} t^i D_i(f) \quad ,$$

where

$$(1.1) \qquad D_i = \begin{cases} 0, & \text{if } i < -(n-r)/2, \\ \sum_{\ell, j} \left[(J_\ell^j)_{i+\frac{n-r}{2}} / \pi_\ell \right]_G , & \text{if } i \geqq -(n-r)/2, \end{cases}$$

with n=dim(G), r=rank(G).

<u>Remark.</u> It may happen that $D_{-\frac{n-r}{2}} = 0$; this is the case if all the J_ℓ^j are without constant terms.

The above proposition enables us to calculate the Fourier transforms $\hat{D}_i$. Namely, all irreducible characters are known for G=SL(2,$\mathbb{R}$) and SL(2,$\mathbb{C}$) (see e.g. Hirai [6]), so we use Prop. 1-4 to obtain D_i . It only remains to carry out the Fourier transformation; we do this explicitly, and get $\hat{D}_i$. Thus diagramatically,

$$\Theta_\pi \Rightarrow \theta_\pi \Rightarrow D_i \Rightarrow \hat{D}_i \quad .$$

We propose another possible way to obtain $\hat{D}_i$ in Chapter 4. There the corresponding diagram is:

$$\Theta_\pi \Rightarrow \theta_\pi \Rightarrow \hat{\theta}_\pi \Rightarrow \hat{D}_i \quad .$$

Regrettably enough, we have not been able to make necessary justification to legalize this latter diagram. If we could, it would certainly clarify an essential aspect of the asymptotic expansion.

Chapter 2. Fourier transforms of higher-order terms.

——the case $G=SL(2,\mathbb{R})$——

In this chapter we give a detailed calculation for the formulas for the Fourier transforms $\hat{D}_i$, for all irreducible representations of $G=SL(2,\mathbb{R})$. The representations consist of the discrete series (D.S. for short), the principal series (P.S. for short), and the finite dimensional ones.

2-1. Definitions on Lie algebra $\mathfrak{g}=sl_2(\mathbb{R})$.

Let

$$e_1=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\quad e_2=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad e_3=\begin{pmatrix}0&-1\\1&0\end{pmatrix}.$$

Then $\mathfrak{g}=\sum_{1\leq i\leq 3}^{\oplus}\mathbb{R}\cdot e_i$.

We denote an element $X\in\mathfrak{g}$ by its coordinates:

$$X=xe_1+ye_2+ze_3=(x,y,z).$$

Then the quadratic form $z^2-x^2-y^2=\det X$ is G-invariant under the adjoint action, and we have a two-fold covering:

$$G\longrightarrow SO_0(2,1).$$

Also we make following definitions for certain subsets of $\mathfrak{g}$:

$$T^+\equiv\{(x,y,z)\mid z^2-x^2-y^2>0,\ z>0\},$$

$$T^-\equiv\{(x,y,z)\mid z^2-x^2-y^2>0,\ z<0\},$$

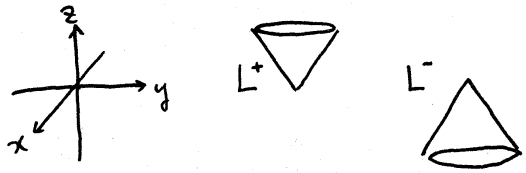
$$S\equiv\{(x,y,z)\mid z^2-x^2-y^2<0\}.$$

We note that there are three nilpotent orbits:

$$L^+ \equiv \{(x,y,z) \mid z^2-x^2-y^2=0,\ z>0\},$$

$$L^- \equiv \{(x,y,z) \mid z^2-x^2-y^2=0,\ z<0\},$$

$$\{0\}.$$



2-2. Explicit formulas of characters.

Given a representation of $G=SL(2,\mathbb{R})$, we pull back its character to $\mathfrak{g} = sl_2(\mathbb{R})$ in the manner prescribed in Introduction. We get an invariant eigendistribution θ_π, whose value on the x- and z-axis is given as follows:

1) For D.S.,

$$\theta((x,0,0)) = e^{(1-2|n|)\cdot x} / x, \quad x>0.$$

$$\theta((0,0,z)) = i\,\varepsilon(n)\cdot e^{i\varepsilon(n)(2|n|-1)\cdot z} / z, \quad z\neq 0.$$

Here $\varepsilon(n)$ =sign(n), and $n\in\mathbb{Z}/2$, $|n|\geq 1$, is the discrete series parameter (Sugiura[12], p.322).

2) For P.S.,

$$\theta((x,0,0)) = (e^{sx}+e^{-sx}) / |x|, \quad x\neq 0.$$

$$\theta((0,0,z)) = 0, \quad z\neq 0.$$

Here $s\in\mathbb{C}$ is the principal series parameter, $s=\sigma-\frac{1}{2}$, $\sigma\in\mathbb{C}$.

3) For finite dimensional representations,

$$\theta((x,0,0)) = -i\,(e^{iqx} - e^{-iqx})/x, \quad x>0,$$

$$\theta((0,0,z)) = (e^{qz} - e^{-qz})/|z|, \quad z \neq 0.$$

Here q is the parameter: $q \in \mathbb{Z}/2$.

2-3. Explicit formulas of distributional coefficients.

We apply Prop. 1-4 to the above character formulas. We thus get the following formulas for D_i :

1) For D.S., we have

$$\theta(f_t) \sim \sum_{j=-1}^{\infty} t^j D_j(f),$$

where

$$D_j = \begin{cases} 1/(j+1)! \cdot (1-2|n|)^{j+1} \cdot (x^2+y^2-z^2)^{\frac{j}{2}} & \text{in } S, \\ 1/(j+1)! \cdot (i\,\varepsilon(n))^{j+2} \cdot (2|n|-1)^{j+1} \cdot (z^2-x^2-y^2)^{\frac{j}{2}} & \text{in } T^+, \\ -1/(j+1)! \cdot (i\,\varepsilon(n))^{j+2} \cdot (2|n|-1)^{j+1} \cdot (z^2-x^2-y^2)^{\frac{j}{2}} & \text{in } T^-. \end{cases}$$

2) For P.S., we have

$$\theta(f_t) \sim \sum_{j=-1}^{\infty} t^j D_j(f),$$

where

$$D_j = \begin{cases} 2/(j+1)! \cdot s^{j+1} \cdot (x^2+y^2-z^2)^{\frac{j}{2}} & \text{in } S,\ j \text{ odd}. \\ 0, & j \text{ even}. \end{cases}$$

$$\Big| \; 0, \qquad \text{in} \quad T^+, T^- .$$

3) For finite dimensional representations, we have

$$\theta(f_t) \sim \sum_{j=0}^{\infty} t^j D_j(f),$$

where

$$D_j = \begin{cases} 2\cdot(i)^j \cdot 1/(j+1)!\cdot q^{j+1}\cdot (x^2+y^2-z^2)^{\frac{j}{2}} \\ \qquad\qquad \text{in } S, \ j \text{ even}. \\ 2/(j+1)!\cdot q^{j+1}\cdot (x^2+y^2-z^2)^{\frac{j}{2}} \\ \qquad\qquad \text{in } T^+, T^-, \ j \text{ even}. \\ 0, \quad j \text{ odd}. \end{cases}$$

Thus we have obtained the distributional coefficients D_i . We now investigate their Fourier transforms.

2-4. Fourier transforms of distributional coefficients.

We are going to carry out the Foutier transformation for the D_i in 2-3. We define the Fourier transform by, for $\phi \in \mathcal{S}(\mathfrak{g})$,

$$(\mathcal{F}\phi)(k_1, k_2, k_3) = (2\pi)^{-\frac{3}{2}} \iiint e^{i(k_1x+k_2y-k_3z)} \cdot \phi(x,y,z)\, dx\,dy\,dz,$$

and for $\phi \in \mathcal{S}(\mathfrak{g})$, $\Phi \in \mathcal{S}'(\mathfrak{g}^*)$,

$$\langle \mathcal{F}\Phi, \phi \rangle = \langle \Phi, \mathcal{F}\phi \rangle .$$

Let, for $\lambda \in \mathbb{C}$, two distributions P_-^λ and P_+^λ be

$$\langle P_-^\lambda, \phi \rangle = \iiint_{P<0} (-P(x,y,z))^\lambda \phi(x,y,z)\, dx\,dy\,dz,$$

$$\langle P_+^\lambda, \phi \rangle = \iiint_{P>0} (P(x,y,z))^\lambda \phi(x,y,z)\, dx\,dy\,dz,$$

where $P = z^2 - x^2 - y^2$, $\phi \in \mathcal{S}(\mathfrak{g})$. These make sense when $\mathrm{Re}\,\lambda$ is large, and are extended to other values of λ by analytic continuation.

<u>Theorem 2-1.</u> 1) Put

$$f = P_-^{-\frac{1}{2}} = \begin{cases} (x^2+y^2-z^2)^{-\frac{1}{2}}, & \text{if } (x,y,z) \text{ is in } S, \\ 0, & \text{otherwise.} \end{cases}$$

Then, as a distribution in $\phi \in \mathcal{S}(\mathfrak{g}^*)$,

$$\langle \mathcal{F}f, \phi \rangle =$$

$$= \sqrt{2\pi} \cdot \int \left[\phi(k_1, k_2, \sqrt{k_1^2+k_2^2}) + \phi(k_1, k_2, -\sqrt{k_1^2+k_2^2}) \right] \Big/ 2\sqrt{k_1^2+k_2^2} \cdot dk_1 dk_2.$$

2) Put

$$g = \mathrm{sign}(z) \cdot P_+^{-\frac{1}{2}} = \begin{cases} (z^2-x^2-y^2)^{-\frac{1}{2}}, & \text{if } (x,y,z) \text{ is in } T^+, \\ -(z^2-x^2-y^2)^{-\frac{1}{2}}, & \text{if } (x,y,z) \text{ is in } T^-, \\ 0, & \text{otherwise.} \end{cases}$$

Then, as a distribution in $\phi \in \mathcal{S}(\mathfrak{g}^*)$,

$$\langle \mathcal{F}g, \phi \rangle =$$
$$= \sqrt{2\pi}\, i \cdot \int \left[\phi(k_1, k_2, \sqrt{k_1^2+k_2^2}) - \phi(k_1, k_2, -\sqrt{k_1^2+k_2^2}) \right] \Big/ 2\sqrt{k_1^2+k_2^2} \cdot dk_1 dk_2 .$$

Proof. We use the results given in Gel'fand-Shilov [4], p.365, on the Fourier transforms of special functions. Note that they use $k_1x+k_2y+k_3z$ for the inner product, while ours is $k_1x+k_2y-k_3z$, and that we put the factor $(2\pi)^{-\frac{3}{2}}$ in the definition of a Fourier transform, so that the formula in their table reads:

$$(2.1) \qquad \mathcal{F}P_-^\lambda = 2^{\frac{1}{2}+2\lambda} i \cdot \pi^{-1} \cdot \Gamma(\lambda+1) \cdot \Gamma(\lambda+\tfrac{3}{2}) \cdot \left[-(Q-i0)^{-\lambda-\frac{3}{2}} + (Q+i0)^{-\lambda-\frac{3}{2}} \right],$$

where in the present case $Q = k_3^2 - k_1^2 - k_2^2$.

Also we have the fundamental identities

$$(2.2) \qquad (Q+i0)^\mu = Q_+^\mu + e^{i\pi\mu} Q_-^\mu ,$$
$$(Q-i0)^\mu = Q_+^\mu + e^{-i\pi\mu} Q_-^\mu ,$$

where for $\mu \in \mathbb{C}$,

$$\langle Q_+^\mu, \phi \rangle = \iiint_{Q>0} (Q(k_1,k_2,k_3))^\mu \cdot \phi(k_1,k_2,k_3)\, dk_1 dk_2 dk_3 ,$$
$$\langle Q_-^\mu, \phi \rangle = \iiint_{Q<0} (-Q(k_1,k_2,k_3))^\mu \cdot \phi(k_1,k_2,k_3)\, dk_1 dk_2 dk_3 .$$

Also we know from [4], p.351, that Q_+^μ and Q_-^μ have simple poles at $\mu = -1$, and that the residues are given by

$$\operatorname*{res}_{\mu=-1} Q_+^\mu = \operatorname*{res}_{\mu=-1} Q_-^\mu = \delta(Q_+) ,$$

where

(2.3) $$\langle \delta(Q_+), \phi \rangle =$$
$$= \int \left[\phi(k_1, k_2, \sqrt{k_1^2+k_2^2}) + \phi(k_1, k_2, -\sqrt{k_1^2+k_2^2}) \right] / 2\sqrt{k_1^2+k_2^2} \cdot dk_1 dk_2 .$$

The formulas (2.1), (2.2) and (2.3) yield, as $\lambda \to -\frac{1}{2}$,

$$\mathcal{F} P_-^{-\frac{1}{2}} = \lim_{\mu \to -1} (2\pi)^{-\frac{1}{2}} \cdot i \cdot \left[(Q_+^\mu + e^{i\pi\mu} Q_-^\mu) - (Q_+^\mu + e^{-i\pi\mu} Q_-^\mu) \right]$$
$$= \lim_{\mu \to -1} (2\pi)^{-\frac{1}{2}} \cdot i \cdot \left[(e^{i\pi\mu} - e^{-i\pi\mu}) Q_-^\mu \right]$$
$$= (2\pi)^{-\frac{1}{2}} \cdot i \cdot (-2\pi i) \cdot \operatorname*{res}_{\mu=-1} Q_-^\mu ,$$

which proves 1).

For g in 2), we work in the opposite direction, that is, we define a distribution Ψ on $\mathfrak{g}^*$ by

$$\langle \Psi, \phi \rangle \equiv \int \left[\phi(k_1, k_2, \sqrt{k_1^2+k_2^2}) - \phi(k_1, k_2, -\sqrt{k_1^2+k_2^2}) \right] / 2\sqrt{k_1^2+k_2^2} \cdot dk_1 dk_2 ,$$

and prove that

$$\mathcal{F}^{-1} \Psi = -(2\pi)^{-\frac{1}{2}} \cdot i \cdot g .$$

We calculate, for $\phi \in \mathcal{S}(\mathfrak{g})$,

$$\langle \mathcal{F}^{-1}\phi, \Psi \rangle = (2\pi)^{-\frac{3}{2}} \int \Big[\int [\exp(-i(k_1 x + k_2 y - z\sqrt{k_1^2+k_2^2})) -$$
$$- \exp(-i(k_1 x + k_2 y + z\sqrt{k_1^2+k_2^2}))] \cdot \phi(x,y,z)\, dx dy dz \Big] \cdot \frac{dk_1 dk_2}{2\sqrt{k_1^2+k_2^2}}$$

(2.4) $$= (2\pi)^{-\frac{3}{2}} \int \lim_{\varepsilon \downarrow 0} \int [\exp(-i(k_1 x + k_2 y - z\sqrt{k_1^2+k_2^2} - i\varepsilon\sqrt{k_1^2+k_2^2})) -$$
$$- \exp(-i(k_1 x + k_2 y + z\sqrt{k_1^2+k_2^2} - i\varepsilon\sqrt{k_1^2+k_2^2})) \cdot \phi(x,y,z)\, dx dy dz \times$$
$$\times \frac{dk_1 dk_2}{2\sqrt{k_1^2+k_2^2}} .$$

By the Lebesgue dominated convergence theorem, (2.4) is equal to

$$(2.5)\qquad (2\pi)^{-\frac{3}{2}} \lim_{\varepsilon\downarrow 0} \int \Big[\int \big[\exp(-i(k_1x+k_2y-z\sqrt{k_1^2+k_2^2}-i\varepsilon\sqrt{k_1^2+k_2^2})) - \\ - \exp(-i(k_1x+k_2y+z\sqrt{k_1^2+k_2^2}-i\varepsilon\sqrt{k_1^2+k_2^2}))\big] \cdot \phi(x,y,z)\,dx\,dy\,dz \times \\ \times \frac{dk_1\,dk_2}{2\sqrt{k_1^2+k_2^2}} .$$

By Fubuni's theorem, (2.5) is equal to

$$(2.6)\qquad (2\pi)^{-\frac{3}{2}} \int \Big[\lim_{\varepsilon\downarrow 0} \int \big[\exp(-i(k_1x+k_2y-z\sqrt{k_1^2+k_2^2}-i\varepsilon\sqrt{k_1^2+k_2^2})) - \\ - \exp(-i(k_1x+k_2y+z\sqrt{k_1^2+k_2^2}-i\varepsilon\sqrt{k_1^2+k_2^2}))\big] \cdot \frac{dk_1\,dk_2}{2\sqrt{k_1^2+k_2^2}} \Big] \cdot \phi(x,y,z)\,dx\,dy\,dz.$$

We pass to the polar coordinates in carrying out the integration with respect to k_1 and k_2 :

$$\int \big[\exp(-i(k_1x+k_2y-z\sqrt{k_1^2+k_2^2}-i\varepsilon\sqrt{k_1^2+k_2^2})) - \\ - \exp(-i(k_1x+k_2y+z\sqrt{k_1^2+k_2^2}-i\varepsilon\sqrt{k_1^2+k_2^2}))\big] \cdot \frac{dk_1\,dk_2}{2\sqrt{k_1^2+k_2^2}}$$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{\infty} \exp(-i(r\cos\theta\cdot x + r\sin\theta\cdot y - rz - i\varepsilon r)) \cdot \frac{1}{2r} \cdot r\,dr\,d\theta - \\ - \int_{\theta=0}^{2\pi}\int_{r=0}^{\infty} \exp(-i(r\cos\theta\cdot x + r\sin\theta\cdot y + rz - i\varepsilon r)) \cdot \frac{1}{2r} \cdot r\,dr\,d\theta$$

$$(2.7)\qquad = \frac{i}{2} \cdot \int_0^{2\pi} \left[\frac{1}{\cos\theta x + \sin\theta y - z - i\varepsilon} - \frac{1}{\cos\theta x + \sin\theta y + z - i\varepsilon} \right] d\theta.$$

If we put $(x,y,z)=(0,0,\pm 1)$, (2.7) becomes

$$\mp \pi i \left[\frac{1}{1-i\varepsilon} + \frac{1}{1+i\varepsilon} \right] \qquad \text{(respectively).}$$

Letting $\varepsilon \downarrow 0$, we see this tends to

$$\mp 2\pi i \qquad \text{(respectively).}$$

Therefore, from G-invariance and homogeneity property of the function under consideration, we get

$$(2.8)\quad \lim_{\varepsilon\downarrow 0}\int\Big\{\exp\big(-i(k_1x+k_2y-z\sqrt{k_1^2+k_2^2}-i\varepsilon\sqrt{k_1^2+k_2^2})\big)-$$
$$-\exp\big(-i(k_1x+k_2y+z\sqrt{k_1^2+k_2^2}-i\varepsilon\sqrt{k_1^2+k_2^2})\big)\cdot\frac{dk_1dk_2}{2\sqrt{k_1^2+k_2^2}}$$
$$=-2\pi i\cdot\mathrm{sign}(z)\cdot(z^2-x^2-y^2)^{-\frac{1}{2}},\quad \text{for } (x,y,z)\in T^+,T^-.$$

For (x,y,z) in $S \equiv$ the outside of $\mathrm{cl}(T^+\cup T^-)$, we have

(2.9) (the left-hand side of (2.8))=0.

In fact, by G-invariance and homogeneity, it suffices to verify that (2.9) is true for (x,y,z)=(1,0,0). But this is clear from (2.7). This proves 2). Q.E.D.

Using the above theorem, we can now give the Fourier transforms of D_i . In fact, the lowest-order term D_{i_0} of a character is a linear combination of f and g above, and the higher order terms are of the form

(a G-invariant polynomial) × (the lowest-order term),

so that their Fourier transforms are of the form

(a G-invariant differential operator)·(the Fourier transform of the lowest term).

In this way we get:

<u>Theorem 2-2.</u> Let $G=SL(2,\mathbb{R})$. Then the Fourier transforms of the distributional coefficients D_i in the asymptotic expansion for irreducible characters are given as follows.

1) For D.S., letting $n>0$ (resp. $n<0$) be the D.S. parameter (see p. 13),

$\hat{D}_{-1}$ = an invariant measure on the upper half L^+ (resp. on the lower half L^-) of the "light cone",

and

$$\hat{D}_{-1+2j} = \frac{1}{(2j)!}\cdot(2|n|-1)^{2j}\cdot(\Box_3)^j\cdot\hat{D}_{-1}, \qquad j\geqq 1.$$

Moreover,

$$\hat{D}_0 = (1-2|n|)\cdot(\text{Dirac measure at the origin}),$$

$$\hat{D}_{2j} = \frac{1}{(2j)!}\cdot(2|n|-1)^{2j}\cdot(\Box_3)^j\cdot\hat{D}_0, \qquad j\geqq 0,$$

where

$$\Box_3 \equiv (\partial/\partial k_3)^2-(\partial/\partial k_1)^2-(\partial/\partial k_2)^2$$

is the 3-dimensional d'Alembertian, and $\Box_3$ acts on $\hat{D}_i$ in distribution sense.

2) For P.S., letting $s\in\mathbb{C}$ be the P.S. parameter, $s=\sigma-\frac{1}{2}$,

$\hat{D}_{-1}$ = an invariant measure even in the variable k_3 on L^+ and L^-,

$$\hat{D}_{-1+2j} = \frac{1}{(2j)!}\cdot s^{2j}\cdot(-\Box_3)^j\cdot\hat{D}_{-1}, \qquad j\geqq 1,$$

$$\hat{D}_{2j}=0, \qquad j\geqq 0.$$

3) For finite dimensional representations, with parameter $q \in \mathbb{Z}/2$,

$$\hat{D}_0 = 2q \times (\text{Dirac measure at the origin}),$$

$$\hat{D}_{2j} = \frac{1}{(2j)!} \cdot q^{2j} \cdot (\Box_3)^j \cdot \hat{D}_0 ,$$

$$\hat{D}_{2j+1} = 0 , \qquad j \geq 0 .$$

<u>Proof.</u> We have only to note that the expressions

$$\int \phi(k_1, k_2, \sqrt{k_1^2+k_2^2}) \cdot \frac{dk_1 dk_2}{2\sqrt{k_1^2+k_2^2}}$$

and

$$\int \phi(k_1, k_2, -\sqrt{k_1^2+k_2^2}) \cdot \frac{dk_1 dk_2}{2\sqrt{k_1^2+k_2^2}}$$

define the nilpotent orbital integrals corresponding to the nilpotent orbits

$$L^+ \equiv \left\{ (k_1, k_2, k_3) \in \mathfrak{g}^* \,\middle|\, k_3^2 - k_1^2 - k_2^2 = 0 ,\ k_3 > 0 \right\}$$

and

$$L^- \equiv \left\{ (k_1, k_2, k_3) \in \mathfrak{g}^* \,\middle|\, k_3^2 - k_1^2 - k_2^2 = 0 ,\ k_3 < 0 \right\} ,$$

so that they are indeed G-invariant measures on L^+ and L^-.

Q.E.D.

We note that in each of above cases the parameter enters as a multiplicative factor. We discuss this in Chapter 4.

Also we notice that all calculation needed to get Theorem 2-2 is mainly combinatorial in nature. For groups of higher

ranks, similar calculation should work as well. But due to the presence of many nilpotent orbits and invariant polynomials, the situation would be much more complicated.

Chapter 3. Fourier transforms of higher-order terms.

—— the case $G=SL(2,\mathbb{C})$ ——

In this chapter we calculate the Fourier transforms of the distributional coefficients in asymptotic expansions for irreducible characters of $G=SL(2,\mathbb{C})$.

3-1. Definitions on Lie algebra $\mathfrak{g} = sl_2(\mathbb{C})$.

Let e_1, e_2, and e_3 be as in Chapter 2:

$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Then $\mathfrak{g} = \sum_{1 \leq j \leq 3}^{\oplus} \mathbb{C} \cdot e_j$.

We fix a Cartan subalgebra $\mathfrak{h}$ as $\mathfrak{h} = \mathbb{C} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Also we note that there are two nilpotent orbits:

1) $N_1 \equiv \{ (x,y,z) \in \mathfrak{g} \mid z^2 - x^2 - y^2 = 0, \ (x,y,z) \neq (0,0,0) \}$,

2) $\{(0,0,0)\}$.

3-2. Explicit formulas of irreducible characters.

The "pull-back" θ_π of irreducible characters are given in the following proposition.

Proposition 3-1. The explicit forms of θ_π are given as

follows.

1) For P.S., with $B, C \in \mathbb{C}$,

$$\theta((x,0,0)) = \left[\exp(B\cdot x + C\cdot \bar{x}) + \exp(-B\cdot x - C\cdot \bar{x})\right] / |x|^2$$

(Harish-Chandra [5], p.511).

2) For finite dimensional representations, with $p, q \in \mathbb{Z}/2$,

$$\theta((x,0,0)) = \left[\exp(px) - \exp(-px)\right] \times \left[\exp(q\bar{x}) - \exp(-q\bar{x})\right] / |x|^2$$

(Warner [13], p.174).

Decomposing θ into homogeneous parts as in Prop. 1-4, we get the distributional coefficients D_i for the asymptotic expansion. We have:

<u>Theorem 3-2.</u> Let $G=SL(2,\mathbb{C})$. Then the distributional coefficients D_i in the asymptotic expansions for irreducible characters are given as follows.

1) For P.S.,

$$D_j = \begin{cases} \dfrac{2}{(j+2)!}\cdot\left[B(x^2+y^2-z^2)^{\frac{1}{2}} + C(\overline{x^2+y^2-z^2})^{-\frac{1}{2}}\right]^{j+2} / |x^2+y^2-z^2|, & j \geqq -2,\ j \text{ even}. \\ 0, & j \text{ odd}. \end{cases}$$

2) For finite dimensional representations,

$$D_j = \begin{cases} 4\cdot \displaystyle\sum_{\substack{u+v=j \\ u,v \text{ even} \\ u,v \geqq 0}} \frac{p^{u+1}\cdot q^{v+1}}{(u+1)!\cdot(v+1)!}\cdot (x^2+y^2-z^2)^{\frac{u}{2}}\cdot \overline{(x^2+y^2-z^2)}^{\frac{v}{2}}, & j \text{ even},\ j \geqq 0. \end{cases}$$

$$\Big\{ \; 0, \qquad j \text{ odd}.$$

3-3. Fourier transforms of distributional coefficients.

We are now going to calculate $\hat{D}_i$. To do so, we first compute the Fourier transform $\mathcal{F}\mu_{N_1}$ of the orbital integral μ_{N_1} for the orbit N_1 (see 3-1).

We define the Fourier transformation as follows:

for $\phi \in \mathcal{S}(\mathfrak{g})$,

$$(\mathcal{F}\phi)(k_1, k_2, k_3) \equiv$$

$$\equiv (2\pi)^{-3} \cdot \int \phi(x,y,z) \cdot \exp(i \cdot \mathrm{Re}(k_1 x + k_2 y - k_3 z)) \cdot dx \wedge d\bar{x} \wedge dy \wedge d\bar{y} \wedge dz \wedge d\bar{z}$$

and for $\phi \in \mathcal{S}(\mathfrak{g}^*)$, $f \in \mathcal{S}'(\mathfrak{g})$,

$$\langle \mathcal{F}f, \phi \rangle \equiv \langle f, \mathcal{F}\phi \rangle,$$

where $\mathfrak{g}^* \equiv \mathrm{Hom}_{\mathbb{R}}(\mathfrak{g}, \mathbb{R})$, with identification $\mathfrak{g}^* \simeq \mathbb{C}^3$ given by

$$\mathbb{C}^3 \ni (k_1, k_2, k_3): \qquad (x, y, z) \longmapsto \mathrm{Re}(k_1 x + k_2 y - k_3 z).$$

Here, as in Chapter 2, we denote by $\mathcal{F}$ (and occasionally by $\wedge$) the Fourier transformation.

3-4. In this subsection we calculate $\mathcal{F}\mu_{N_1}$.

Theorem 3-3. Let μ_{N_1} be the nilpotent orbital integral corresponding to the orbit N_1 . Then

$$\mathcal{F}\mu_{N_1} = A \cdot |k_3^2 - k_1^2 - k_2^2|^{-1}$$

for some constant $A \neq 0$.

Remark. Of course, the above constant A is something to be explicitly determined. For our purpose, however, it is not strictly necessary, since our primary interest lies in examining what sort of distributions appear in higher-order terms.

Proof of Theorem 3-3. μ_{N_1} is given by the following volume element on N_1 :

$$\frac{1}{|z|^2} \cdot dx \wedge d\bar{x} \wedge dy \wedge d\bar{y} \ .$$

Since the Fourier transform of a nilpotent orbital integral is an invariant eigendistribution with trivial eigenvalue (or infinitesimal character), $\mathcal{F}\mu_{N_1}$ is a locally summable function on $\mathfrak{g}^*$. By Coad(G)-invariance, Theorem 3-3 is proved if we verify the following two points:

1) $\mathcal{F}\mu_{N_1}$ is invariant under the phase transformation

$$(k_1, k_2, k_3) \longmapsto e^{i\theta}(k_1, k_2, k_3), \quad \theta \in \mathbb{R},$$

2) $\mathcal{F}\mu_{N_1}$ is homogeneous of degree -2:

$$\mathcal{F}\mu_{N_1}(t\cdot(k_1, k_2, k_3)) = t^{-2}\cdot \mathcal{F}\mu_{N_1}((k_1, k_2, k_3)), \quad t \geq 0.$$

And these are easily observed from the definition of a Fourier transform and the above-mentioned form of the volume element defining μ_{N_1} . Q.E.D.

3-5. The above Theorem 3-3 allows us to calculate the Fourier transforms $\hat{D}_i$. In fact, Theorem 3-3 takes care of the lowest-order term, and the Fourier transforms of higher-order terms are obtained by applying suitable invariant differential operators on it. In this way we get:

<u>Theorem 3-4.</u> Let G=SL(2,C). Then the Fourier transforms of the distributional coefficients D_i in the asymptotic expansions for irreducible characters are given as follows.

1) For P.S., with parameters B, $C \in \mathbb{C}$,

$$\theta(f_t) \sim \sum_{j=-2}^{\infty} t^j D_j(f),$$

where

$\hat{D}_{-2}$ = an invariant measure on N_1,

$$\hat{D}_{-2+2j} = \Xi_{1,j} \cdot \hat{D}_{-2} + \Xi_{2,j} \cdot \delta_c^3,$$

where the differential operators $\Xi_{1,j}$ and $\Xi_{2,j}$ are given by

$$\Xi_{1,j} \equiv \frac{1}{(j+2)!} \sum_{e \text{ even}} \binom{j+2}{e} B^e \cdot C^{j-e+2} \cdot (\Box_c)^{\frac{e}{2}} \cdot (\bar{\Box}_c)^{\frac{j-e+2}{2}},$$

$$\Xi_{2,j} \equiv \frac{1}{(j+2)!} \sum_{e \text{ odd}} \binom{j+2}{e} B^e \cdot C^{j-e+2} \cdot (\Box_c)^{\frac{e-1}{2}} \cdot (\bar{\Box}_c)^{\frac{j-e+1}{2}}.$$

$\hat{D}_{2j-1} = 0, \qquad j \geq 0.$

2) For finite dimensional representations, with p, $q \in \mathbb{Z}/2$,

$$\theta(f_t) \sim \sum_{j=0}^{\infty} t^j D_j(f),$$

where

$\hat{D}_0 = 4pq \cdot \delta_c^3$,

$\hat{D}_{2j} = \Xi_{3,j} \cdot \delta_c^3, \qquad , j \geq 1,$

where the differential operator $\Xi_{3,j}$ is given by

$$\Xi_{3,j} \equiv 4 \cdot \sum_{\substack{u+v=2j, \\ u,v \text{ even}, \\ u,v \geq 0.}} \frac{p^{u+1} \cdot q^{v+1}}{(u+1)!\,(v+1)!} \cdot (\Box_c)^{\frac{u}{2}} \cdot (\bar{\Box}_c)^{\frac{v}{2}}.$$

and

$$\hat{D}_{2j+1} = 0, \qquad j > 0 .$$

In 1) and 2) above, δ_c^3 denotes the Dirac measure at the origin of $\mathfrak{g}^* \simeq \mathbb{R}^6$, and the complex d'Alembertians $\Box_c$ and $\overline{\Box}_c$ are defined by

$$\Box_c \equiv \frac{\partial^2}{\partial k_1^2} + \frac{\partial^2}{\partial k_2^2} - \frac{\partial^2}{\partial k_3^2} ,$$

$$\overline{\Box}_c \equiv \frac{\partial^2}{\partial \bar{k}_1^2} + \frac{\partial^2}{\partial \bar{k}_2^2} - \frac{\partial^2}{\partial \bar{k}_3^2} .$$

Here in this theorem again, the parameter specifying the representations appears as multiplicative factors. We will give a discussion on this point in Chapter 4.

Chapter 4. Some observation on the asymptotic expansion.

In this chapter we make some observation on the connection between Kirillov character formula and the asymptotic expansion.

Let us consider the Kirillov character formula, for instance, for the unitary principal series representations of G= =SL(2,$\mathbb{R}$):

$$\theta = \mathcal{F}^{-1} \cdot \mu_{-c} , \qquad c > 0 , \tag{4.1}$$

where μ_{-c} is a G-invariant measure on $\mathfrak{g}^*$ with support on the hypersurface defined by $P \equiv z^2 - x^2 - y^2 = -c$. We may alternatively denote it as $\delta(P+c)$, where δ is the Dirac measure in one variable. Note that (4.1) is valid in a neighborhood of 0 in $\mathfrak{g}$ (Rossmann [11]).

In accordance with f_t defined in Introduction, we have

$$(\mathcal{F}\theta)_t = \frac{1}{t^3} \cdot \delta(P(k_1/t, k_2/t, k_3/t) + c)$$

$$= \frac{1}{t} \cdot \delta(P(k_1, k_2, k_3) + t^2 c).$$

So we are naturally led to consider the following formal expansion:

$$(4.2) \qquad \delta(P(k_1, k_2, k_3) + t^2 c) = \delta(P(k_1, k_2, k_3)) + t^2 c \cdot \delta'(P(k_1, k_2, k_3)) + \frac{1}{2} \cdot (t^2 c)^2 \cdot \delta''(P(k_1, k_2, k_3)) + \cdots .$$

Notice that c is essentially the P.S. parameter: $c = -(\sigma - \frac{1}{2})^2$, $\sigma = i\rho + \frac{1}{2}$, $\rho \in \mathbb{R}$, and it enters in each term in (4.2) as a multiplicative factor. This is in agreement with Theorem 2-2:

$$(4.3) \qquad \hat{D}_{-1+2j} = \delta^{(j)}(P(k_1, k_2, k_3)) \quad \text{(formally).}$$

Thus we have the following problem:

<u>Problem.</u> 1) Justify the expansion (4.2). More precisely, justify the substitution $\xi = P$ into $\delta^{(j)}(\xi)$,

$$\delta^{(j)}(P),$$

where δ denotes the Dirac delta function, and the surface $P = 0$ is singular at the origin. Also justify the formal Taylor series expansion in (4.2).

2) Verify the formula (4.3), so that (4.2) becomes another way of expressing the Fourier transforms of higher-order terms (see diagrams at the end of Chapter 1).

3) Generalize these to groups of higher ranks.

Solving the above problems may serve our purpose of obtaining from the asymptotic expansion some information about the original representation π.

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