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CONTINUED FRACTIONS AND ERGODIC THEORY

H. Jager

Let \( x \) be an irrational number between \( 0 \) and \( 1 \),

\[
(1) \quad x = \left\{ 0; a_1, a_2, a_3, \ldots \right\}
\]

its expansion as a regular continued fraction and \( \left( \frac{p_n}{q_n} \right)_{n=1}^{\infty} \) the corresponding sequence of convergents. As is well-known

\[
(2) \quad \left| x - \frac{p_n}{q_n} \right| < \frac{1}{q_n^2}.
\]

Define \( \theta_n(x) := q_n \left| q_n x - p_n \right| \). Hence, by (2), the sequence \( \theta_n(x) \), \( n = 0, 1, 2, \ldots \) is for every fixed \( x \) a sequence in the unit interval. The well-known theorem of Vahlen states that for every \( x \) and every \( n \in \mathbb{N} \)

\[
\min \{ \theta_{n-1}(x), \theta_n(x) \} < 0.5.
\]

It was conjectured by H. W. Lenstra Jr. and proved by Bosma, Jager and Wiedijk [1] that for almost all \( x \), in the sense of Lebesgue, the sequence \( \theta_n(x) \), \( n = 0, 1, 2, \ldots \) is distributed in the unit interval according to the density function \( f \), where \( f \) is given by

\[
f(a) = \begin{cases} 
\frac{\log 2}{2}, & 0 \leq a \leq 0.5 \\
\frac{\log 2}{2} (a^{-1} - 1), & 0.5 \leq a \leq 1 
\end{cases}
\]

In [2] a theorem was proved which contains the two above mentioned results as special cases. It reads
THEOREM 1

For all $x$ the two-dimensional sequence, $(\theta_{n-1}(x), \theta_n(x))$, $n=1,2,\ldots$ is a sequence in the triangle with vertices $(0,0), (1,0)$ and $(0,1)$. For almost all $x$ this sequence is distributed over this triangle according to the density function $f$, where

$$f(a,b) = (\log 2)^{-1} (1 - 4ab)^{-1/2}.$$

The main purpose of this paper is to give a shorter and simpler proof of this theorem than the original one in [2]. At the basis is again the following fundamental result of Sh. Ito, H. Nakada and S. Tanaka, [4] and [5].

THEOREM 2 (Ito, Nakada, Tanaka)

Denote the set of irrational numbers between 0 and 1 by $\Omega$ and put $\Omega := \Omega [0,1]$. Let $B$ be the set of all Borel subsets of $\Omega$, and $\mu$ the measure induced on $B$ by the density function $(\log 2)^{-1} (1 + xy)^{-2}$. Finally, let the operator $T : \Omega \rightarrow \Omega$ be defined by

$$T(x,y) = (Tx, (a_1 + y)^{-1})$$

where, if $x$ is given by (1), $Tx$ is defined as

$$Tx := [0; a_2, a_3, \ldots] = \frac{1}{x} - \left[ \frac{1}{x} \right].$$

Then $(\Omega, B, \mu, T)$ forms an ergodic system.

From this theorem we derive the following

THEOREM 3

For almost all irrational numbers $x$ the two-dimensional sequence

$$\left( T^n x, \frac{q_n}{q_{n-1}} \right)_{n=1}^{\infty}$$

is distributed over the unit square according to the density function $g$, with $g(x,y) = (\log 2)^{-1} (1 + xy)^{-2}$. Here $T^n x$ is defined inductively by $T(T^{n-1}x)$ with the $T$ from (3).
Proof. Denote by \( A \) that set of numbers \( x \in \Omega \) for which the sequence
\[
\left( T^n x, \frac{q_{n-1}}{q_n} \right)_{n=1}^{\infty}
\]
is not distributed according to the density function \((\log 2)^{-1} (1 + xy)^{-2}\).

In view of the well-known relation
\[
\frac{q_{n-1}}{q_n} = [0; a_n, a_{n-1}, a_{n-2}, \ldots, a_1] \quad \text{we see that} \quad T^n(x, 0) = \left( T^n x, \frac{q_{n-1}}{q_n} \right).
\]

Further it follows from the definition of \( T \) that for all \( x \in \Omega \) and all pairs \( y \) and \( y' \in [0,1] \), the sequence \( \left( T^n(x, y) - T^n(x, y') \right) \), \( n = 1, 2, 3, \ldots \) is a null-sequence. Hence, if \( A := A \times [0,1] \), then for every pair \( (x, y) \in A \), the sequence \( T^n(x, y) \) is not distributed according to the density function \((\log 2)^{-1} (1 + xy)^{-2}\). Now if \( A \) had, as a one-dimensional set, a positive Lebesgue measure, so had \( A \) as a two-dimensional set. But this would be in conflict with theorem 2.

With this simple consequence of the theorem of Ito, Nakada and Tanaka it is now easy to prove theorem 1 as follows. We have, see [2], (2.1) and (2.2):

\[
(4) \quad \theta_{n-1}(x) = \frac{q_{n-1}}{q_n} \left( 1 + \frac{q_{n-1}}{q_n} T^n x \right)^{-1}, \quad \theta_n(x) = T^n x \left( 1 + \frac{q_{n-1}}{q_n} T^n x \right)^{-1}.
\]

In view of this we consider the function

\[
F : (x, y) \to \left( \frac{y}{1 + xy}, \frac{x}{1 + xy} \right), \quad xy \neq -1.
\]

It is easily verified that \( F \) maps the interior of the unit square bijectively onto the interior of the triangle from theorem 1. Put \( a := y (1 + xy)^{-1} \), \( b := x (1 + xy)^{-1} \). The determinant of Jacobi, \( J \), of \( F \) equals \((xy - 1)(1 + xy)^{-3}\).

For almost all \( x \) the sequence \( \left( T^n x, \frac{q_{n-1}}{q_n} \right)_{n=1}^{\infty} \) is distributed according to the density function \( g \) from theorem 3. Hence, for almost all \( x \) the sequence \( \left( F(T^n x, \frac{q_{n-1}}{q_n}) \right)_{n=1}^{\infty} \)
which is in view of (4) the sequence \( \theta_{n-1}(x), \theta_n(x) \), \( n = 1, 2, 3, \ldots \) is distributed over the interior of the triangle with vertices \((0,0), (1,0)\) and \((0,1)\) according to the density function \( g(J)^{-1} \).

Now

\[
g(x,y) | J |^{-1} = (\log 2)^{-1} \frac{1 + xy}{1 - xy} = (\log 2)^{-1} \left( \frac{(1 - xy)^2}{1 + xy} \right)^{-1/2} =
\]

\[
= (\log 2)^{-1} \left( \frac{(1 + xy)^2 - 4xy}{(1 + xy)^2} \right)^{-1/2} = (\log 2)^{-1} (1 - 4ab)^{-1/2} = f(a,b).
\]

In [2], several properties of the sequence \( \theta_n(x) \), \( n = 1, 2, 3, \ldots \) were given as corollaries of theorem 1. We mention one more.

**COROLLARY.** Let \( \lambda \geq 0 \). Then for almost all \( x \) one has

\[
\lim_{n \to \infty} n^{-1} \# \{ j; 1 \leq j \leq n, \theta_{j-1}(x) < \lambda \theta_j(x) \} = \begin{cases} 
(2 \log 2)^{-1} \log (1 + \lambda), & 0 \leq \lambda \leq 1 \\
1 - (2 \log 2)^{-1} \log (1 + \lambda^{-1}), & 1 \leq \lambda
\end{cases}.
\]

**Proof.**

It follows from (4) that the condition \( \theta_{j-1}(x) < \lambda \theta_j(x) \) is equivalent with

\[
\frac{q_{n-1}}{q_n} < \lambda \, T^n x. \quad \text{Hence, by theorem 3, we have, when } 0 \leq \lambda \leq 1, \text{ for almost all } x
\]

\[
\lim_{n \to \infty} n^{-1} \# \{ j; 1 \leq j \leq n, \theta_{j-1}(x) < \lambda \theta_j(x) \} = \frac{1}{\log 2} \int_0^1 \frac{\lambda x}{(1 + xy)^2} dx = \frac{1}{2 \log 2} \int_0^1 \frac{2 \lambda x}{1 + \lambda x^2} dx = \frac{1}{2 \log 2} \log(1 + \lambda).
\]

The case \( 1 \leq \lambda \) follows immediately from the case \( 0 \leq \lambda \leq 1 \).
Final remark.

In [3], C. Kraaikamp extended the method of [2] to the nearest integer continued fraction and Hurwitz’s singular continued fraction and obtained several interesting results. This author has now also obtained the results from [3] by the method of the present paper (oral communication). He has also applied the method to general $\alpha$-expansions. His results will be published in due course.

REFERENCES


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