<table>
<thead>
<tr>
<th>Title</th>
<th>CONTINUED FRACTIONS AND ERGODIC THEORY (Transcendental Numbers and Related Topics)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Jager, H.</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 1986(1986), 599: 55-59</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1986-10</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/99589">http://hdl.handle.net/2433/99589</a></td>
</tr>
<tr>
<td>Right</td>
<td></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
<tr>
<td>Kyoto University</td>
<td></td>
</tr>
</tbody>
</table>
CONTINUED FRACTIONS AND ERGODIC THEORY

H. Jager

Let \( x \) be an irrational number between 0 and 1,

\[
x = \left[ 0; a_1, a_2, a_3, \ldots \right]
\]

its expansion as a regular continued fraction and \( \left( \frac{p_n}{q_n} \right)_{n=1}^{\infty} \) the corresponding sequence of convergents. As is well-known

\[
\left| x - \frac{p_n}{q_n} \right| < \frac{1}{q_n^2}.
\]

Define \( \theta_n(x) := q_n | q_n x - p_n | \). Hence, by (2), the sequence \( \theta_n(x) \), \( n = 0, 1, 2, \ldots \) is for every fixed \( x \) a sequence in the unit interval. The well-known theorem of Vahlen states that for every \( x \) and every \( n \in \mathbb{N} \)

\[
\min \{ \theta_{n-1}(x), \theta_n(x) \} < 0.5.
\]

It was conjectured by H. W. Lenstra Jr. and proved by Bosma, Jager and Wiedijk [1] that for almost all \( x \), in the sense of Lebesgue, the sequence \( \theta_n(x) \), \( n = 0, 1, 2, \ldots \) is distributed in the unit interval according to the density function \( f \), where \( f \) is given by

\[
f(a) = \begin{cases} 
(\log 2)^{-1} & , 0 \leq a \leq 0.5 \\
(\log 2)^{-1} (a^{-1} - 1) & , 0.5 \leq a \leq 1
\end{cases}
\]

In [2] a theorem was proved which contains the two above mentioned results as special cases. It reads
THEOREM 1

For all \( x \) the two-dimensional sequence \( (\theta_{n-1}(x), \theta_n(x)) \), \( n=1,2,\ldots \) is a sequence in the triangle with vertices \((0,0),(1,0)\) and \((0,1)\). For almost all \( x \) this sequence is distributed over this triangle according to the density function \( f \), where

\[
f(a,b) = (\log 2)^{-1} (1 - 4ab)^{-1/2}.
\]

The main purpose of this paper is to give a shorter and simpler proof of this theorem than the original one in [2]. At the basis is again the following fundamental result of Sh. Ito, H. Nakada and S. Tanaka, [4] and [5].

THEOREM 2 (Ito, Nakada, Tanaka)

Denote the set of irrational numbers between 0 and 1 by \( \Omega \) and put \( \Omega := \Omega \times [0,1] \). Let \( B \) be the set of all Borel subsets of \( \Omega \), and \( \mu \) the measure induced on \( B \) by the density function \( (\log 2)^{-1} (1 + xy)^{-2} \).

Finally, let the operator \( T : \Omega \to \Omega \) be defined by

\[
T(x,y) = (Tx, (a_1 + y)^{-1})
\]

where, if \( x \) is given by \((1), Tx \) is defined as

\[
Tx := \lfloor 0; a_2, a_3, \ldots \rfloor = \frac{1}{x} - \lfloor \frac{1}{x} \rfloor.
\]

Then \((\Omega, B, \mu, T)\) forms an ergodic system.

From this theorem we derive the following

THEOREM 3

For almost all irrational numbers \( x \) the two-dimensional sequence

\[
(\frac{T^nx}{q_n}, \frac{q_{n-1}}{q_n})^\infty_{n=1}
\]

is distributed over the unit square according to the density function \( g \), with \( g(x,y) = (\log 2)^{-1} (1 + xy)^{-2} \).

Here \( T^n x \) is defined inductively by \( T(T^{n-1} x) \) with the \( T \) from \((3)\).
Proof.

Denote by $A$ that set of numbers $x \in \Omega$ for which the sequence
\[(T^n x, \frac{q_{n-1}}{q_n})_{n=1}^{\infty}\]
is not distributed according to the density function $(\log 2)^{-1} (1 + xy)^{-2}$.

In view of the well-known relation
\[\frac{q_{n-1}}{q_n} = [0; a_n, a_{n-1}, a_{n-2}, \ldots, a_1]\]
we see that $T^n(0,0) = (T^n x, \frac{q_{n-1}}{q_n})$.

Further it follows from the definition of $T$ that for all $x \in \Omega$ and all pairs $y$ and $y' \in [0,1]$, the sequence $(T^n(x,y) - T^n(x,y'))$, $n = 1, 2, 3, \ldots$ is a null-sequence. Hence, if $A := A \times [0,1]$, then for every pair $(x,y) \in A$, the sequence $T^n(x,y)$ is not distributed according to the density function $(\log 2)^{-1} (1 + xy)^{-2}$. Now if $A$ had, as a one-dimensional set, a positive Lebesgue measure, so had $A$ as a two-dimensional set. But this would be in conflict with theorem 2.

With this simple consequence of the theorem of Ito, Nakada and Tanaka it is now easy to prove theorem 1 as follows. We have, see [2], (2.1) and (2.2):

\[\tag{4} \theta_{n-1}(x) = \frac{q_{n-1}}{q_n} \left( 1 + \frac{q_{n-1}}{q_n} T^n x \right)^{-1}, \quad \theta_n(x) = T^n x \left( 1 + \frac{q_{n-1}}{q_n} T^n x \right)^{-1}.\]

In view of this we consider the function

\[F : (x,y) \rightarrow \left( \frac{y}{1 + xy}, \frac{x}{1 + xy} \right), \quad xy \neq -1.\]

It is easily verified that $F$ maps the interior of the unit square bijectively onto the interior of the triangle from theorem 1. Put $a := y (1 + xy)^{-1}$, $b := x (1 + xy)^{-1}$. The determinant of Jacobi, $J$, of $F$ equals $(xy - 1)(1 + xy)^{-3}$.

For almost all $x$ the sequence $(T^n x, \frac{q_{n-1}}{q_n})_{n=1}^{\infty}$ is distributed according to the density function $g$ from theorem 3. Hence, for almost all $x$ the sequence $(F(T^n x, \frac{q_{n-1}}{q_n}))_{n=1}^{\infty}$
which is in view of (4) the sequence \( (\theta_{n-1}(x), \theta_n(x)) \), \( n = 1, 2, 3, \ldots \) is distributed over the interior of the triangle with vertices \((0,0), (1,0)\) and \((0,1)\) according to the density function \( g \big| J \big|^{-1} \).

Now

\[
g(x,y) \big| J \big|^{-1} = (\log 2)^{-1} \, \frac{1 + xy}{1 - xy} = (\log 2)^{-1} \, \left( \frac{1 - xy}{1 + xy} \right)^{1/2} =
\]

\[
= (\log 2)^{-1} \, \left( \frac{(1 + xy)^2 - 4xy}{(1 + xy)^2} \right)^{1/2} = (\log 2)^{-1} \, (1 - 4ab)^{-1/2} = f(a,b).
\]

In [2], several properties of the sequence \((\theta_n(x)), n = 1,2,3, \ldots\) were given as corollaries of theorem 1. We mention one more.

**COROLLARY.** Let \( \lambda \geq 0 \). Then for almost all \( x \) one has

\[
\lim_{n \to \infty} n^{-1} \# \{ j; 1 \leq j \leq n, \theta_{j-1}(x) < \lambda \theta_j(x) \} = \begin{cases} 
(2\log 2)^{-1} \log (1 + \lambda), & 0 \leq \lambda \leq 1 \\
1 - (2\log 2)^{-1} \log (1 + \lambda^{-1}), & 1 \leq \lambda
\end{cases}.
\]

**Proof.**

It follows from (4) that the condition \( \theta_{j-1}(x) < \lambda \theta_j(x) \) is equivalent with

\[
\frac{q_{n-1}}{q_n} < \lambda \, T^n x. \quad \text{Hence, by theorem 3, we have, when } 0 \leq \lambda \leq 1, \text{ for almost all } x
\]

\[
\lim_{n \to \infty} n^{-1} \# \{ j; 1 \leq j \leq n, \theta_{j-1}(x) < \lambda \theta_j(x) \} =
\]

\[
= \frac{1}{\log 2} \int_0^1 \frac{\lambda x}{(1 + xy)^2} \, dx = \frac{1}{2\log 2} \int_0^1 \frac{2\lambda x}{1 + \lambda x^2} \, dx = \frac{1}{2\log 2} \log(1 + \lambda).
\]

The case \( 1 \leq \lambda \) follows immediately from the case \( 0 \leq \lambda \leq 1. \)
Final remark.

In [3], C. Kraaikamp extended the method of [2] to the nearest integer continued fraction and Hurwitz's singular continued fraction and obtained several interesting results. This author has now also obtained the results from [3] by the method of the present paper (oral communication). He has also applied the method to general $\alpha$-expansions. His results will be published in due course.

REFERENCES


H. Jager
Mathematical Institute of the University of Amsterdam
Roetersstraat 15
1018 WB Amsterdam
the Netherlands