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Normal Forms of Piecewise Linear Vector Fields

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ABSTRACT

This paper provides normal forms of continuous piecewise linear vector fields in R^n under linear conjugacy. It is proved that linearly conjugate classes are uniquely determined by eigenvalues of linear vector field in each region if the piecewise linear vector fields are proper.

0. Introduction.

Since a strange attractor has been studied by E.Lorenz[1] in 1963, it has been well known that nonlinear autonomous ordinary differential equations on R^n ($n \geqq 3$) have generally many kinds of strange attractors. In [3] and [4], for example, a lot of equations with strange attractors are reported from physics, chemistry, ecology, electrical engineering and other fields, including the four prototype equations proposed by O.E.Rossler[2]. These equations are non-integrable systems including smooth nonlinear terms, as x^2,xy or x^2y, and they are mainly studied by the method of numerical or experimental.

In 1981, O.E.Rossler has studied in [5] an equation with nonlinear term described by a piecewise linear function, and shown that it has a strange attractor as same as in the case of smooth nonlinear term. The piecewise linear system is considered as a system glued two linear systems each other, thus the solution is explicitly written by analytical form even though partially. B.Uehleke and O.E.Rossler have derived in [6] and [7] an analytical expresion of Poincaré half-return map for a feedback system with a piecewise linear feedback function, and have asserted that piecewise linear systems are typical systems to investigate strange attractors analytically. C.Sparrow has studied in [8] bifurcation problem of strange attractors of a piecewise linear system by one-dimensional approximation of Poincaré map.

L.O.Chua, T.Matsumoto and the auther study in [9] a piecewise linear system derived from an electric circuit, and prove rigorously that the system has a Shilnikov homoclinic orbit at some parameter values. This means that the system includes countable many periodic orbits and uncountable many non-periodic orbits by Shilnikov's theorem, thus, in this sense, the system is chaotic. Differently from usual method which has studied an individual piecewise linear system, they consider a wide class of 3-dimensional piecewise linear vector fields (the class of continuous proper 3-region systems with point symmetry; see section 5 for definition) which includes the objective system, and prove that for this class,

(1) linearly conjugate class is uniquely determined by eigenvalues in each linear region;

(2) Poincaré half-return map is explicitly expressed by the parametric equation with the half-return time as a parameter.

The existance of Shilnikov's homoclinic orbit is proved from the above statements, and this method is valid to various piecewise linear systems. So it is important for the future study

(i) to extend this result to more general class of piecewise linear vector fields (of n-dimensional, with many regions),

(ii) to investigate strange attractors in piecewise linear system in a viewpoint of geometric structure, statistic property or bifurcations, and

(iii) to clarify the relation between the strange attractors in piecewise linear systems and the one in smooth nonlinear systems.

In this paper we will give an normal form for linearly conjugate class of n-dimensional 2-region systems, which are the most fundamental piecewise linear vector fields. More general piecewise linear systems are considered as combinations of 2-region systems. Indeed, as example, we will derive a normal form of 3-dimensional proper system having 3 regions and point symmetry, and having 4 regions and axial symmetry.

1. Definition of piecewise linear vector fields and the continuous condition.

Suppose n-1 dimensional hyperplanes $U_1, U_2, \dots, U_k$ in R^n, which divide R^n into the regions $R_1, \dots, R_l$. A mapping $f: R^n \longrightarrow R^n$ is a piecewise linear mapping with linear regions $R_1, \dots, R_l$ if f is differentiable at all points which does not belong the set $B = U_1 \dots U_k$, and if the derivative Df is constant in the interior of R_j for each j=1,...,l, i.e.

$$Df(x) = M_j \quad \text{if } x \in \text{int } R_j \ (j=1,\ldots,l)$$

where M_j is n×n matrix. A piecewise linear map f may be discontinuous at points belonging to B. If f is continuous at each point on B, thus at all points in R^n, it is called a continuous piecewise linear mapping.

If $f:R^n \longrightarrow R^n$ is continuous piecewise linear, a vector field X_f on R^n defined by an ordinary differential equation

$$\frac{dx}{dt} = f(x) \qquad (x \in R^n) \tag{1.1}$$

i.e. $X_f:R^n \longrightarrow R^n$; $X_f(x) = f(x) \quad (x \in R^n)$ (1.2)

is called a continuous piecewise linear vector field, or simply a PL-system. Two PL-systems X_f and X_g are linearly conjugate if there is a nonsingular affine transformation $h:R^n \longrightarrow R^n$; $h(x) = Hx + p$ ($H \in GL(n,R)$, $p \in R^n$) which satisfies

$$HX_f(x) = X_g(h(x)) \qquad (x \in R^n) \tag{1.3}$$

where GL(n,R) denotes the set of all nonsingular n×n real matrices.

Suppose a hyperplane U defined by

$$U = \{ x \in R^n : \langle\alpha,x\rangle = \beta \}$$

where $\alpha \in R^n$, $\beta \in R$, and $\langle\ ,\ \rangle$ denotes the usual inner product, and suppose two regions;

$$\begin{aligned} R^+ &= \{ x \in R^n: \langle\alpha,x\rangle - \beta > 0 \} \\ R^- &= \{ x \in R^n: \langle\alpha,x\rangle - \beta < 0 \} \end{aligned} \tag{1.4b}$$

We define a 2-region PL-map $f:R^+ \cup R^- \longrightarrow R^n$ by

$$f(x) = f(x;A,B,q_A,q_B,\alpha,\beta) = \begin{cases} Ax + q_A & (x \in R^+) \\ Bx + q_B & (x \in R^-) \end{cases} \tag{1.5}$$

where $A, B \in M(n,R)$ (the set of all n×n real matrices) and $q_A, q_B \in R^n$.

Theorem 1 (Continuous Condition) The 2-region PL-map $f(x) = f(x; A, B, q_A, q_B, \alpha, \beta)$ defined above can be extended to R^n as a continuous map, if and only if there exists an $m \in R^n$ with $\langle m,\alpha\rangle = 1$ which satisfies

$$B - A = (B - A)\, m\, {}^T\alpha \tag{1.6a}$$

$$q_B - q_A = -\beta(B - A)\, m \tag{1.6b}$$

where $m, \alpha \in R^n$ are column vectors and ${}^T\alpha$ denotes the transposition of α.

Corollary 1.1. Let $f(x) = f(x; A, B, q_A, q_B, \alpha, \beta)$ defined by (1.5) be continuous , and assume that $q_B \neq 0$ and $\beta \neq 0$. Then f is expressed by the form

$$f(x) = Bx - \frac{1}{2\beta}\, q\{\langle\alpha,x\rangle - \beta + |\langle\alpha,x\rangle - \beta|\} \qquad (x \in R^n)$$

where $q = q_A \in R^n$, and $|\cdot|$ denotes the absolute value.

Fróm now on, we will consider only continuous piesewise linear vector fields, unless otherwise stated.

2. Normal Forms of Linear Vector Fields with Section.

Suppose a linear vector field

$$X_A : R^n \longrightarrow R^n \;;\; X_A(x) = Ax \qquad (x \in R^n)$$

where $A \in M(n,R)$, and an n-1 dimensional hyperplane (called a section)

$$U = U(\alpha,\beta) = \{x \in R^n : \langle\alpha,x\rangle = \beta\}$$

where $\alpha \in R^n$ and $\beta \in R$. The pair (X_A, U) is called a linear vector field with a section. Two linear vector fields with a section, (X_A, U_A) and (X_B, U_B), are linearly conjugate if there exists $G \in GL(n,R)$ such that for any $x \in R^n$

$$GX_A(x) = X_B(Gx) \quad \text{and} \quad G(U_A) = U_B .$$

A section $U = U(\alpha,\beta)$ is regular if U does not pass through the origin, i.e. $\beta \neq 0$. To state normal forms for linearly conjugate class of linear vector fields with a section, we prepare notation of real Jordan normal forms for matrices.

Definition 2.1 (Real Jordan Matrices)

Real Jordan blocks are denoted by

$$J(\lambda;k) = \begin{bmatrix} \lambda 1 & & 0 \\ & \ddots & 1 \\ 0 & & \lambda \end{bmatrix} \in M(k,R),$$

$$J(a,b;k) = \begin{bmatrix} A I & & 0 \\ & \ddots & I \\ 0 & & A \end{bmatrix} \in M(2k,R)$$

where $A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, and $\lambda, a, b \in R$ $(b \neq 0)$, and $k \geqq 1$ is an integer. Real Jordan matrix is denoted as a direct sum of real Jordan blocks by

$$J = J_1 + J_2 + \dots = \begin{bmatrix} J_1 & & 0 \\ & J_2 & \\ 0 & & \ddots \end{bmatrix}$$

where $J_i = J(\lambda_i;k_i)$, $J(a_i,b_i;k_i)$ $(i=1,2,\dots)$. Any linear vector field X_A is transformed by suitable linear transformation $G:R^n \longrightarrow R^n$ $(G \in GL(n,R))$ into a vector field X_J defined by a real Jordan matrix J;

$$GX_AG^{-1} = X_{GAG^{-1}} = X_J .$$

Therefore, in order to consider normal form of a linear vector field with a section, (X_A,U), we may assume that A is a real Jordan matrix without loss of generality. We will identify a linear transformation and an element of $GL(n,R)$ with respect to a natural basis of R^n, if there is no confusion.

Theorem 2 (Normal Forms of Linear Vector Fields with a Section)

Let X_J be a linear vector field defined by a real Jordan matrix $J \in M(n,R)$;

$$J = \sum_{i=1}^{r} J(\lambda_i;k_i) + \sum_{j=1}^{s} J(a_j,b_j;l_j) \qquad (2.1)$$

and U a regular section defined by

$$U = \{ x \in R^n : \langle\alpha,x\rangle = \beta \} \qquad (\beta \neq 0).$$

For the linear vector field with a section, (X_J,U), there exists a linear transformation $G \in GL(n,R)$ which satisfies the following (1) - (5);

(1) $JG = G J$

(2) $G(U) = \{ x \in R^n: \langle\alpha',x\rangle = 1 \}$

(3) $\alpha' \in R^n$ in (2) has following form;

$$\alpha' = {}^T(u_1,\ldots,u_r,v_1,\ldots,v_s) \neq 0 \qquad (2.3)$$

where $u_i \in R^{1\times k_i}$, $v_j \in R^{1\times 2l_j}$ are column vectors.

(4) each $u_i \in R^{1\times k_i}$ in (3) is one of the following;

$$u_i = (1,0,\ldots,0),\ (0,1,0,\ldots,0),\ldots.,$$
$$(0,\ldots,0,1),\ (0,\ldots.,0) \qquad (2.4)$$

(5) each $v_j \in R^{1\times 2l_j}$ in (3) is one of the following;

$$v_j = (\underline{1},\underline{0},\ldots,\underline{0}),\ (\underline{0},\underline{1},\underline{0},\ldots,\underline{0}),\ldots.,$$
$$(\underline{0},\ldots,\underline{0},\underline{1}),\ (\underline{0},\ldots,\underline{0}) \qquad (2.5)$$

where $\underline{0} = (0,0)$ and $\underline{1} = (1,0)$.

Moreover, for a linearly conjugate class of (X_j,U), the representation of α' which satisfies (3) - (5) is uniqly determind except change of u_i or v_j corresponding to change of real Jordan blocks.

Definition 2.2. A vector $\alpha' \in R^n$ having the form (2.3) in

Theorem 2 is called a canonical normal vector corresponding to a real Jordan matrix (2.1), and a section

$$U' = \{ x \in R^n : <\alpha', x> = 1 \}$$

defined by α' is called a canonical section.

Example 2.3. (Normal Forms of 3-Dimensional Linear Vector Fields with Section)

3×3 real Jordan matrices are classefied into the following 4 cases;

$$(1) \begin{bmatrix} \lambda & & \\ & \mu & \\ & & \nu \end{bmatrix} \quad (2) \begin{bmatrix} \lambda & 1 & \\ & \lambda & \\ & & \mu \end{bmatrix} \quad (3) \begin{bmatrix} \lambda & 1 & \\ & \lambda & 1 \\ & & \lambda \end{bmatrix} \quad (4) \begin{bmatrix} \lambda & & \\ & a & -b \\ & b & a \end{bmatrix}$$

In each case, canonical vectors α R^n are following;

Case of (1):

$$\alpha = {}^T(1,1,1),\ {}^T(0,1,1),\ {}^T(0,0,1).$$

Case of (2):

$$\alpha = {}^T(1,0,1),\ {}^T(0,1,1),\ {}^T(0,0,1),\ {}^T(1,0,0),\ {}^T(0,1,0).$$

Case of (3):

$$\alpha = {}^T(1,0,0),\ {}^T(0,1,0),\ {}^T(0,0,1).$$

Case of (4):

$$\alpha = {}^T(1,1,0),\ {}^T(0,1,0),\ {}^T(1,0,0).$$

3. Normal Forms of Regular 2-Region Piecewise Linear Vector Fields.

Recall the 2-region piecewise linear map f defined by (1.4) and (1.5);

$$f(x) = \begin{cases} Ax + q_A, & x \in R^+ \\ Bx + q_B, & x \in R^- \end{cases} \qquad (3.1)$$

For f, a piecewise linear map $\bar{f}$ defined by

$$\bar{f}(x) = \begin{cases} Bx + q_B, & x \in R^+ \\ Ax + q_A, & x \in R^- \end{cases} \qquad (3.2)$$

is called the <u>complement</u> of f. For a piecewise linear vector field X_f defined by f in (3.1), the piecewise linear vector field $X_{\bar{f}}$ defined by $\bar{f}$ in (3.2) is called the <u>complement</u> of X_f. A 2-region piecewise linear map $f(x) = f(x;A,B,q_A,q_B,\alpha,\beta)$ defined by (1.4) and (1.5) is regular if there exists x_0 R^n such that

(1) $\langle\alpha,x_0\rangle \neq \beta$ i.e. $x_0 \notin U$, and

(2) $f(x_0) = 0$ or $\bar{f}(x_0) = 0$.

A piecewise linear vector field X_f defined by f is <u>regular</u> if f is regular. For a 2-region piecewise linear map $f(x;A,B,q_A,q_B,\alpha,\beta)$, to consider $f(x;A,B,q_A,q_B,-\alpha,-\beta)$ is called to <u>change the signe of regions</u>.

Suppose X_f is regular. Then, taking a parallel translation : $x \longrightarrow x-x_0$, and if necessary, taking the complement and changing the signe of regions, X_f reduces to the following form;

$$X_f(x) = f(x) = \begin{cases} Ax + q_A & ,x \in R^+ \\ Bx & ,x \in R^- \end{cases} \qquad (3.3a)$$

$$R^{\pm} = \{ x \in R^n : \pm(\langle\alpha,x\rangle - \beta) > 0\}, \quad \beta > 0 \qquad (3.3b)$$

Moreover, by Theorem 2, we can suppose that B is a real Jordan matrix, α is a canonical normal vector and $\beta = 1$, without loss of generality;

$$X_f(x) = f(x) = \begin{cases} Ax + q_A & ,x \in R^+ \\ Jx & ,x \in R^- \end{cases} \qquad (3.4a)$$

$$R^{\pm} = \{ x \in R^n : \pm(\langle\alpha,x\rangle - 1) > 0\} \qquad (3.4b)$$

where α is a canonical normal vector corresponding to a real Jordan matrix J.

Theorem 3 (Normal Forms of Regular 2-Region Piecewise Linear Vector Fields) If necessary, taking the complement and changing the signe of regions, any continuous n-dimensional regular 2-resion piecewise linear vector field is linearly conjugate to the vector field with the following form;

$$X_f(x) = f(x) = \begin{cases} (J - q{}^T\alpha)x + q & , x \in R^+ \\ Jx & , x \in R^- \end{cases} \qquad (3.5a)$$

$$= Jx - \frac{1}{2} q \{ |\langle\alpha,x\rangle - 1| + \langle\alpha,x\rangle - 1 \} \qquad (3.5b)$$

$$R^{\pm} = \{ x \in R^n : \pm(\langle\alpha,x\rangle - 1) > 0\},\ q \in R^n \qquad (3.5c)$$

where J is a real Jordan matrix and $\alpha \in R^n$ is a canonical normal vector corresponding to J.

Remark. Suppose a continuous regular 2-region piecewise linear map $f(x;A,B,q_A,q_B,\alpha,\beta)$ of (3.1). If we choose a real Jordan matrix J_A of A as J in (3.5a), α amd q are uniquely determined corresponding to it, and we obtain one representation of f by (3.5). However if we choose a real Jordan matrix J_B of B as J in (3.5a), α amd q are uniquely determined corresponding to it, and we obtain another representation of f by (3.5). No representation of f by (3.5) exists except these two representations. In this sence, (3.5) gives us a normal form for continuous regular 2-region piecewise linear vector fields under linear conjugacy, which is determined by (J, α, q).

Example 3.1 (Normal Forms of 3-Dimensional Regular 2-Region Piecwise Linear Vector Fields) Define $(c_1,c_2,c_3) = -(1/2){}^T q \in R^3$.

(1) Case of $J = \begin{bmatrix} \lambda & & \\ & \mu & \\ & & \nu \end{bmatrix}$ and $\alpha = {}^T(1,1,1)$:

$$\dot{x} = \lambda x - c_1 \{| x + y + z - 1 | + (x + y + z - 1)\}$$
$$\dot{y} = \mu y - c_2 \{| x + y + z - 1 | + (x + y + z - 1)\}$$
$$\dot{z} = \nu z - c_3 \{| x + y + z - 1 | + (x + y + z - 1)\}$$

(2) Case of $J = \begin{bmatrix} \lambda & 1 & \\ & \lambda & \\ & & \mu \end{bmatrix}$ and $\alpha = {}^T(1,0,1)$:

$$\dot{x} = \lambda x + y - c_1 \{| x + z - 1 | + (x + z - 1)\}$$
$$\dot{y} = \lambda y \quad - c_2 \{| x + z - 1 | + (x + z - 1)\}$$
$$\dot{z} = \mu z \quad - c_3 \{| x + z - 1 | + (x + z - 1)\}$$

(3) Case of $J = \begin{bmatrix} \lambda & & \\ & a & -b \\ & b & a \end{bmatrix}$ and $\alpha = {}^T(1,1,0)$:

$$\dot{x} = \lambda x \quad - c_1 \{| x + y - 1 | + (x + y - 1)\}$$
$$\dot{y} = ay - bz - c_2 \{| x + y - 1 | + (x + y - 1)\}$$
$$\dot{z} = by + az - c_3 \{| x + y - 1 | + (x + y - 1)\}$$

As stated in Example 2.3, corresponding to choice of J and α, there exist 14 different ordinary differential equations including the above three equations.

4. Normal Forms of Proper 2-Region Piecewise Linear Vector Fields

As stated in section 3, any regular 2-region piecewise linear vector field is transformed, by a parallel translation, taking the complement and changing the signe of region, into the following vector field;

$$X_f(x) = f(x) = \begin{cases} Ax + q_A & ,x \in R^+ \\ Bx & ,x \in R^- \end{cases} \qquad (4.1a)$$

$$R^{\pm} = \{ x \in R^n : \pm(<\alpha,x> - \beta) > 0\}, \quad \beta \neq 0 \qquad (4.1b)$$

A linear subspace E in R^n is B-invariant if $B(E) \subset E$. A regular 2-region piecewise linear vector field is proper if there exists no linear subspace $E \subset R^n$ $(0 < \dim E < n)$ which is parallel to $U = \{ x \in R^n : <\alpha,x> = \beta \}$ and is B-invariant. In this section, we consider normal forms of continuous proper 2-region piecewise linear vector fields.

By taking a linear transformation and the complement of the vector field, without loss of generality, we may assume that (4.1) has the following form (see (3.4));

$$X_f(x) = f(x) = \begin{cases} Ax + q_A & ,x \in R^+ \\ Jx & ,x \in R^- \end{cases} \qquad (4.2a)$$

$$R^{\pm} = \{ x \in R^n : \pm(<\alpha,x> - 1) > 0\} \qquad (4.2b)$$

$$J = \sum_{i=1}^{r} J(\lambda_i;k_i) + \sum_{j=1}^{s} J(a_j,b_j;l_j) \qquad (4.2c)$$

α is a canonical normal vector corresponding to J (4.2d)

Proposition 4.1. Suppose X_f in (4.2) is proper, then the following holds;

(i) the eigen values of J, λ_i $(1 \leqq i \leqq r)$, $a_j \pm ib_j$ $(1 \leqq j \leqq s)$ are distinct.

(ii) $$\alpha = {}^T(u_1,\ldots,u_r,v_1,\ldots,v_s) \qquad (4.3)$$

$$u_i = (1,0,\ldots,0) \in R^{1\times k_i},$$

$$v_j = (\underline{1},\underline{0},\ldots,\underline{0}) \in R^{1\times 2l_j}, \quad \underline{1} = (1,0), \ \underline{0} = (0,0)$$

Proposition 4.2. (Sylvester Matrix) For an n×n matrix defined by

$$S = \begin{bmatrix} 0 & 1 & & & 0 \\ 0 & 0 & 1 & & \\ \vdots & \vdots & & \ddots & \\ \vdots & \vdots & & & 1 \\ a_n & a_{n-1} & \cdots & & a_1 \end{bmatrix} \quad (a_1,\ldots,a_n \in R)$$

(called Sylvester matrix), the following hold.

(i) If λ is a real eigenvalue of S, the real eigenspace belonging to λ, $E(\lambda) = \{x \in R^n : (S-\lambda I)x = 0\}$, is a 1-dimensional linear subspace spanning by

$$u_\lambda^1 = {}^T(1,\lambda,\ldots,\lambda^{n-1}) \in R^n, \quad \text{i.e. } E(\lambda) = \text{span}\{ u_\lambda^1 \}$$

(ii) Moreover, if λ has multiplicity k, the generalized real eigenspace belonging to λ, $E_w(\lambda) = \{x \in R^n : (S-\lambda I)^k x = 0\}$, is a k-dimensional linear subspace spanning by

$$u_\lambda^j = {}^T(u_{j1},u_{j2},\ldots,u_{jn}) \in R^n \quad (1 \leqq j \leqq k) \qquad (4.5a)$$

where

$$u_{ji} = {}_{i-1}C_{j-1}\,\lambda^{i-j} \qquad (1 \leqq i \leqq n,\ 1 \leqq j \leqq k) \qquad (4.5b)$$

$${}_iC_j = \begin{cases} i!/\{j!\,(i-j)!\} & \text{if } i \geqq j \\ 0 & \text{if } i<j \end{cases} \qquad (4.5c)$$

i.e. $E_w(\lambda) = \text{span}\{ u_\lambda^1,\ldots,u_\lambda^k \}$.

(iii) If ω and $\bar{\omega}$ are complex eigenvalues of S, the real eigenspace belonging to ω and $\bar{\omega}$, $E(\omega,\bar{\omega}) = \text{span}\{\text{Re}(z), \text{Im}(z) \in R^n : (S-\omega I)z = 0,\ z \in C^n \}$, is a 2-dimensional linear subspace spanning by $\text{Re}(v_\omega^1)$ and $\text{Im}(v_\omega^1)$ where

$$v_\omega^1 = {}^T(1,\omega,\ldots,\omega^{n-1}) \in C^n$$

i.e. $E(\omega,\bar{\omega}) = \text{span}\{ \text{Re}(v_\omega^1), \text{Im}(v_\omega^1) \}$.

(iv) Moreover, if ω and $\bar{\omega}$ have multiplicity 1, the generalized real eigenspace belonging to ω and $\bar{\omega}$, $E_w(\omega,\bar{\omega})$ = span{Re(z), Im(z): $(S-\omega I)^l z = 0$, $z \in C^n$ }, is a 2l-dimensional linear subspace spanning by $Re(v_\omega^j)$ and $Im(v_\omega^j) \in R^n$ $(1 \leqq j \leqq l)$ where

$$v_\omega^j = {}^T(v_{j1}, v_{j2}, \ldots, v_{jn}) \in C^n \quad (1 \leqq j \leqq k) \qquad (4.6a)$$

$$v_{ji} = {}_{i-1}C_{j-1}\ \omega^{i-j} \qquad (1 \leqq i \leqq n,\ 1 \leqq j \leqq k) \qquad (4.6b)$$

$${}_iC_j = \begin{cases} i!/\{j!\ (i-j)!\} & \text{if} \quad i \geqq j \\ 0 & \text{if} \quad i<j \end{cases} \qquad (4.6c)$$

i.e. $E_w(\omega,\bar{\omega})$ = span{ $Re(v_\omega^j), Im(v_\omega^j)$: $1 \leqq j \leqq l$ }.

(v) Let $\mu_1, \ldots, \mu_n \in C$ be all eigenvalues of S including multiplicity, then

$$a_k = (-1)^{k-1} \sum \mu_{i_1} \mu_{i_2} \cdots \mu_{i_k} \qquad (1 \leqq k \leqq n) \qquad (4.7)$$

where $\sum$ takes all $i_1, i_2, \ldots, i_k$ such that $i_1 < i_2 < \ldots < i_k$. In particular, a_1 = trace S and $a_n = (-1)^{n-1}$det S.

If the piesewise linear vector field X_f in (4.2) is proper, number of real Jordan block belonging to one eigenvalue is one by Proposition 4.1. Thus, J is transformed into a Sylvester matrix S under some linear transformation by Proposition 4.2. Indeed, using column vectors $u_\lambda^i \in R^n$ and $v_\omega^j \in C^n$ in Proposition 4.2(ii) and (iv), define a matrix G by

$$G = [\ u_1^{\ 1}, \ldots u_1^{\ k_1}, u_2^{\ 1}, \ldots, u_r^{\ k_r}, Re\ v_1^{\ 1}, Im\ v_1^{\ 1}, \ldots, Re\ v_1^{\ l_1},$$
$$Im\ v_1^{\ l_1}, Re\ v_2^{\ 1}, Im\ v_2^{\ 1}, \ldots, Re\ v_s^{\ l_s}, Im\ v_s^{\ l_s}\ ]$$

where $u_i^{\,j} = u_{\lambda_i}^{\,j}$, $v_i^{\,j} = v_{\omega_i}^{\,j}$ and $\omega_i = a_i + ib_i$.

Then we can verify that

$$S = GJG^{-1} \tag{4.10a}$$

$$G(U) = \{ x \in R^n : \langle \alpha', x \rangle = 1 \} \tag{4.10b}$$

$$\alpha' = {}^T(1,0,\ldots,0) \tag{4.10c}$$

where $U = \{ x \in R^n : \langle \alpha, x \rangle = 1 \}$ and α is a normal vector in (4.3). Under considering the above argument, we have the following.

Theorem 4 (Normal Forms of Proper 2-Region Piecewise Linear Vector Fields) If necessary, taking the complement and changing the signe of regions, any continuous n-dimensional proper 2-region piecewise linear vector field is linearly conjugate to the vector field with the following form;

$$X_f(x) = f(x) = \begin{cases} Cx + q & , x \in R^+ \\ Sx & , x \in R^- \end{cases} \tag{4.11}$$

where

$$R^{\pm} = \{ x \in R^n : \pm(\langle \alpha, x \rangle - 1) > 0 \} \tag{4.12a}$$

$$\alpha = {}^T(1,0,\ldots,0), \quad q = -{}^T(c_1,\ldots,c_n) \tag{4.12b}$$

$$S = \begin{bmatrix} 0 & 1 & & & \\ 0 & 0 & 1 & & \\ \cdot & \cdot & & \ddots & \\ \cdot & \cdot & & & 1 \\ a_n & a_{n-1} & \cdots & & a_1 \end{bmatrix} \tag{4.12c}$$

$$C = \begin{bmatrix} c_1 & 1 & & & \\ c_2 & 0 & 1 & & \\ \vdots & \cdot & & \ddots & \\ c_{n-1} & \cdot & & & 1 \\ c_n + a_n & a_{n-1} & \cdots & & a_1 \end{bmatrix} \tag{4.12d}$$

Moreover, the following hold.

(i) Put $p = {}^T(c_1,\dots,c_n) \in R^n$, then (4.11) is written as

$$X_f(x) = Sx + (1/2)p\{|\langle\alpha,x\rangle - 1|+(\langle\alpha,x\rangle - 1)\} \quad (4.13)$$

(ii) Let $\mu_1,\dots,\mu_n \in C$ be eigenvalues of S including the multiplicity, and define

$$a_k = (-1)^{k-1}\sum \mu_{i_1}\mu_{i_2}\cdots\mu_{i_k} \quad (1 \leqq k \leqq n) \quad (4.14)$$

where $\sum$ takes all $i_1,i_2,\dots,i_k$ such that $i_1<i_2<\dots<i_k$.

(iii) Let $\nu_1,\dots,\nu_n \in C$ be eigenvalues of C including the multiplicity, and define

$$b_k = (-1)^{k-1}\sum \nu_{i_1}\nu_{i_2}\cdots\nu_{i_k} \quad (1 \leqq k \leqq n) \quad (4.15)$$

where $\sum$ takes all $i_1,i_2,\dots,i_k$ such that $i_1<i_2<\dots<i_k$. Then

$$c_k = b_k - a_k + \sum_{i=1}^{k-1} a_i c_{k-i} \quad (1 \leqq k \leqq n) \quad (4.16)$$

(iv) Assume C^{-1} exists, and put $Q = -C^{-1}q \in R^n$. Then (4.11) is written as

$$X_f(x) = \begin{cases} C(x + Q) & ,x \in R^+ \\ S\,x & ,x \in R^- \end{cases} \quad (4.17)$$

and

$$Q = {}^T(1-a_n/b_n,\ c_1a_n/b_n, c_2a_n/b_n,\dots,c_{n-1}a_n/b_n) \quad (4.18)$$

Example 4.3. (Normal Forms of 3-Dimensional Proper 2-Region Piecewise Linear Vector Fields)

$$\dot{x} = y + (1/2)\, c_1\{|x-1| + (x-1)\}$$

$$\dot{y} = z + (1/2)\, c_2\{|x-1| + (x-1)\}$$

$$\dot{z} = a_3 x + a_2 y + a_1 z + (1/2)\, c_3\{|x-1| + (x-1)\}$$

Remark 4.4. Theorem 4 means that the linearly conjugate classes of proper 2-region systems are determined by the fundamental symmetric expression of eigenvalues for linear vector fields in each region, except the complement vector fields. To take the complement is needed for only 2-region systems. For proper system having many region (more than 2, but not greater than 2^n), the linearly conjugate classes are determined by the fundamental symmetric expression of eigenvalues in each region. The condition for proper (i.e. invariant subspaces are not parallel to the boundary of regions) is generic. Thus, if we want to know the global bifurcation, it is important to study it for the proper piesewise linear vector fields. In this case, the problem of global bifucation can be stated by values of fundamental symmetric expression of eigenvalues. In this sense, the values of fundamental symmetric expression of eigenvalues in each region can be thought as an univarsal bifurcation parameter for proper 2-region piecewise linear vector fields.

5. Three Dimensional Many Region Systems and Chaotic Attractors.

In this section, as an application of Theorem 4, we will derive normal forms of 3-dimensional 3-region systems with point symmetry and normal forms of 3-dimensional 4-region systems with axial symmetry. We will deal with only 3-dimensional systems, while the method of derivation of normal forms is valid for n-dimensional systems with many regions.

<u>Definition 5.1.</u> Let X_f be a PL-vector field having many regions. If each 2-region system deriving from adjacent two regions of X_f is regular (resp. proper), X_f is regular (resp. proper).

A vector field $X_f : R^n \longrightarrow R^n$ is symmetric with respect to the origin if $X_f(-x) = -X_f(x)$ holds for all x R^n. Then 3-region PL-vector fields with the symmetry with respect to the origin are expressed as follows;

$$X_f(x) = f(x) = \begin{cases} Ax + q\ , & x \in R^+ \\ Bx & x \in R^0 \\ Ax + q\ , & x \in R^- \end{cases} \qquad (5.1)$$

$R^{\pm} = \{ x \in R^n : \pm(\langle\alpha,x\rangle - \beta) > 0\},$

$R^0 = \{ x \in R^n : \ .\langle\alpha,x\rangle. < \beta \}, \quad \beta > 0 .$

<u>Theorem 5 (3-Dimensional 3-Region System with Origin Symmetry)</u>

Any continuous 3-dimensional proper 3-region system with symmetry with respect to the origin is linearly conjugate to the vector field with the following form;

$$\begin{aligned} \dot{x} &= c_1 x + y + (1/2)c_1\{|x-1| - |x+1\ \} \\ \dot{y} &= c_2 x + z + (1/2)c_2\{|x-1| - |x+1|\} \qquad (5.2) \\ \dot{z} &= (c_3+a_3)x + a_2 y + a_1 z + (1/2)c_3\{|x-1| - |x+1|\} \end{aligned}$$

Moreover the following holds;

(i) (5.2) is written as

$$X_f(x,y,z) = \begin{cases} M_1 \, {}^T(x,y,z) + q & , \quad x \geqq 1 \\ M_0 \, {}^T(x,y,z) & , \quad |x| < 1 \\ M_1 \, {}^T(x,y,z) - q & , \quad x \leqq -1 \end{cases} \qquad (5.3a)$$

$$M_0 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a_3 & a_2 & a_1 \end{bmatrix}, \quad M_1 = \begin{bmatrix} c_1 & 1 & 0 \\ c_2 & 0 & 1 \\ c_3+a_3 & a_2 & a_1 \end{bmatrix}, \quad q = \begin{bmatrix} -c_1 \\ -c_2 \\ -c_3 \end{bmatrix} \qquad (5.3b)$$

(ii) Let μ_i and ν_i (i=1,2,3) be eigenvalues of M_0 and M_1 respectively, then

$$a_1 = \mu_1+\mu_2+\mu_3 \ , \ a_2 = -(\mu_1\mu_2+\mu_2\mu_3+\mu_3\mu_1) \ , \ a_3=\mu_1\mu_2\mu_3,$$
$$b_1 = \nu_1+\nu_2+\nu_3 \ , \ b_2 = -(\nu_1\nu_2+\nu_2\nu_3+\nu_3\nu_1) \ , \ b_3=\nu_1\nu_2\nu_3, \qquad (5.4)$$
$$c_1=b_1-a_1 \ , \ c_2=b_2-a_2+a_1c_1 \ , \ c_3=b_3-a_3+a_2c_1+a_1c_2$$

(iii) Linearly conjugate classes are uniquely determined by the values of a_i and b_i (i = 1,2,3).

(iv) If $b_3 \neq 0$ and $a_3/b_3 < 0$, singular points of X_f are three points P^+, P^- and O which are given by

$$O = {}^T(0,0,0), \quad P^{\pm} = \pm\,{}^T(1-a_3/b_3,\ c_1a_3/b_3, c_2a_3/b_3) \qquad (5.5)$$

Definition 5.2. A 3-dimensional vector field X_f: $R^3 \longrightarrow R^3$ defined by

$$X_f(x,y,z) = (f_1(x,y,z),\ f_2(x,y,z),\ f_3(x,y,z))$$

is symmetric with respect to z-axis if

$$X_f(-x,-y,z) = (-f_1(x,y,z),\ -f_2(x,y,z),\ f_3(x,y,z))$$

is satisfied for all $(x,y,z) \in R^3$. Two plaines in R^3 with general position devide R^3 into four regions. We consider 3-dimensional 4-region system with axial symmetry. In the following theorem. notice that α denotes a real number.

Theorem 6 (3-Dimensional Proper 4-Region System with z-Axial Symmetry) Any continuous 3-dimensional proper 4-region piesewise linear vector field with z-axial symmetry such that the boundaries of regions are not parallel to z-axis is linearly conjugate to the vector field with the following form;

$$\begin{aligned} \dot{x} &= ((\alpha_1+\alpha_0)/(2\alpha_0))y + ((\alpha_1-\alpha_0)/(4\alpha_0))\{|y+z| - |y-z|\} \\ \dot{y} &= \alpha_0 x + ((\beta_1+\beta_0)/2)y + ((\beta_1-\beta_0)/4)\{|y+z| - |y-z|\} \\ \dot{z} &= ((\gamma_1+\gamma_0)/2)z + ((\gamma_1-\gamma_0)/4)\{|y+z| - |y-z|\} - 1 \end{aligned} \tag{5.6}$$

Moreover the following holed;

(i) (5.6) is equivalent to the following.

$$X_f(x,y,z) = \begin{cases} M_0{}^T(x,y,z) + q & ,(x,y,z) \in R_0 \\ M_1{}^T(x,y,z) + q & ,(x,y,z) \in R_1 \\ M_2^+{}^T(x,y,z) + q & ,(x,y,z) \in R_2^+ \\ M_2^-{}^T(x,y,z) + q & ,(x,y,z) \in R_2^- \end{cases} \tag{5.7a}$$

$$M_0 = \begin{bmatrix} 0 & 1 & 0 \\ \alpha_0 & \beta_0 & 0 \\ 0 & 0 & \gamma_0 \end{bmatrix}, \qquad M_1 = \begin{bmatrix} 0 & \alpha_1/\alpha_0 & 0 \\ \alpha_0 & \beta_1 & 0 \\ 0 & 0 & \gamma_1 \end{bmatrix} \tag{5.7b}$$

$$M_2^{\pm} = \begin{bmatrix} 0 & (\alpha_1+\alpha_0)/2\alpha_0 & \pm(\alpha_1-\alpha_0)/2\alpha_0 \\ \alpha_0 & (\beta_1+\beta_0)/2 & \pm(\beta_1-\beta_0)/2 \\ 0 & \pm(\gamma_1-\gamma_0)/2 & (\gamma_1+\gamma_0)/2 \end{bmatrix}, \quad q = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \tag{5.7c}$$

$$R_0 = \{(x,y,z) \in R^3 : y+z \leqq 0,\ y-z \geqq 0\},$$

$$R_1 = \{(x,y,z) \in R^3 : y+z \geqq 0,\ y-z \leqq 0\},$$

$$R_2^+ = \{(x,y,z) \in R^3 : y+z \geqq 0,\ y-z \geqq 0\}, \qquad (5.7d)$$

$$R_2^- = \{(x,y,z) \in R^3 : y+z \leqq 0,\ y-z \leqq 0\},$$

(ii) Define

$$P_i = -M_i^{-1}q = \begin{bmatrix} 0 \\ 0 \\ 1/\gamma_i \end{bmatrix} \ (i = 0,1) \qquad (5.8a)$$

$$Q^{\pm} = \frac{1}{\alpha_0\gamma_1+\alpha_1\gamma_0} \begin{bmatrix} \pm(\alpha_1\beta_0 - \alpha_0\beta_1)/\alpha_0 \\ \pm(\alpha_0 - \alpha_1) \\ (\alpha_0 + \alpha_1) \end{bmatrix} \qquad (5.8b)$$

If $P_i \in R_i$ $(i = 0,1)$ (resp. $Q^{\pm} \in R_2^{\pm}$), then P_i (resp. $Q^{\pm}$) are singular points of X_f.

(iii) If

$$\alpha_i + \beta_i\gamma_i - \gamma_i^2 \neq 0 \quad (i = 0,1)$$

holds, (5.7) is proper. Then linearly conjugate classes of (5.7) are uniquely determined by the values of α_i, β_i and γ_i $(i = 0,1)$.

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