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Large-time behavior of solutions for
the equations of a viscous gas

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1. Introduction

We consider one-dimensional flow of a compressible fluid. In the
Lagrange mass coordinate \((t,x)\), the motion of the fluid is described by
the following equations.

\[
\begin{align*}
&v_t - u_x = 0 , \\
&u_t + p_x = (\mu u_x/v)_x , \\
&\left( e + u^2/2 \right)_t + (pu)_x = (\kappa \theta_x/v + \mu uu_x/v)_x .
\end{align*}
\]

Here \(v > 0\) is the specific volume, \(u\) the velocity, \(\theta > 0\) the absolute
temperature, \(e\) the internal energy, \(\mu\) the coefficient of viscosity and \(\kappa\)
the coefficient of heat-conductivity. Let us denote the entropy by \(s\). It
is known that among five thermodynamic variables \(v, \theta, p, e \) and \( s \), only
two of them are independent. In fact they may all be considered as smooth
functions of \((v, \theta), (v, s), (v, e)\) or \((p, s)\). We write \(p = p(v, \theta) =
\hat{p}(v, s)\) and \(e = e(v, \theta)\) and assume that

\[
\begin{align*}
&(1.2)_1 \quad \partial p(v, \theta)/\partial v < 0 , \quad \partial p(v, \theta)/\partial \theta > 0 , \quad \partial e(v, \theta)/\partial \theta > 0 , \\
&(1.2)_2 \quad \partial^2 \hat{p}(v, s)/\partial v^2 > 0 .
\end{align*}
\]
Notice that these conditions are satisfied for the case of an ideal polytropic gas:

\begin{equation}
(1.3) \quad p = R \varrho / v = \hat{R} v^{(\gamma - 1)} s / R, \quad e = R \varrho / (\gamma - 1) + \text{constant},
\end{equation}

where \( R > 0 \) is the gas constant, \( \gamma > 1 \) is the adiabatic exponent and \( \hat{R} \) is a positive constant. We also assume that \( \mu \) and \( \kappa \) are smooth functions of two independent thermodynamic variables and satisfy one of the following two conditions.

\begin{align}
(1.4)_1 & \quad \mu > 0, \quad \kappa > 0 \quad (\text{viscous heat-conductive fluid}), \\
(1.4)_2 & \quad \mu \equiv 0, \quad \kappa > 0 \quad (\text{inviscid heat-conductive fluid}).
\end{align}

We shall study the large-time behavior of solutions to the initial value problem for (1.1). Our main result is as follows: If the initial data are close to a given constant state, then a unique smooth solution of (1.1) exists for all time \( t \geq 0 \) and approaches the superposition of the nonlinear and linear diffusion waves constructed in terms of the self-similar solutions of the Burgers equation and the linear heat equation as \( t \to \infty \).

We remark that the same asymptotic result has been obtained in [5] for a wide class of systems including (1.1).

\textbf{Notations}

We introduce several function spaces. Let \( p \in [1, \infty], \beta \in \mathbb{R} \) and \( s \geq 0 \). \( L^p \) denotes the usual Lebesgue space on \( \mathbb{R} \), with the norm \( \| \cdot \|_p \). \( L^p_\beta \) denotes the space of functions \( f = f(x) \) such that \( (1 + |x|)^\beta f \in L^p \), with the norm \( \| \cdot \|_{p, \beta} \). \( H^s \) denotes the space of functions \( f = f(x) \) such that \( \partial_x^\ell f \in L^2 \) for \( 0 \leq \ell \leq s \), with the norm \( \| \cdot \|_s \). Note that \( H^0 = L^2 \)
and \( \| \cdot \|_0 = | \cdot |_2 \). \( C^0([0,\infty); H^5) \) is the space of continuous functions on \([0,\infty)\) with values in \( H^5 \).

2. Preliminaries

We first choose \( v \) and \( \theta \) as independent thermodynamic variables and write \( p = p(v,\theta) \), \( e = e(v,\theta) \) and \( s = s(v,\theta) \). The thermodynamic law \( \text{de} = \partial s \text{ds} - p \text{dv} \) gives

\[
(2.1) \quad e_v = -(p - \theta p_\theta), \quad s_v = p_\theta, \quad s_\theta = e_\theta/\theta,
\]

where we used abbreviations such as \( e_v = \partial e(v,\theta)/\partial v \). When \( v \) and \( s \) are regarded as independent variables, we write \( \theta = \hat{\theta}(v,s) \), \( p = \hat{p}(v,s) \) and \( e = \hat{e}(v,s) \). Using (2.1), we obtain

\[
(2.2) \quad \hat{\theta}_v = \theta/p_\theta/e_\theta, \quad \hat{p}_v = p_v - \theta p^2_\theta/e_\theta, \quad \hat{e}_v = -p,
\]

\[
\hat{\theta}_s = \theta/e_\theta, \quad \hat{p}_s = \theta p_\theta/e_\theta, \quad \hat{e}_s = \theta,
\]

where \( \hat{\theta}_v = \partial \hat{\theta}(v,s)/\partial v \), etc. In particular, we have \( \hat{p}_v < 0 \) by (1.2). Similarly, choosing \( v \) and \( e \) as independent variables and writing \( \theta = \tilde{\theta}(v,e) \), \( p = \tilde{p}(v,e) \) and \( s = \tilde{s}(v,e) \), we obtain

\[
(2.3) \quad \tilde{\theta}_v = (p - \theta p_\theta)/e_\theta, \quad \tilde{p}_v = (p_v - \theta p^2_\theta/e_\theta) + p p_\theta/e_\theta, \quad \tilde{s}_v = p/\theta,
\]

\[
\tilde{e}_v = 1/e_\theta, \quad \tilde{p}_e = p_\theta/e_\theta, \quad \tilde{s}_e = 1/\theta,
\]

where \( \tilde{\theta}_v = \partial \tilde{\theta}(v,e)/\partial v \), etc. In particular, we have \( \tilde{p}_v - \tilde{p}_e = \hat{p}_v \).

3. Vector form of the system

Put \( E = e + u^2/2 \). Then (1.1) is regarded as a system for \((v,u,E)\)
and is rewritten in the vector form

\[(3.1) \quad w_t + f(w)_x = (G(w)w)_x,\]

where \( w = (v,u,E)^T, f(w) = (-u,p,pu)^T, \) and \( G(w) \) is the matrix given by (3.3) below. We denote by \( A(w) \) the Jacobian of \( f(w) \) with respect to \( w \). Then (3.1) is equivalent to

\[(3.1') \quad w_t + A(w)w_x = (G(w)w)_x.\]

\( A(w) \) and \( G(w) \) are given explicitly as follows.

\[
(3.2) \quad A(w) = \begin{pmatrix}
0 & -1 & 0 \\
\tilde{p}_v & -u\tilde{p}_e & \tilde{p}_e \\
u\tilde{p}_v & p - u^2\tilde{p}_e & u\tilde{p}_e 
\end{pmatrix},
\]

\[
(3.3) \quad G(w) = \begin{pmatrix}
0 & 0 & 0 \\
0 & \mu/v & 0 \\
k\tilde{\theta}_v/v & \mu u/v - ku\tilde{\theta}_e/v & k\tilde{\theta}_e/v 
\end{pmatrix}.
\]

By straightforward calculations, using (2.2) and (2.3), we know that the eigenvalues of \( A(w) \) are given by

\[
(3.4) \quad \lambda_1(w) = -(-\hat{p}_v)^{1/2}, \quad \lambda_2(w) = 0, \quad \lambda_3(w) = (-\hat{p}_v)^{1/2}.
\]

These are all real and distinct since \( \hat{p}_v < 0 \) by (1.2)\textsuperscript{1}. This means that the inviscid system \( w_t + f(w)_x = 0 \) is strictly hyperbolic. The corresponding right and left eigenvectors, \( r_j(w) \) and \( l_j(w) \), are

\[
(3.5) \quad r_j(w) = a_j(1, -\lambda_j, -u\lambda_j - p)^T, \quad j = 1, 3,
\]

\[
(3.5) \quad r_2(w) = a_2(\tilde{p}_e, 0, -\tilde{p}_v)^T,
\]

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\[ \ell_j^2(w) = b_j(-\tilde{p}_v, -\lambda_j + \omega \tilde{p}_e, -\tilde{p}_e), \quad j = 1, 3, \]
\[ \ell_2^2(w) = b_2(p, -u, 1), \]
where \( a_j b_j \not= 0, j = 1, 2, 3 \). We choose \( a_j \) and \( b_j \) such that \( 2a_j b_j = 1/(-\tilde{p}_v), j = 1, 3, \) and \( a_2 b_2 = 1/(-\tilde{p}_v) \). In this case we have
\[ \langle \ell_j^2(w), r_k^2(w) \rangle = \delta_{jk}, \quad j, k = 1, 2, 3, \]
where \( \langle , \rangle \) denotes the standard inner product of \( \mathbb{R}^3 \). When (1.2)2 is assumed, we determine \( a_j \) such that \( a_j = -2\lambda_j \tilde{p}_{vv}, j = 1, 3, \) and \( a_2 = \partial(\tilde{p}_v), \) where \( \tilde{p}_{vv} = \partial^2 p(v, s)/\partial v^2 \). Then we have
\[ \langle \nabla \lambda_j^2(w), r_j^2(w) \rangle = 1, \quad j = 1, 3, \]
\[ \langle \nabla s(w), r_2^2(w) \rangle = 1. \]
Here the gradient \( \nabla \) is with respect to \( w \), and \( s = s(w) \) is the entropy. Since \( \langle \nabla \lambda_j^2(w), r_j^2(w) \rangle \not= 0, j = 1, 3, \) the first and the third characteristic fields are genuinely nonlinear in the sense of Lax [7]. While the second field is linearly degenerate ([7]) because we have \( \langle \nabla \lambda_2^2(w), r_2^2(w) \rangle = 0 \) by \( \lambda_2^2(w) = 0 \).

4. Global existence and decay of solution

We consider (3.1) with the initial condition
\[ w(0, x) = w_0(x), \]
where \( w_0 = (v_0, u_0, E_0)^T \) with \( E_0 = e_0 + u_0^2/2 \). We seek a solution of (3.1), (4.1) in a neighborhood of a constant state \( \bar{w} = (\bar{v}, \bar{u}, \bar{E})^T \), where \( \bar{v} > 0, \bar{u} \in \mathbb{R}, \) and \( \bar{E} = \bar{e} + \bar{u}^2/2 \) with \( \bar{e} = \bar{e}(\bar{v}, \bar{e}) > 0 \). We have the following
global existence result.

**Theorem 4.1.** ([6], see also [4]) Assume (1.2)$_1$, and (1.4)$_1$ or (1.4)$_2$. If $w_0(x) - \bar{w}$ is small in $H^S$, $s \geq 2$, then the initial value problem (3.1),(4.1) has a unique global solution $w(t,x)$ in an appropriate function space. In particular, we have $w - \bar{w} \in C^0([0,\infty); H^S)$ and $\|w(t) - \bar{w}\|_S \leq C \|w_0 - \bar{w}\|_S$ for $t \in [0,\infty)$, where $C$ is a constant.

Moreover, the solution $w(t,x)$ converges to the constant state $\bar{w}$ uniformly in $x \in \mathbb{R}$ as $t \to \infty$.

This result is proved by an energy method which makes use of the following properties: The system (3.1) has an entropy function and is transformed into a symmetric system of hyperbolic-parabolic type which satisfies the stability condition. We refer the reader to [4],[5] for the details. See also [1],[2].

Next we study a decay rate of the difference $w(t,x) - \bar{w}$ for $t \to \infty$. The linearized system of (3.1) around the constant state $\bar{w}$ is

$$w_t^l + A(\bar{w})w_x^l = G(\bar{w})w_{xx}^l.$$  \hspace{1cm} (4.2)

Denote by $e^{tR}$ the semigroup of (4.2). We have

$$|a_x^l(e^{tR}f)|_2 \leq Ce^{-ct}|a_x^Sf|_2 + C(1+t)^{-1/2} |a_x^k|_1,$$  \hspace{1cm} (4.3)

where $0 \leq k \leq l$, $C$ and $c$ are positive constants, and $f = f(x)$ is a function such that the norms on the right hand side of (4.3) are finite (see [9]). Making use of (4.3), we obtain the following

**Theorem 4.2.** ([4]) Assume (1.2)$_1$, and (1.4)$_1$ or (1.4)$_2$. If $w_0(x) - \bar{w}$ is small in $H^S \cap L^1$, $s \geq 3$, then the solution $w(t,x)$ of (3.1) constructed in Theorem 4.1 satisfies
(4.4) \[ |a_x^*(w(t) - \overline{w})|_2 \leq CN_s (1+t)^{-1/2 + \beta}/2, \quad t \in [0, \infty), \]

where \( \beta \geq 0, 3\beta \leq s - 2, C \) is a constant and \( N_s = \|w_0 - \overline{w}\|_s + \|w_0 - \overline{w}\|_1. \)

5. Approximation by uniformly parabolic system

We first note that the matrix \( A(w) \) has the spectral resolution \( A(w) = \sum \lambda_j(w) P_j(w), \) where \( P_j(w) = r_j(w) l_j(w) \) and the summation is taken over all \( j = 1, 2, 3. \) We then define the matrix \( D(w) \) by

(5.1) \[ D(w) = \sum_{j=1}^{3} \kappa_j(w) P_j(w), \]

where \( \kappa_j(w) = \langle l_j(w), G(w) r_j(w) \rangle \) with \( G(w) \) given by (3.3). By straightforward calculations we have

(5.2) \[
\kappa_j(w) = (-\mu p_v + \kappa p_s^2/\theta)/(-2vp_v), \quad j = 1, 3, \\
\kappa_2(w) = (-\kappa p_v)/(-vp_v). 
\]

Note that these coefficients are all positive by (1.2)_1, and (1.4)_1 or (1.4)_2.

Now we consider the system

(5.3) \[ z_t + f(z)_x = D(w)z_{xx}, \]

with the initial condition \( z(0,x) = w_0(x). \) The system (5.3) is semilinear and uniformly parabolic, and hence has a unique global solution \( z(t,x), \) provided that \( w_0(x) - \overline{w} \) is small in \( H^s, s \geq 1. \) The linearized system of (5.3) around the constant state \( \overline{w} \) is

(5.4) \[ z_t^i + A(\overline{w}) z_{xx}^i = D(\overline{w})z_{xx}^i. \]

Denote by \( e^{tS} \) the semigroup of (5.4). We easily obtain the estimate
(5.5) \[ |\partial_x^k(e^{tS}f)|_2 \leq Ce^{-ct}|\partial_x^k f|_2 + C(1+t)^{-\frac{1}{2} + \frac{\kappa}{2} - k/2}|\partial_x^k f|_1, \]

where \(0 \leq k \leq \kappa, C \) and \(c \) are positive constants. Making use of (5.5), we know that if \(w_0(x) - \bar{w} \) is small in \(H^s \cap L^1, s \geq 1\), then the solution \(z(t,x)\) of (5.3) satisfies

(5.6) \[ |\partial_x^\kappa(z(t) - \bar{w})|_2 \leq CN_s(1+t)^{-\frac{1}{2} + \frac{\kappa}{2}}/2, \quad t \in [0,\infty), \]

where \(0 \leq \kappa \leq s\) and \(C\) is a constant.

Furthermore, we can show that for \(t \to \infty\), the solution \(w(t,x)\) of (3.1) is well approximated by the solution \(z(t,x)\) of (5.3). More precisely, we have the following

Theorem 5.1. (\([5]\)) Assume (1.2)_1 and (1.4)_1 or (1.4)_2. If \(w_0(x) - \bar{w} \) is small in \(H^s \cap L^1, s \geq 5\), then we have

(5.7) \[ |\partial_x^\kappa(w(t) - z(t))|_2 \leq CN_s(1+t)^{-\frac{3}{2} + \frac{\kappa}{2}} + \alpha, \quad t \in [0,\infty), \]

where \(\kappa \geq 0, 3\kappa \leq s - 5, C\) is a constant, and \(\alpha > 0\) is a small fixed constant.

This approximation result is based on the following better decay estimate for the difference between the semigroups \(e^{tR}\) and \(e^{tS}\).

(5.8) \[ |\partial_x^\kappa(e^{tR} - e^{tS})f|_2 \leq Ce^{-ct}|\partial_x^\kappa f|_2 + C(1+t)^{-\frac{3}{2} + \frac{\kappa}{2} - k/2}|\partial_x^\kappa f|_1, \]

where \(0 \leq k \leq \kappa, C \) and \(c \) are positive constants.

6. Diffusion waves

Following Liu [8], we shall construct the diffusion waves. First we determine the coefficients \(\delta_j(w), j=1,2,3\), by
\[
(6.1) \quad \int_{-\infty}^{\infty} (w_0(x) - \bar{w}) \, dx = \frac{3}{4} \delta_j(w) r_j(\bar{w}) .
\]

Put \( \delta(w) = (\delta_1(w), \delta_2(w), \delta_3(w)) \) and assume that \( \delta(w) \neq 0 \). Next we introduce the Riemann invariant. A function of \( w \) which are constant in the direction of \( r_j(w) \) is called \( j \)-Riemann invariant. For each \( j \), we have two independent \( j \)-Riemann invariants given below.

\[
(6.2) \quad s \text{ and } u + \int_{s_0}^{s} \lambda_j(v,s)dv \quad \text{ for } j=1,3, \\
p \text{ and } u \quad \text{ for } j=2.
\]

Here the eigenvalue \( \lambda_j(w) \) is regarded as a function of \( v \) and \( s \).

Now, for the genuinely nonlinear field \( \lambda_j(w) \), \( j=1 \) or \( j=3 \), we define \( j \)-diffusion wave \( W_j(t,x) \), \( W_j = (v_j, u_j, E_j)^T \) with \( E_j = e_j + u_j^2/2 \), by

\[
(6.3) \quad \\
\lambda_j(v_j(t,x), \bar{s}) - \lambda_j(\bar{v}, \bar{s}) = Y(t+1, x - \lambda_j(\bar{v}, \bar{s})(t+1); \kappa_j(w), \delta_j(w)) .
\]

Here \( \bar{s} = \bar{s}(\bar{v}, \bar{e}), \kappa_j(w) \) and \( \delta_j(w) \) are given by (5.2) and (6.1), respectively, and

\[
(6.4) \quad Y(t,x; \kappa, \delta) = \sqrt{\kappa} t^{-1/2} \frac{(e^{\delta/2\kappa} - 1)e^{-\xi^2}}{\sqrt{\pi} + (e^{\delta/2\kappa} - 1)\int_{\xi}^{\infty} \frac{e^{-\eta^2}}{\eta} d\eta} , \quad \xi = x/\sqrt{4\kappa t} .
\]

The function \( Y \) in (6.4) is the self-similar solution of the Burgers equation \( y_t + yy_x = \kappa y_{xx} \) and satisfies

\[
(6.5) \quad \int_{-\infty}^{\infty} Y(t,x; \kappa, \delta) \, dx = \delta , \quad t \in (0, \infty).
\]

See [3],[8]. Note that \( W_j(t,x) \) lies on the curve \( R_j(w) \) defined by \( dw/dt = r_j(w) \) and \( w = \bar{w} \) at \( t = 0 \). Since \( \lambda_j(w) \) is monotone along \( R_j(w) \) by (3.8), the relations in (6.3) uniquely determine \( v_j(t,x) \) and...
and therefore all other thermodynamic variables.

For the linearly degenerate field $\lambda_2(w) = 0$, we define 2-diffusion wave $W_2(t,x)$, $W_2 = (v_2, u_2, E_2)^T$ with $E_2 = e_2 + u_2^2/2$, by

$$
\begin{align*}
& p_2(t,x) = \bar{p}, \quad u_2(t,x) = \bar{u}, \\
& s_2(t,x) - \bar{s} = Y(t+1, x; \kappa_2(w), \delta_2(w)),
\end{align*}
$$

(6.3)

where $\bar{p} = \hat{p}(\bar{v}, \bar{e})$, etc., and

$$
Y(t,x; \kappa, \delta) = \delta(4\pi kt)^{-1/2} e^{-\xi^2}, \quad \xi = x/\sqrt{4\pi t}.
$$

(6.4)

This $Y$ is the self-similar solution of the linear heat equation $\partial_t y = \kappa \partial_{xx} y$ and satisfies (6.5). Notice that $W_2(t,x)$ lies on the curve $\mathcal{R}_2(w)$. The relations in (6.3) define $p_2(t,x)$ and $s_2(t,x)$ and therefore all other thermodynamic variables.

Finally, we define $W(t,x)$, the superposition of the diffusion waves, by

$$
W(t,x) - \bar{w} = \sum_{j=1}^3 (W_j(t,x) - \bar{w}).
$$

(6.6)

By straightforward calculations, using (6.3)$_{1,2}$ and (6.4)$_{1,2}$, we have

$$
W_t + f(W)_x = D(w)W_{xx} + r_x(t,x) - q(t,x),
$$

(6.7)

where $r(t,x)$ and $q(t,x)$ are known functions such that

$$
\begin{align*}
|\partial_x^\varphi r(t,x)| &\leq C|\delta(w)| e^{-c(t+|x|)}, \\
|\partial_x^\varphi q(t,x)| &\leq C|\delta(w)|^2 (1+t)^{-1} e^{-c\xi_j^2},
\end{align*}
$$

(6.8)

where $\xi_j = (x - \lambda_j(w)(t+1))/\sqrt{t+1}$, and $C$ and $c$ are positive constants. For the details, see [5],[8].
7. Large-time behavior

We shall show that $W(t,x)$ defined by (6.6) is an asymptotic solution for $t \to \infty$ of the uniformly parabolic system (5.3). To this end we construct the linear hyperbolic wave $\zeta(t,x)$ as the solution of

$$
(7.1) \quad \zeta_t + A(\overline{w})\zeta_x = q(t,x),
$$

with the following condition: $\zeta(t,x) \to 0$ uniformly in $x \in \mathbb{R}$ as $t \to \infty$. Here $q(t,x)$ is the function in (6.7). By the characteristic method, we have a unique smooth solution $\zeta(t,x)$ satisfying

$$
(7.2) \quad |\partial_x^\ell \zeta(t,x)| \leq C |\delta(\overline{w})|^2 \sum_{j=1}^{3} \left\{ \frac{(t+1+|x-\lambda_j(\overline{w})(t+1)|)^{-(2+\ell)/2} + (t+1+|x-\lambda_j(\overline{w})(t+1)|)^{-(3+\ell)/2}}{\sum_{j=1}^{3} \delta_j(\overline{w}) r_j(\overline{w})} \right\},
$$

where $\ell \geq 0$ and $C$ is a constant. Also, it is shown that for $t \in [0,\infty)$,

$$
(7.3) \quad \int_{-\infty}^{\infty} (W(t,x) - \overline{w} + \zeta(t,x)) dx = \sum_{j=1}^{3} \delta_j(\overline{w}) r_j(\overline{w}).
$$

From (6.1) and (7.3) we know that $w_0(x) - W(t,x) - \zeta(t,x)$ has zero integral for each $t \in [0,\infty)$. By virtue of this property, we have a desired conclusion.

**Theorem 7.1.** ([5]) Assume (1.2)$_{1,2}$ and (1.4)$_{1}$ or (1.4)$_{2}$. Suppose that $w_0(x) - \overline{w}$ is small in $H^s \cap L^1_B$, $s > 1$ and $B \geq 1/2$. Let $z(t,x)$ be the solution of (5.3) and let $W(t,x)$ be the superposition of the diffusion waves defined by (6.6). Then we have

$$
(7.4) \quad |\partial_x^\ell (z(t) - W(t))|_2 \leq CM_s (1+t)^{-(1+\ell)/2 + \alpha}, \quad t \in [0,\infty),
$$

where $0 \leq \ell \leq s$, $C$ is a constant, $M_s = \|w_0 - \overline{w}\|_s + |w_0 - \overline{w}|_{1,1/2}$, and $\alpha > 0$ is a small fixed constant.
In the proof of this theorem, the following estimate for the semigroup $e^{tS}$ plays an essential role: If $f \in L^1_{\beta}$, $\beta \in [0,1]$, and $f(x)$ has zero integral, then we have

$$
|a_x^\beta(e^{tS}f)|_p \leq Ct^{-(1-1/p+\beta+\varepsilon)/2}|f|_{1,\beta}, \quad t \in (0,\infty),
$$

where $\varepsilon \geq 0$, $p \in [1,\infty]$ and $C$ is a constant.

We remark that (7.4) is a meaningful asymptotic relation for $t \to \infty$, because for large $t$, the $L^2$-norm of $a_x^\beta(W(t,x) - \overline{W})$ is bounded from below by $c|\delta(W)|^{-(1/2+\varepsilon)/2}$ with a positive constant $c$.

A combination of Theorems 5.1 and 7.1 gives the main result of this paper.

**Theorem 7.2.** ([5]) We assume the conditions of Theorem 7.1 with $s \geq 1$ replaced by $s \geq 5$. Then the solution $w(t,x)$ of (3.1) satisfies

$$
|a_x^\beta(w(t) - W(t))|_2 \leq CM_\delta(1+t)^{-(1+\varepsilon)/2+\alpha}, \quad t \in [0,\infty),
$$

where $\varepsilon \geq 0$, $3\varepsilon \leq s-5$ and $C$ is a constant; $M_\delta$ and $\alpha$ are the same as those in Theorem 7.1.

This theorem means that the superposition of the diffusion waves defined by (6.6) is also an asymptotic solution for $t \to \infty$ of the system (3.1).

References


