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On the Uniqueness of Einstein Kähler Metrics

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One of the most important results in Kähler geometry would be the resolution of Calabi Conjecture by Aubin and Yau. It says in particular any compact Kähler manifold of null or negative first Chern class admits an Einstein Kähler metric. Since the existence of an Einstein Kähler metric implies positive, null, or negative first Chern class, people's attentions are focused to the remaining positive first Chern class case. Contrary to the previous cases, we do not always have Einstein Kähler metrics. Two obstructions to existence are known. One is a theorem of Matsushima : if a compact Kähler manifold X admits an Einstein Kähler metric, then its complex automorphism group $\mathrm{Aut}(X)$ is reductive. Another is the Futaki invariant, which we shall explane latter. So far we do not know these obstructions are only ones or not, besides the case of homogenous or almost homogenous with two exceptional orbits. The latter case was treated by Sakane and Koiso who showed that the vanishing of the Futaki invariant is sufficient. Putting aside the question of existence, let us turn to another face of the subject, a matter of uniqueness. In the null or negative first Chern class case, an Einstein Kähler metric is uniquely determined by the cohomology class of its Kähler form in $H^{1,1}(X,R)$, as easily seen by the maximum principle of elliptic equations of second order. As to the positive first Chern class case, we again encounter a difficulty, it can not be unique if X admits non trivial holomorphic vector fields. But fortunately it turned out that it was the only difficulty, as we shall see in the folloing. This is a joint work with Mabuchi.

§0. Notation.

We fix an n-dimensional compact Kähler manifold X with positive first Chern class $C_1(X) > 0$. We do not distinguish a Kähler metric and its Kähler form. Set

$$K = \{ \omega : \text{positive definite d-closed real } (1,1)\text{-forms s.t. } \omega \in 2\pi C_1(X) \} .$$

For $\omega \in K$, we associate its Ricci form γ_ω , scalar curvature σ_ω , the Laplacian Δ_ω acting on the space of functions $C^\infty(X)$, and a function f_ω such that $\gamma_\omega - \omega = \sqrt{-1}\partial\bar{\partial} f_\omega$, which is determined up to a constant. Put

$$K^+ = \{ \omega \in K \mid \gamma_\omega > 0 \} ,$$

$$E = \{ \omega \in K \mid \gamma_\omega = \omega \} ,$$

Aut(X) : the group of biholomophic automorphisms of X ,

$Aut^0(X)$: the identity component of Aut(X) .

Since we frequently use a smooth path $\omega_t \in K$, and associated notions like γ_{ω_t} , σ_{ω_t} , $\cdots$, we will denote them as γ_t , σ_t , $\cdots$, or simply γ , σ , $\cdots$, for the sake of brevity. Let us introduce functionals I , J , M , and μ after Aubin and Mabuchi.

Def.

For ω_0 , $\omega_1 \in K$, choose a path $\omega_t = \omega_0 + \sqrt{-1}\partial\bar{\partial}u_t \in K$ connecting ω_0 and ω_1 , where $u_t \in C^\infty(X)$, $t \in [0,1]$. Let us define functionals I , J , and M on $K\times K$ as follows ;

$$I(\omega_0,\omega_1) = \int_X u_1(\omega_0{}^n - \omega_1{}^n)/V ,$$

$$J(\omega_0,\omega_1) = \int_0^1 dt \int_X \frac{du}{dt} (\omega_0{}^n - \omega_t{}^n)/V ,$$

$$M(\omega_0,\omega_1) = -\int_0^1 dt \int_X \frac{du}{dt} (\sigma_t - n)\, \omega_t{}^n/V , \text{ where } V = \int_X \omega^n .$$

For Kähler manifold (X, ω_0) $\omega_0 \in K$, we define the K-energy map $\mu = \mu_{\omega_0}$ from K to R by $\mu(\omega) = M(\omega_0,\omega)$.

For the well-definedness and the proof of the following properties of I, J, M, and μ, see (2), (12), and (13).

Proposition 1.

For ω_0, ω_1, and $\omega_2 \in K$, we have that

i) $I(\omega_0, \omega_1) = I(\omega_1, \omega_0) \geq 0$,

and "=" holds if and only if $\omega_0 = \omega_1$,

ii) $J(\omega_0, \omega_1) + J(\omega_1, \omega_0) = I(\omega_0, \omega_1)$,

$\frac{1}{n+1} I(\omega_0, \omega_1) \leq J(\omega_0, \omega_1) \leq \frac{n}{n+1} I(\omega_0, \omega_1)$,

iii) $M(\omega_0, \omega_1) + M(\omega_1, \omega_2) + M(\omega_2, \omega_0) = 0$,

thus $\mu_{\omega_1}(\omega) = \mu_{\omega_0}(\omega) + M(\omega_1, \omega_0)$,

iv) I, J, and M are invariant under the action of Aut(X),

for example $M(\Phi^* \omega_0, \Phi^* \omega_1) = M(\omega_0, \omega_1)$ where $\Phi \in \mathrm{Aut}(X)$.

§1. Theorems.

Theorem 1. (Mabuchi)

For $\omega \in K$, $\omega \in E$ if and only if ω is a critical point of the K-energy map μ.

Note that by iii) in the Proposition 1, the notion of critical points are well-defined free from the choice of the base point $\omega_0 \in K$.

Def.

Define a group homomorphism $C : \mathrm{Aut}(X) \longrightarrow R$ as follows, with fixed $\omega_0 \in K$, $C(\Phi) = \mu_{\omega_0}(\Phi^* \omega_0) = M(\omega_0, \Phi^* \omega_0)$.

Theorem 2. (Mabuchi)

C is a well-defined gruop homomorphism and its differential coinsides with the Futaki invariant up to a multiple constant.

Corollary. (Futaki)

If $E \neq \phi$, then the Futaki invariant vanishes.

Theorem 3.

If $E \neq \phi$, $Aut^0(X)$ acts on E transitively.

A standerd method shows that each connected component of E is an $Aut^0(X)$ orbit. Thus what we have to prove is that E is connected.

Remark.

1) Theorem 3 says that for any ω_0 , $\omega_1 \in E$, we can find $\Phi \in Aut^0(X)$ s.t. $\Phi^* \omega_0 = \omega_1$. In the view of Kähler geometry it means the uniqueness of Einstein Kähler metrics, since Φ is an isometric biholomorphism between (X, ω_0) and (X, ω_1) .

2) If $Aut^0(X) = \{id\}$ i.e. there exist no nonzero holomorphic vector fields, then E is actually a point.

Theorem 4.

If $E \neq \phi$, μ attains its minimum value exactly on E . In particular μ is bounded from below.

In course of the proof of Theorem 3 , 4 , we notice a weak inverse of Theorem 4 holds.

Theorem 5.

If μ is bounded from below, then we can solve Aubin's equation on $[0,1)$.

Refering the definition of Aubin's equation to the next § , we are content to point out that a solution of the equation at $t = 1$ is an Einstein Kähler metric. Thus we can regard Theorem 4 as an existence result of almost Einstein Kähler metrics. In this direction we can prove with the aid of Hamilton's equation

Theorem 6.

If μ is bounded from below, for any $\varepsilon > 0$ we can find $\omega \in K$ s.t. $|\sigma - n| < \varepsilon$.

Letting $\varepsilon \longrightarrow 0$, we get,

Theorem 7.

If μ is bounded from below, the following inequality of

Miyaoka-Yau type holds.

$$2(n+1)C_2(X)C_1(X)^{n-2} \geq nC_1(X)^n ,$$

where $C_2(X)$ is second Chern class.

Theorem 3 shows E is a symmetric space, on which Mabuchi constructed a natural metric (Mabuchi metric). Because the Mabuchi metric is invariant under Aut(X) ,

Theorem 8.

For any compact subgroup G (may not be connected) of Aut(X) , we can find $\omega \in E$, which is G-invariant.

In the folloeing sections we shall explane how to prove Theorem 3.

§2. Idea of proof of Theorem 3.

Remember the continuity method to solve an equation $F(x) = 0$. First we embedd the equation into a family of equations $F_t(x) = 0$, $t \in [0,1]$ s.t. $F_1 = F$, and verify the following properties.

i) $F_0(x) = 0$ has an unique solution.
ii) $dF_t(x)$ is invertible on $\{ (x,t) \mid F_t(x) = 0 \}$.
iii) $\{ (x,t) \mid F_t(x) = 0 \}$ is precompact.

Then we conclude $F(x) = 0$ has an unique solution.

Emphasis is usually made on the existence part, but we can also use it to prove the uniqueness. So let us try to apply it in our situation. The equation we want to treat is

$$\gamma_\omega = \omega \ , \ \omega \in K \ .$$

We embedd it in, with a fixed $\omega_b \in K$,

$$\gamma_\omega = t\omega + (1 - t)\omega_b \ , \quad \text{(Aubin's equation).}$$

Then i) is done by the resolution of Calabi Conjecture. But unfortunately we can not prove ii) at $t = 1$ and iii) , that was expected, because we do not have the uniqueness. So we have to make an adjustment. Since what we want to prove is not the uniqueness but the connectedness of $E = \{ x \mid F(x) = 0 \}$, we replace ii) and iii) as follows.

ii)' $dF_t(x)$ is invertible on $\{ (x,t) \mid F_t(x) = 0 \ , \ t \in [0,1) \}$.

iii)' Let x_t , $t \in (a,b] \subset [0,1)$ be any solution of $F_t(x) = 0$ which smoothly depend in t , then $\{ x_t \mid t \in (a,b] \}$ is precompact.

iv) For any connected component E_0 of E , we can find $x_1 \in E_0$ s.t. for some $\varepsilon > 0$, we can extend x_1 to x_t , $t \in [1 - \varepsilon, 1]$ smoothly so that $F_t(x_t) = 0$.

Then we conclude E is connected.

New version of continuity method works good in our situation.

ii)′ is already shown by Aubin [2], using the fact that for $t \in (0,1)$, $\gamma_\omega > t\omega$, thus $\lambda_1 > t$ holds, where λ_1 is the first eigenvalue of Δ_ω.

iii)′ is a consequence of Gallot's sharp isoperimetric inequality and a heat kernel estimate of Cheng - Li.

iv) is done by the bifurcation method (to be precise we have to perturb $\omega_b \in K$ to be generic).

§3. A priori estimate.

For the solution of Aubin's equation, we associate $u_t \in C^\infty(X)$ s.t. $\omega_t = \omega_b + \sqrt{-1}\partial\bar{\partial}u_t$, which is determined up to a constant.

Proposition 2.

There exists a constant A s.t. for $t \in (0,1)$,

$$\operatorname{Osc} u_t = \operatorname{Max} u_t - \operatorname{Min} u_t \le I(\omega_b, \omega_t) + A(1 + t^{-1}).$$

Proof.

Combination of Gallot's sharp isoperimetric inequality and a heat kernel estimate of Cheng - Li implies a lower bound of Green functions. Let G_ω be the Green function of the Laplacian Δ_ω, then there exists a constant $B > 0$ such that

$$\inf G_t \geq -Bt^{-1}, \qquad \text{and} \qquad \inf G_{\omega_b} \geq -B .$$

Since

$$-\Delta_t u_t \geq -n, \qquad \text{and} \qquad -\Delta_{\omega_b} u_t \leq n,$$

we have that

$$\begin{aligned} u_t &= \int_X u_t\, \omega_t{}^n/V + \int_X (-\Delta_t u_t)\, G_t\, \omega_t{}^n \\ &= \int_X u_t\, \omega_t{}^n/V + \int_X (-\Delta_t u_t)\, (G_t + Bt^{-1})\, \omega_t{}^n \\ &\geq \int_X u_t\, \omega_t{}^n/V + \int_X (-n)\, (G_t + Bt^{-1})\, \omega_t{}^n \\ &= \int_X u_t\, \omega_t{}^n/V - nBt^{-1}, \end{aligned}$$

similarly

$$u_t \leq \int_X u_t\, \omega_b{}^n/V + nBV .$$

Thus

$$\mathrm{Osc}\, u_t \leq I(\omega_b, \omega_t) + nBV(1 + t^{-1}) .$$

Lemma 1.

For a solution u_t smoothly depending in t,

$$\frac{d}{dt}(I - J)(\omega_b, \omega_t) = -\frac{1}{1-t}\frac{d}{dt}\mu(\omega_t) \geq 0 .$$

Proof.

We can have u_t smoothly depending in t. Differentiating the equation in t, we get that

$$(\Delta_t + t)\frac{d}{dt}u_t + u_t + a_t = 0 , \quad a_t \in R .$$

Also the equation implies that $\sigma_t = n - (1 - t)\Delta_t u_t$.

Lemma 2.

iii)′ holds.

Proof.

By the proof of Calabi Conjecture by Yau, it is enough to have an uniform estimate on Osc u_t . If $a \neq 0$, it is clear from Proposition 2 and Lemma 1. If $a = 0$, we have an estimate of Osc tu_t . Choosing suitably the indetermined constant of u_t we can rewrite the equation as

$$\omega_t{}^n = (\omega_b + \sqrt{-1}\partial\bar{\partial}u_t)^n = \exp(-tu_t + f_{\omega_b})\,\omega_b{}^n .$$

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Because

$$\int_X \omega_t^{\,n} = \int_X \omega_b^{\,n} ,$$

we have $tu_t = f_{\omega_b}$ somewhere on X. Since $\mathrm{Osc}\ tu_t$ is bounded it implies $|tu_t|_{C^0(X)}$ is bounded. Again by the proof of Calabi Conjecture we conclude that $\mathrm{Osc}\ u_t$ is bounded.

Lemma 3.

$\iota = (I - J)(\omega_b, \cdot)|_E$ is a proper function.

§4. Bifurcation.

Lemma 4.

If H_t satisfies

a) $H_1 = 0$,

b) $\frac{\partial}{\partial t} H_t = 0$, at $(x_1, 1)$,

c) $d\ \frac{\partial}{\partial t} H_t$ is invertible at $(x_1, 1)$,

then x_1 is smoothly extendable to x_t , $t \in (1 - \varepsilon, 1]$ for some $\varepsilon > 0$ s.t. $H_t(x_t) = 0$.

Let E_0 be any connected component of E , and $\omega_1 \in E_0$. We project Aubin's equation on $\mathrm{Ker}(dF_1)$, then the resulting equation $H_t = 0$ satisfies a) . Curiously, the condition b) is equivalent to saying that ω_1 is a critical point of ι , and in this case, the linear map in c) coincides with the Hessian of ι regarded as a linear map. Because of Lemma 3 , we can always get a minimum in E_0 , then at the point ω_1 cearly b) holds. And we can assume ι has the positive definite Hessian at ω_1 , if necessary, taking a samll perturbation of ω_b . This finishes with iv) . For a detail of the proof, refer to (4) .

Remark.

The above proof also shows Theorem 5 and a part of Theorem 4. The former is immediate, because Lemma 1 implies that if μ is bounded from below, $(I - J)(\omega_b, \omega_t)$ is also bounded on $(0, 1)$. The latter is as follows. Since as ω_0 one can take any element in K^+ , and μ is non increasing on the solution of Aubin's equation, we get that if $E \neq \phi$, $\mu|_{K^+}$ takes its minimum

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