A Note on the Modified Zero-Fourth-Order

Cumulant Approximation

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The modified zero-fourth-order cumulant approximation
(MZCA) has been developed by Tatsumi and his collaborators
with extensive applications to various turbulent fields.[1]
In the present note the necessary and sufficient condition
for the validity of the method is given.

For illustration, we first consider the following exactly
solvable model:

$$\dot{q} + ibq = 0 \quad , \tag{1}$$

where b is a Gaussian random variable with zero mean and take
constant value in any one realization. Namely

$$q = \exp(-ibt)$$

and its average is obtained as

$$\langle q \rangle = \exp(-\sigma^2 t^2/4) \tag{2}$$

in which $\sigma^2/2 = <b^2>$, and any moment can be calculated exactly.

If eq.(1) is solved by closure, one has

$$<\dot{q}> + i<bq> = 0 \quad , \tag{3}$$

$$\frac{d}{dt}<bq> + i<b^2q> = 0 \quad , \tag{4}$$

while $<b^2q> = <b^2q>_c + <b^2><q>$. The formal solution of eq.(4) is given by

$$<bq> = -\int_0^t G(t-t')i<b^2q(t')>dt'$$

where $G(t)$ is the Green's function equal to 1. Following the procedure of the modified zero-fourth order cumulant approximation, one may solve (4) as

$$<bq> = -\int_0^t G(t-t')dt' \; i<b^2><q(t)>$$

$$= -it<b^2><q> \quad . \tag{5}$$

Introducing eq.(5) into eq.(3) we obtain the exact solution (2). On the other hand $<bq>$ and $<b^2q>$ can be calculated exactly, which are expressed as

$$<bq> = -it<b^2><q> \quad , \tag{6}$$

$$<b^2q> = -it<b^2><bq> + <b^2><q> \quad . \tag{7}$$

Equation (6) demonstrates that eq.(5) given by the MZCA yields the exact solution for $<bq>$, while eq.(7) shows that $<b^2q>_c = -it<b^2><bq>$. Substituting eqs.(6) and (7) into eqs.(3) and

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(4) we have

$$\frac{d}{dt}<q> + <b^2>t<q> = 0 \quad,$$

$$\frac{d}{dt}<bq> + <b^2>t<bq> = -i<b^2><q> \quad,$$

hence

$$-i<b^2q>_c = [<\dot{q}>/<q>]<bq> \quad. \tag{8}$$

In other words, in eq.(4) written as

$$\frac{d}{dt}<bq> + i<b^2q>_c = -i<b^2><q> \quad, \tag{4}'$$

if (8) is valid, it takes the form

$$\frac{d}{dt}<bq> - [<\dot{q}>/<q>]<bq> = -i<b^2><q> \quad,$$

consequently, $<q>$ is a solution of the homogeneous equation
so that the Green's function $G(t-t')$ becomes $<q(t)>/<q(t')>$ to
give

$$<bq> = \int_0^t G(t-t')[-i<b^2><q(t')>]dt'$$

$$= \int_0^t <q(t)>/<q(t')>[-i<b^2><q(t')>]dt'$$

$$= -it<b^2><q>$$

that is the result obtained by the procedure of the MACA. On
the other hand, if the solution of eq.(4)' is given by eq.(5),
substituting eq.(5) for $<bq>$ in eq.(4)' yields eq.(8). Hence for

eq.(1), eq.(8) is the necessary and the sufficient condition

for the exact validity of the MZCA. Here the two points should

be marked.

(i)   In view of eq.(3), the MZCA is exact, if and only if the

cumulant $<b^2q>_c$ is given by eq.(8) as a certain combination

of $<q>$ and $<bq>$.

(ii) Under eq.(8), eq.(4)' becomes a differential equation of

resonance type so that the solution becomes secular type.

It is also noticed that the crucial point of the above

argument on the MZCA is in solving eq.(4). In this regard, in

order to extend the result to general case it will be of help

to show the following theorem for the equation

$$(\frac{d}{dt} + \nu)X + gX = f \quad .\tag{9}$$

Theorem:

if X is solved as

$$X = \int_0^t \exp[-\nu(t-t')]dt' \; f(t)\tag{10}$$

then

$$g = -i\dot{f}/f \quad , \tag{11}$$

conversely, if (11) is valid, we get (10). The proof is

straightforward.

We now consider the Navier-Stokes equation. Using the

same notation as was used by Tatsumi in Ref.1), we have the

cumulants hierarchy for the homogeneous turbulence:

$$[\frac{\partial}{\partial t} + \nu(k^2+k'^2)]C_{pq}^{(2)}(k, k', t)$$

$$= i \sum_{[k,k']}^{[p,q]} k_\alpha \Delta_{p\beta}(k) \int C_{\alpha q\beta}^{(3)}(k-h, k', h, t)dh \quad,$$

$$(k + k' = 0) \tag{12}$$

$$[\frac{\partial}{\partial t} + \nu(k^2+k'^2+k''^2)]C_{pqr}^{(3)}(k, k', k'', t)$$

$$= i \sum_{[k,k',k'']}^{[p,q,r]} k_\alpha \Delta_{p\beta}(k) [\frac{1}{2} \int C_{\alpha qr\beta}^{(4)}(k-h, k', k'', h, t)dh$$

$$- C_{\alpha q}^{(2)}(-k', k', t)C_{r\beta}^{(2)}(k'', -k'', t)] \quad,$$

$$(k + k' + k'' = 0) \quad. \tag{13}$$

Here $C_{pq}^{(2)}$ $C_{pqr}^{(3)}$ are the second and third cumulants respectively, the suffices refere to components of velocity vectors, k, k' and k" are the 3d wave-number vectors and the summations are taken over all simultaneous permutations of (p, q, r) and (k, k', k") while

$$\Delta_{p\beta}(k) = \delta_{p\beta} - k_p k_\beta/k^2 \quad.$$

Introduce $f_{pqr}(k, k', k'', t)$ and $g_{pqr}(k, k', k'', t)$ by

$$f_{pqr}(k, k', k'', t)$$

$$= -i \sum_{[k,k',k'']}^{[p,q,r]} k_\alpha \Delta_{p\beta} C_{\alpha q}^{(2)}(-k', k', t)C_{r\beta}^{(2)}(k'', -k'', t) \quad,$$

$$-g_{pqr}(k,\ k',\ k'',\ t)C^{(3)}_{pqr}(k,\ k',\ k'',\ t)$$

$$= -i\ \sum_{[k,k',k'']}^{[p,q,r]} k_\alpha \Delta_{p\beta} \int C^{(4)}_{\alpha qr\beta}(k-h,\ k',\ k'',\ h,\ t)dh\ \ .$$

Then eq.(13) is written as

$$[\frac{\partial}{\partial t} + \nu(k^2+k'^2+k''^2)]C^{(3)} + gC^{(3)} = f \tag{13}'$$

for p, q, r and k, k', k" with k + k' + k" = 0, which takes
the same form as eq.(9); therefore, if and only if

$$g_{pqr}(k,\ k',\ k'',\ t) = -\dot{f}_{pqr}(k,\ k',\ k'',\ t)/f_{pqr}(k,\ k',\ k'',\ t)\ \ ,$$

$C^{(3)}$ is given by the MZCA, that is,

$$C^{(3)}_{pqr} = \int_0^t \exp[-\nu(k^2+k'^2+k''^2)(t-t')]dt'f_{pqr}(k,\ k',\ k'',\ t)\ \ .$$

Reference

1) T. Tatsumi: Advances in Applied Mechanics, ed, N.C. Freeman
   (Academic Press, New York, 1980), Vol.20, pp.39-133.