Metric Diophantine Approximation

on some Fuchsian Groups

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Let $\Gamma$ be a finitely generated Fuchsian group acting on the upper half complex plane $\mathbb{H}^2$, $L$ the set of limit points of $\Gamma$ and $P$ the set of parabolic cusps. We assume that $\alpha \in P$.

An element $g \in \Gamma$ can be viewed as a $2 \times 2$ real matrix
\[
\begin{bmatrix}
    a & b \\
    c & d
\end{bmatrix}
\]
of determinant $1$. We write $a = a(g), b = b(g), c = c(g)$ and $d = d(g)$ for convenience.

In Lehner [4], he proved that there exists a positive number $k$ depending on $\Gamma$ such that

$$\# \{ g(\alpha) : |\alpha - g(\alpha)| < \frac{k}{c^x(g)} , g \in \Gamma \} = \infty$$

for any $\alpha \in L \setminus P$. He also proved that if $\Gamma$ is of the first kind ($L = \mathbb{R}$), then for any sequence $\{\varepsilon_n\}$ of positive numbers and almost all $\alpha \in L \setminus P$, there exists a sequence $\{g_n\} \subset \Gamma$ such that

$$|\alpha - g_n(\alpha)| < \frac{\varepsilon_n}{c^x(g_n)} .$$

Moreover, Patterson [7] proved a kind of Khintchine theorem when $\Gamma$ is of the first kind: for example, his result implies that

$$\# \{ g(\alpha) : |\alpha - g(\alpha)| < \frac{1}{c^x(g) \cdot \log |c(g)|} , g \in \Gamma \} = \infty$$

for almost all $\alpha \in \mathbb{R} \setminus P$.

In this note, we shall calculate the asymptotic number of
\[ g(\omega) : |\alpha - g(\omega)| < \frac{k}{c^2(g)} , \; g \in \Gamma \]

for some positive real number \( k \) and almost all \( \alpha \in \mathbb{R} \setminus \mathbb{P} \). To do this, we consider a relation among the Diophantine inequality, geodesics of \( H^2 \) and geodesics of \( H^2/\Gamma \). We show that the ergodicity of the geodesic flow on \( H^2/\Gamma \) with the hyperbolic measure is closely related to the quantitative theory of the Diophantine approximation on \( \Gamma \). The relation between the Diophantine inequality and geodesics of \( H^2 \) also have been considered by Haas [2] and Haas and Series [3] to determine Lagrange spectrum of the approximations on \( \Gamma \). They have pointed out that the spectrum is related to the "height" of a \( \Gamma \)-congruent family of geodesics.

1. Main Theorem

We assume that \( \Gamma \) is of the first kind and \( \omega \in \mathbb{P} \). Since \( \omega \in \mathbb{P} \), there exists

\[ U_\lambda = \begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix} \in \Gamma, \quad \lambda \in \mathbb{R}_+, \]

such that

\[ \{ U^k : k \in \mathbb{Z} \} = \Gamma_\omega \]

where \( \Gamma_\omega \) denotes the subgroup of \( \Gamma \) that fixes \( \omega \). We define the fundamental region \( F \) of \( \Gamma \) by

\[ F = \{ z = x + iy : \frac{-\lambda}{2} < x < \frac{\lambda}{2} , \; y > 0 \} \]

\[ \bigcap_{g \in \Gamma \setminus \Gamma_\omega} \bigcap \{ z : |c(g) \cdot z + d(g)| > 1 \}. \]

It is well-known that the hyperbolic metric \( d\sigma = \sqrt{dx^2 + dy^2} / y \) and the hyperbolic measure \( d\mu = dx \, dy / y^2 \) on \( H^2 \) are invariant.
under $\Gamma$-action over $H^2$.

**Theorem.** Let

$$k_0 = \frac{1}{2} \min_{g \in \Gamma \setminus \Gamma_\infty} |c(g)|,$$

then we have for almost all $a \in \mathbb{R} \setminus \mathbb{P}$,

$$\lim_{N \to \infty} \frac{\# \{ g(\infty) : |a - g(\infty)| < \frac{k}{c^2(g)}, |c(g)| \leq N, g \in \Gamma \}}{\log N} = \frac{2\lambda \cdot k}{\pi \cdot \mu(\Gamma)}$$

for any $k$, $0 < k < k_0$.

In the sequel, we outline the proof of this theorem.

2. **Some lemmas** We denote by $\gamma(a, \beta)$ the geodesic curve which starts from $a$ and ends at $\beta$ for $(a, \beta) \in (\mathbb{R} \cup \{\infty\})^2 \setminus \{\text{diagonal}\}$. We also denote by

$$F_k(g(\infty)), \ k > 0,$$

the circle tangent to the real line at $\frac{a(g)}{c(g)}$ with the radius $\frac{k}{c^2(g)}$ for $g \in \Gamma_\infty$ and $\{ x + iy : y = 1/2k \}$ for $g \in \Gamma_\infty$.

![Diagram](image)

It is possible to show the following:

**Lemma 1.** If we fix $k > 0$, then

$$g'(F_k(g(\infty))) = F_k(g'g(\infty))$$

for any $g'$ and $g \in \Gamma$. 


This lemma implies that \( \{ F_k(g(\alpha)) : g \in \Gamma \} \) is an invariant family of circles under \( \Gamma \)-action. The next lemma is essential.

**Lemma 2.** For any \( k > 0 \),
\[
|\alpha - g(\alpha)| < \frac{k}{c^2(g)}
\]
holds if and only if
\[
\gamma(\alpha, \alpha) \cap F_k(g(\alpha)) \neq \emptyset.
\]

If \( k < k_0 \), then we see that \( \{ F_k(g(\alpha)) \} \) is a disjoint family of circles, that is,
\[
F_k(g(\alpha)) \cap F_k(g'(\alpha)) = \emptyset
\]
if \( g(\alpha) \neq g'(\alpha) \). Thus we have the following:

**Lemma 3.** If \( 0 < k < k_0 \), then every point of \( F_k(g(\alpha)) \setminus (R \cup \{ \infty \}) \) is congruent to some point of \( F_k(\alpha) \cap (F \setminus \{ \infty \}) \), that is, if \( p \in F_k(g(\alpha)) \setminus (R \cup \{ \infty \}) \), then there exists \( g' \in \Gamma \) such that \( g'(p) = x + \lambda y, \ -\frac{\lambda}{2} < x < \frac{\lambda}{2} \) and \( y = 1/2k \).

3. **Sketch of the proof** Let \( T(H^2) \) and \( T(F) \) be the unit tangent bundles of \( H^2 \) and \( F \), respectively. We consider the geodesic flows \( f_s \) and \( \hat{f}_s \) on \( T(H^2) \) and \( T(F) \), respectively. For \( \omega \in T(H^2) \), there is a unique geodesic \((\alpha, \beta)\) passing tangentially through \( \omega \). If \( \alpha \neq \infty \) and \( \beta \neq \infty \), then we denote by \( \delta \) the (directed)
hyperbolic length from the top of the geodesic arc \((a, b)\) to \(\omega\), which is the base point of \(\omega^*\). If \(a=\infty\) (or \(b=\infty\)), then we denote by \(s\) the hyperbolic length from the point \(b + i\) (or \(a + i\)) to \(\omega\), (respectively). Thus we can parametrize \(\omega^* \in T(H^2)\) by \((a, b, s)\) \((R \cup \{\infty\})^2 \setminus \{\text{diagonal}\} \times R\). So if \(0 < k < k_0\), we see from lemmas 2 and 3 that

\[
\begin{align*}
\#\{ g(\infty) : |a - g(\infty)| < \frac{k}{c_2(g)}, |c(g)| \leq N, g \in \Gamma \} \\
= \#\left\{ s : f_\infty(\omega^*, a, -\log (k_0+1)) \text{ crosses a circle } F_k(g(\infty)) \right\} \\
= \#\left\{ s : \text{ from outside at time } s, 0 < s \leq \log(k_0+1) - \log k + 2 \log N \right\} \\
= \#\left\{ s : \hat{f}_s(\omega^*) \text{ crosses } F_k(\infty) \text{ from below at a time } s, \right\} \\
0 < s \leq \log(k_0+1) - \log k + 2 \log N
\end{align*}
\]

where \(\omega^* \in T(F)\) is the congruent point to \((\infty, a, -\log (k_0+1)) \in T(H^2)\).

Now we apply the individual ergodic theorem for \((T(F), \hat{f}, \hat{\mu})\) to our problem. Here, the hyperbolic measure \(\hat{\mu}\) on \(T(F)\) induced from \(\mu\) is defined by

\[
\hat{\mu} = \frac{da \, d\beta \, ds}{(\alpha - \beta)^2}
\]

if we parametrize a point in \(T(F)\) by \((a, b, s)\).

**Proposition 4.** If we fix \(k, 0 < k < k_0\), then

\[
\lim_{u \to \infty} \frac{\#\{ s : f_\infty(\omega^*) \text{ crosses } F_k(\infty) \text{ from below, } 0 < s < u \}}{u} = \frac{\mu\{ x + iy \in F : y > 1/2k \}}{2\pi \cdot \mu(F)}
\]

for almost all \(\omega^* \in T(F)\).
Moreover, by using an approximation method on $k$, we have

**Proposition 4'.** For almost all $\omega^* \in \mathcal{T}(F)$,

$$
\lim_{u \to \infty} \frac{\# \{ \delta : \hat{\psi}_{\delta}(\omega^*) \text{ crosses } \mathcal{F}_k(\infty) \text{ from below}, 0 < \delta < u \}}{u}
$$

$$
= \frac{\mu \{ x + iy \in F : y > 1/2k \}}{2\pi \cdot \mu(F)}
$$

for any $k$, $0 < k < k_0$.

Furthermore, it is possible to show that if $\omega^* = (\alpha, \beta, \delta) \in \mathcal{T}(F)$ has the above property, then for any $\alpha' \in \mathcal{R} \cup \{ \infty \}$ and $\delta' \in \mathcal{R}$, $\omega^{**} = (\alpha', \beta', \delta')$ also has the same property. Since the hyperbolic length between $\alpha + (k_0 + 1)i$ and $\alpha + \frac{1}{N}i$ is equal to $\log N + \log (k_0 + 1)$, we have

$$
\lim_{N \to \infty} \frac{\# \{ g(\infty) : |\alpha - g(\infty)| < \frac{k}{c^2(g)}, |c(g)| \leq N, g \in \Gamma \}}{\log N}
$$

$$
= 2 \cdot \frac{\mu \{ x + iy \in F : y > 1/2k \}}{2\pi \cdot \mu(F)}
$$

$$
= \frac{2\lambda \cdot k}{\pi \cdot \mu(F)}
$$

for any $k$, $0 < k < k_0$ and almost all $\alpha \in \mathcal{R} \setminus \mathcal{P}$.

#### 4. Some remarks

It is possible to apply the theorem to Hecke group $G_n$, $n \geq 3$, and its principal congruence subgroups $G_n(m)$. Let $G_n$ be the group generated by

$$
\begin{bmatrix}
0 & 1 \\
-1 & 0
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
1 & \lambda_n \\
0 & 1
\end{bmatrix}
$$

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where $\lambda_n = 2 \cdot \cos \frac{\pi}{n}$ for $n \geq 3$, and $G_n(m)$ be the subgroup of $G_n$ defined by

$$G_n(m) = \{ g \in G_n : g \equiv \begin{bmatrix} r_1 & 0 \\ 0 & r_1 \end{bmatrix} \pmod{(m \cdot \lambda_n)} \}$$

where $(m \cdot \lambda_n)$ denotes the ideal generated by $m \cdot \lambda_n$ with positive integer $m$.

A fundamental region $F_n$ of $G_n$ is given by

$$F_n = \{ x + iy : -\cos \frac{\pi}{n} < x \leq \cos \frac{\pi}{n}, \quad x^2 + y^2 > 1, \quad y > 0 \}.$$ 

Thus we see that $G_n$ is of the first kind and

$$P = P_n = G_n(\infty) = \{ g(\infty) : g \in G_n \}$$

if $n \neq \infty$. In this case, we have

$$\lim_{N \to \infty} \frac{\# \{ g(\infty) : g \in G_n, \quad |\alpha - g(\infty)| < \frac{k}{c^2(g)}, \quad |c(g)| \leq N \}}{\log N}$$

$$= \frac{2 \cdot n \cdot \lambda_n \cdot k}{(n-2) \cdot \pi^2}$$

for any $k, \ 0 < k < 1/2$ and almost all $\alpha \in \mathbb{R} \setminus P_n$. By using the normality of $G_n(m)$, we can prove that the above inequality holds for all $k > 0$.

It is also possible to prove a theorem of the same type for some Kleinian groups of the first kind acting on

$$H^3 = \{ (x, y, z) : x, y \in \mathbb{R}, \quad z > 0 \}.$$ 

For example, we can prove the following: Let $d$ be a square free negative integer and $\mathcal{O}(d)$ the set of integers in $\mathbb{Q}(\sqrt{d})$. Then we have for almost all complex number $\alpha$,

$$\lim_{N \to \infty} \frac{\# \{ \frac{p}{q} : |\alpha - \frac{p}{q}| < \frac{k}{|q|^2}, \quad p, q \in \mathcal{O}(d), \quad |q| \leq N, \quad (p, q)=1 \}}{\log N}$$

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\[ c_d \cdot k^2 \]

for all \( k > 0 \), where \( c_d \) is a constant depending only on \( d \).

[References]


