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Vorticity and viscosity

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This is a resume of my joint work with T. Miyakawa and H. Osada [36].

We consider the Navier-Stokes system

(1) $$\frac{\partial u}{\partial t} - \nu\Delta u + (u\cdot\nabla)u + \nabla p = 0, \quad \nabla\cdot u = 0$$

on the whole plane R^2, where u and p represents unknown velocity and pressure, respectively and $\nu > 0$ is the kinematic viscosity. Since the space dimension is two, the vorticity $v = \nabla \times u = \partial u^2/\partial x_1 - \partial u^1/\partial x_2$ is scalar. Moreover, v solves

(2a) $$\frac{\partial v}{\partial t} - \nu\Delta v + (u\cdot\nabla)v = 0$$

(2b) $$u(x,t) = \iint_{R^2} \nabla^{\perp}E(x-y)v(x,t)dx$$

where $\nabla^{\perp} = (-\partial/\partial x_2, \partial/\partial x_1)$ and $E(x) = (2\pi)^{-1}\log|x|$. These equations are formally obtained by taking $\nabla\times$ of (1) and using the condition $\nabla\cdot u = 0$. As is well known the vorticity equation (2a)(2b) is formally equivalent to the Navier-Stokes system provided that u is assumed to decay to zero at space infinity.

We consider the initial value preblem for (1) or (2a)(2b) assuming only that initial vorticity $v(x,0)$ is a finite Radon measure. A typical example is N-point sources of votex, i.e.,

$$(3) \qquad v(x,0) = \sum_{j=1}^{N}\alpha_j\delta(x-z_j).$$

Here z_j is a point on which j-th point source is located and α_j is a real number describing the strength of the source; δ is a Dirac measure supported at zero. One naive question is whether such point sources of vortex are smoothed out because of viscosity. In other words do solutions for (1) or (2a),(2b) exist globally-in-time and smooth for $t > 0$ even if $v(x,0)$ is a finite measure? When initial vorticity consists only one point source carried at zero (i.e. $N = 1$, $z_1 = 0$), we know an exact solution of (2a),(2b)

$$v(x,t) = \frac{\alpha_1}{4\pi\nu t}\exp\left(\frac{-|x|^2}{4\nu t}\right)$$

which is a constant multiple of the fundamental solution of the heat equation. For a general initial data we claim that a smooth solution exists globally in time. As anticipated, the viscosity smoothes singular vorticities.

Theorem ([36]). *Suppose that* $v(x,0)$ *is a finite Radon measure on* R^2. *Then there is a global solution* $v(x,t)$, $u(x,t)$ *to* (2a),(2b) *or* (1) *such that* v *and* u *are smooth for* $t > 0$ *and* $v(x,t)$ *converges to* $v(x,0)$ *under the weak topology of measures as* t *tends to zero*.

In [3] Benfatto, Esposito and Pulvirenti prove similar results under more stringent assumptions. They assume $v(x,0)$ is expressed by (3) and $|\alpha_j|$ is small compared with ν. Our results need no assumptions on particular forms or smallness of initial vorticity.

The main mathematical difficulty is that the initial energy on D

$$\iint_D |u(x,0)|^2 \, dx$$

is not necessarily finite even if D is a bounded domain. If the initial energy is finite, it is classical that there is a global classical solutions to (1) (cf. [16,17,30]).

To construct such a solution we approximate initial vorticity by smooth functions and solve (2a),(2b) with approximate initial data. It is not difficult to construct a global solution for smooth data. We expect that solutions with approximate initial data converge to a true solution for the original problem. To carry out this process we need a priori estimates.

Lemma ([36]). *Suppose that* $v(x,0)$ *is smooth and*

$\iint_{R^2}|v(x,0)|dx \leq m$. *Let* $\Gamma_u(x,t;y,s)$ *is a fundamental solution to* (2a), *regarding* u *is a known function*. *Then*,

$$c(t-s)^{-1}\exp\left(\frac{-|x-y|^2}{c(t-s)}\right) \leq \Gamma_u(x,t;y,s) \leq C(t-s)^{-1}\exp\left(\frac{-|x-y|^2}{C(t-s)}\right)$$

with c *and* C > 0 *depending only on* m.

Estimates of fundamental solutions independent of the regularity of coefficients are obtained by Aronson [1] for linear parabolic equations of divergence form (see also [2]). Osada [25] extends the estimate for non-divengence form which includes (2a) as a typical example. The above a priori estimates enable us to carry out our original idea.

For uniqueness of the solution we do not know much. We show the uniqueness when v(x,0) is small. In particular, if v(x,0) is absolutely continuous with respect to Lebesgue measure, we can assert the uniqueness.

Our references include those of the paper [36] for the reader's convenience.

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