数理解析研究所講究録
605巻 1987年 16-24

On the existence of infinitely many closed geodesics

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(Note by Ozawa)

We denoted by $\Lambda(S^n)$ or by Λ the space of free loops on a sphere S^n with a fixed riemannian metric, which defines an energy function E on Λ, as a modification of the length function.

E satisfies the conditions (C) of Smale ([K]) with respect to a manifold structure on Λ so that we can develope a Morse theory on Λ relative to the gradient flow ψ of E.
Since E is invariant under the S^1 action induced by the paramater shift of loops, ψ is equivariant and is critical not only at a closed geodesic c, but also on the 1 dimensional family $\{\theta\cdot c / t_1 \leq \theta \leq t_2\}$ of closed geodesics. We generalize this situation, starting with an equivariant flow φ in of a certain non degeneracy condition, called the strong non degeneracy, we attach index to each critical point and to the 1 family of critical points, called a ridge, and furthermore we define the boundary of them so as to make the sum of the set of critical points and ridges into a complex, whose homology is that of Λ.

The quotient space of Λ by the S^1 action is denoted by Π and there the ridges obviously vanish.

Let φ be a strongly non degenerate S^1 equivarient flow on Λ. A sequence of critical points $\{c_i, i=1,\cdots,n\}$ for φ is said to be a train of length n if every pair c_i, c_{i+1} of successive critical points satisfies one of the following conditions:

1) The order $m(c_{i+1})$ of the isotropy group called the multiplicity of c_{i+1} devides that of c_i

$$m(c_{i+1}) \mid m(c_i)$$

and ∂c_{i+1} contains c_i with intersection multiplicity 1, thus $\mathrm{ind}(c_{i+1}) = \mathrm{ind}(c_i) + 1$, in particular.

2) $$m(c_{i+1}) \mid m(c_i)$$

and ∂c_i contains c_{i+1} or a ridge over c_i.

The cases 1), 2) above correspond to the cases 1), 2) of Modified Divisibility lemma. ([S, K.])*

Thus we have from [S.K.]

<u>Lemma</u> Starting from any critical points we can construct a train over the critical point of length ≥ 2.

A train of length n is said to be terminated at n if ~~there occurs only~~ the last pair c_{n-1}, c_n is only the pair with the same multiplicity and with the intersection multipticity ± 1 (of case 1), that is,

$$m(c_{n-1}) = m(c_n),\ m(c_{i+1}) < m(c_i)\ (i<n),\ [\partial c_n, c_{n-1}] = \pm 1$$
$$\cancel{(\text{or } [c_n, \partial c_{n-1}] = \pm 1)}$$

<u>Corollary</u> For any train T, there exists a terminated train T of bounded index containing T.

In principle, T is obtained from the lemma above by

<u>Lemma</u> Any terminated train starting from a critical point c_1 (of finite multiplicity and index) has bounded index and length.

* Footnote p.9

In fact for a pair c_i , c_{i+1} of case 1), $m(c_{i+1})$ looses at least one divisior from $m(c_i)$ for $i < n$, therefore the increment of index in the train should not be greater them the number of divisors of $m(c_1)$ and if the length is infinite then the possibility for case 2) should get infinity which is a contradiction.

<u>Corollary</u> Starting from any critical point c_1 , there can be constructed a terminated train over c_1 of bounded index and length.

A train $\{ c_i , i = 1 .. n \}$ is said to be oneway, if the index $ind(c_i)$ of c_i satisfies that

$$ind(c_i) < ind(c_{i+1}) , \quad i \leq n-2 .$$

An equivarient version of the basis theorem in [M] yields that

<u>Lemma</u> If a train $\{ c_i , i = 1 .. n \}$ for φ has a triple of critical points c_i , c_{i+1} , c_{i+2} such that

$$ind(c_i) = ind(c_{i+2})$$
$$ind(c_{i+1}) < ind(c_i)$$

then it is possible to modify φ equivariantly so as to cancell c_{i+1} from ∂c_i . Hence a finite repetition give a pair c_i, c'_{i+1} of case 1).

Here it should be noted that the critical points of the modified flow remain essentially same as φ .

The proof is essentially obtained by a careful check of the basis theorem in the equivariant case and will be published somewhere together with that for the cancelling theorem

<u>Proposition</u> Starting from any critical point c_1, we can construct a terminated oneway train over c_1 by an equivariant

modification of φ without any essential change of critical points. In fact, we can modify any terminated train over c_1 which is not oneway by above lemma into eneway. If c_i, c_{i+1} in the train satisfy that

$$ind(c_i) > ind(c_{i+1})$$

then there exists $j \geq i$ so that

$$ind(c_j) = ind(c_{j+2})$$
$$ind(c_{j+1}) < ind(c_j)$$

as it is terminated. We apply the lemma to such a triple of minimal j, fixing i first, and repeat the process until we come back to i. Further repetition of the process above shifting i yields the proposition.

As in the basis theorem, the cancelling theorem of Smale ([M]) extends to the equivariant case. The original tedeous proof is very much simplified ~~again~~ and improved by Keuper

<u>Theorem</u> Let c, c' be critical points for an equivaiant non degenerate flow φ such that $\partial c'$ cantains c with the intersection multiplicity one and the multiplicity of them are the same;

$$m(c') = m(c), \quad [\partial c', c] = \pm 1$$

then there exists an equivariant strongly non degenerate flow φ' so that φ' remains to be same as φ outside of an open set U containing θc, $\theta c'$ $(\theta \in S^1)$ and φ' is not critical in U, not adding any critical point to that of φ only cancelling c, c' equivariantly.

Making use of the cancelling theorem above, we can have infinitely many critical points of bounded index.

as a first step we see by a direct application of the theorem to the terminal that

<u>Lemma</u> Let $\{ c_i, i= 1,\cdots n \}$ be a terminated oneway train over c_1 for φ then if $n \geq 3$ we can contine the train $\{ c_i, i= 1,\cdots n-2 \}$ to a terminated oneway train $\{c_1, c_{n-2}, c'_{n-1}, c'_{n'} \}$ having new critical points $c'_{n-1} \cdots c'_n$, for φ not equal to any of $\{ c_i \}$.

Hence we have as many distinct critical points for φ as the number of the terminated oneway trains over c_1 thus obtained until it reduces length 2. For this case we have

<u>Lemma</u> Suppose for any critical point c of index k, the train in above lemma reduces to that of length 2 after a finite repetition of the process, then every k-th homology of Λ should project down to zero in Π.

The lemma is a direct consequence of the fact that $H_*(\Lambda)$ can be written by the Morse complex for any equivariant flow.

Combining two lemmas above, we have

<u>Corollary</u> Suppose there exists a sequence $\{ x_k \in H_k(\Lambda), k \uparrow \infty \}$ so that x_k has non zero image in $H_k(\Pi)$ under the quotient projection, then there exists infinitely many critical points of bounded index especially from below by k starting from a critical point c_1 of index k.

We refer them as critical points of k-th cluster.

Thus Bott, Gromoll, Meyer type index formula ([B]) ~~the m-fold covering of a crifical point c~~, which estimates ~~ind(c^m)~~, ~~as follows~~ the index $\mathrm{ind}(c^m)$ of m-fold cover of a crtical point c as

$$m\alpha_c - \beta_c \leq \mathrm{ind}(c^m) \leq m\alpha_c - \beta_c$$

implies that

<u>Theorem</u> There exist infinitely many simple closed geodesics in $\Lambda = \Lambda(S^n)$ $(n \geq 3)$.

In fact, the result by Švarc ([S]) and Sulivan classes assure that Λ satisfies all the homology, homotopy conditions to apply every lemma, proposition and theorem above on it, thus we have infinitely many critical points of bounded index of the gradient flow of the energy function. Thus if there were only finitely many simple closed geodesics expressing the critical points as m_i-fold covering of them, then the index formula yields $\alpha_c = 0$ for these simple geodesics c which appears infinitely many times in the k-th cluster for some k. Then consider ~~the set~~ B = max of β_c for these critical points and take k > B to have the contradiction.

References

[B] R. Bott On the interation of closed geodesics, Comm. Pure Appl. Math, 9, 171-206 (1956).

[K] W. Klingenberg Lectures on Closed Geodesics.

[M] J, Milnor Lectures on h-cobordism.

[Š] A. Svarc Homology of the space of closed curves Trud Moscow 9, 3-44 (1960).

[S.K] Y.Shikata, W.Klingenberg On a proof of Divisibility Lemma I, Nagoya Math. J. 100, 65-81 (1985).

Remark 1 Since it is necessary in general to predeform the flow so as to have the non twisting property and the strong non degeneracy with multiplicity 1 intersection for case 1) of Divisibility Lemma, the set of critical points of the flow we started here may contain some other points than closed geodesics.

A close examination of the discussion of these deformations which will be given in part II of [S, K] shows that these superfluous critical points are harmless, because we need to add only finite number of critical points of at most index 2 to perform the tunnel killing process without assuming non twisting property, for instance.

Remark 2 It is not strictly neccessary to have oneway trains in order only to have infinite existence of closed geodesics. But from the existence of the oneway train starting at any given critical point, we see an original form of the divisibility lemma of Klingenberg that is, the existence of a crtical point of one higher index whose multiplicity divedes that of given one.

Remark 3 (-- Hyperbolic Case-----)

In this case we have a simple relation between the multplicity of the summit and its index:

$$\mathrm{index}(\alpha^m) = m\ \mathrm{index}(\alpha)$$

Therefore the modified divisibility yields directly the existence of infinitely many closed geodesics independent of the discussion of this paper as is reported partly by Kamiya in his master thesis. In fact, application of the modified divisibility lemma to a critical point c yields a critical point c' of index 1 higher or 1 or 2 lower than that of c satisfying the divisibility, therefore we see the index equality

$$\alpha + q\ m'\ k = m'k',\ q\ m'k = \mathrm{ind}(c),\ m'k' = \mathrm{ind}(c').$$

for the multiplicity m' of $\overleftrightarrow{c'}$, q, k, k' $\in$ Z and $\alpha = \pm 1$ or -2, indicating that m' = 1 or 2.

Hence we have simple closed geodesic of index m' or m'/2.

Thus increasing the index of c, we have infinitely many simple closed geodesics.

* Footnote

We note that the multiplicity m(c) used in Part I is defined to be the order of the isotropy group of a summit c in Morse complex and therefore it may differ by multiple of 2 from that which is defined geometrically, that is, from the minimum integer $m = m_\alpha$ satisfying

$$1/m_\alpha \cdot \alpha(t) = \alpha(t)$$

for a closed geodesic α which is a summit at the same time with a fixed orientation in the unstable manifold $U(\alpha)$.

This is the reason why we had to consider a multiple of 2 in the modified divisibility lemma in Part I ([S.K] p.78 Prop 2).

If we use m(c) only, instead of mixed use of m(c) and m , we have naturally a finer expression of the modified divisibility lemma suppressing the ambiguous mulitiple of 2.

We refer this expression as the modified divisibility lemma in what follows.