

KNOTTED ATTRACTING PERIODIC ORBITS
IN SILNIKOV BIFURCATIONS

東京理科大 戸川 美郎

(Yoshio Togawa)

1. INTRODUCTION

Let X be a C^1 -vector field on \mathbb{R}^3 which satisfies the following conditions;

- (i) $X(0)=0$,
- (ii) $DX(0)$ has eigenvalues $\lambda, -\mu \pm i\omega$ with $0 < \mu < \lambda, \omega \neq 0$,
- (iii) $W^s(0)$ and $W^u(0)$ intersect at a homoclinic orbit Γ .

We call such a vector field a Silnikov system. A Silnikov system has infinitely many periodic orbits arbitrarily close to the homoclinic orbit Γ ³⁾. Moreover the knot types of these periodic orbits and their linking numbers around Γ determines μ/λ . Therefore μ/λ is a modulus, that is, if X and X' are two Silnikov systems with $\mu/\lambda \neq \mu'/\lambda'$, then they are not topologically equivalent.⁴⁾⁵⁾ If a Silnikov system X has eigenvalues $\lambda, -\mu \pm i\omega$ with $0 < \mu < \lambda < 2\mu$ and $\omega \neq 0$, then one can perturb X to X' so that X' has an attracting periodic orbit arbitrarily close to Γ . Attracting periodic orbits in \mathbb{R}^3 are the simplest attractors which can have non trivial embedding types. So

it would be important to ask whether μ/λ gives any restrictions on the possible embedding types of attracting periodic orbits of perturbed systems. In this paper, we prove that:

THEOREM *Let X and X' be Silnikov systems with homoclinic orbits Γ and Γ' and eigenvalues $\lambda, -\mu \pm i\omega$ and $\lambda', -\mu' \pm i\omega'$ respectively. Suppose that $0 < \mu < \lambda < 2\mu$ and $\mu/\lambda < \mu'/\lambda'$.*

Let X_s be a 1-parameter family of vector fields with $X_0 = X$. Let X_s satisfies the transversality condition defined later. Then there exists a $\delta > 0$ such that for any $\varepsilon > 0$ there exists a positive $s < \varepsilon$ such that

- (i) X_s has an attracting periodic orbit γ in the ε -neighbourhood of Γ , and
- (ii) any δ -perturbation X'' of X' has no periodic orbit in the δ -neighbourhood of Γ' whose embedding type into the δ -neighbourhood is the same of that of γ .

To prove the theorem we use the attracting periodic orbits which trips three times along Γ .

2. ANALYTIC STUDY OF ATTRACTING PERIODIC ORBITS

Let S be the unit cylinder;

$$S = \{(x_1, x_2, x_3) \mid x_1^2 + x_2^2 \leq 1, |x_3| \leq 1\}.$$

We use (r, θ, z) -coordinate in the unit cylinder S ;

$$x_1 = r \cos \theta, \quad x_2 = r \sin \theta, \quad x_3 = z.$$

Let SS be the side surface of S ;

$$SS = \{(x_1, x_2, x_3) \mid x_1^2 + x_2^2 = 1, |x_3| \leq 1\}.$$

Let X_s be a 1-parameter family of C^1 -vector fields on \mathbb{R}^3 with X_0 Silnikov. To avoid the technical difficulties, we suppose the following;

(*) there exists a 1-parameter family of diffeomorphisms which linearizes X_s in the unit cylinder.

Take a sufficiently small r_1 so that any X_s which starts at a point p in

$$\Sigma^2 = \{(r, \theta, 1) \mid r \leq r_1\}$$

crosses SS at a point $f_s(p)$, and the mapping

$$f_s: p \in \Sigma^2 \rightarrow f_s(p) \in SS$$

is a diffeomorphism onto its image. For each s let $\text{height}(s)$ denote the z -coordinate of the point at which $W^u(0)$ crosses SS . We suppose the transversality condition;

$$\frac{d}{dt} \text{height}(s) \Big|_{s=0} \neq 0.$$

Let L_s be the map $SS \rightarrow \{(r, \theta, 1) \mid r \leq 1\}$ induced by the linear vector field $X_s|_S$. Let $AP(s, \varepsilon)$ be the set of attracting periodic orbit γ of X_s which satisfies

- (i) γ is in the ε -neighbourhood of Γ ,
- (ii) γ crosses SS at three points, say $P_1(\gamma)$, $P_2(\gamma)$ and $P_3(\gamma)$ with z -coordinates $m_1(\gamma)$, $m_2(\gamma)$ and $m_3(\gamma)$,
- (iii) $f_s \circ L_s(p_i) = P_{i+1} \pmod{3}$ and $m_1(\gamma) < m_2(\gamma) < m_3(\gamma)$.

Note that $AP(s, \varepsilon)$ may be empty. Let

$$\eta(s, \varepsilon) = \inf \frac{\log m_3(\gamma)}{\log m_2(\gamma)}$$

where infimum is taken over all γ 's in $AP(s', \varepsilon)$ with $0 < s' \leq s$.

PROPOSITION 1. For any $\varepsilon_0 > 0$ there exists a $\delta > 0$ such that

$$\eta(s, \delta) > \mu/\lambda - \varepsilon_0$$

for all $|s| < \delta$.

PROPOSITION 2. $\eta(s_0, \varepsilon_0) \leq \mu/\lambda$ for sufficiently small $s_0 > 0$ and ε_0 .

2. PROOF OF THE PROPOSITIONS.

Proof of proposition 1. Let $c(s)=\mu/\lambda$. If $\delta_0 > 0$ is sufficiently small, then there exists a constant K such that

$$\text{hight}(s) - K(m_1(\gamma))^{c(s)} \leq m_2(\gamma),$$

$$m_3(\gamma) \leq \text{hight}(s) + K(m_2(\gamma))^{c(s)}$$

for all γ in $AP(s, \delta_0)$ with $0 < s \leq \delta_0$. Since $m_1(\gamma) < m_2(\gamma)$, we get

$$m_3(\gamma) \leq (1+2K)(m_2(\gamma))^{c(s)}.$$

Taking the logarizm, we get

$$\frac{\log m_3(\gamma)}{\log m_2(\gamma)} \geq c(s) + \frac{1+2K}{\log m_2(\gamma)}.$$

Taking a smaller δ , we get the required inequality.

Proof of proposition 2. The proposition 2 would be proved by a standard but very messy arguments.

3. TOPOLOGICAL INVARIANT

The braid of the periodic orbit in $AP(s, \varepsilon)$ is indicated in the fig 1, where β is an unknown but fixed braid. The numbers N_1 and N_2 goes to infinity as ε goes to zero. The ratio of N_1 and N_2 is estimated by the propositions 1 and 2, so the remaining task is to relate the numbers N_1 and N_2 to some topological invariants. Hence the following proposition completes the proof of the theorem.

Let $\Delta_\gamma(t)$ be the Alexander polynomial of the knot γ^2 . Let $n(\gamma)$ be the number of the non zero terms of the polynomial $\Delta_\gamma(t)$.

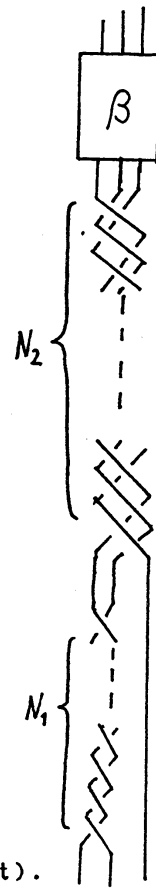


Fig 1

PROPOSITION 3. *There exists a constant C such that*

$$| m(t) - N_2 | < C$$

$$| \text{degree of } \Delta_\gamma(t) - (6N_1 + N_2) | < C,$$

for all N_1 and N_2 .

Proof. Let $\begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix}$ be the Peron Mugnus representaion of the braid β^1 . Then the Alexander polynomial $\Delta_\gamma(t)$ is

$$\begin{aligned} & (-1)^m t^{-(6n+m)} (a(t)d(t) - b(t)c(t)) + t^{-3n} b(t) \sum_{k=1}^m (-t)^{-k} \\ & - (-1)^m t^{-(3n+m)} a(t) - t^{-3n} d(t) + 1, \end{aligned}$$

where $n=N_1$ and $m=N_2$. Now the proposition is clear, since neither $b(t)$ nor $a(t)d(t) - b(t)c(t)$ is a zero polynomial. (Note that neither $a(1)d(1) - b(1)c(1)$ nor $b(-1)$ is zero).

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