数理解析研究所講究録
607巻 1987年 5-21

SOME RESULTS ON TWOFOLD DESIGNS WITH BLOCK SIZE THREE AND FOUR

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0. Introduction. Twofold designs are balanced incomplete block designs of index $\lambda=2$. Although they have been somewhat less popular than those of index $\lambda=1$, there are, in our opinion, several reasons why twofold designs deserve attention. For small block sizes, their spectrum is, roughly speaking, twice as large as that of designs with $\lambda=1$. Many problems that can either be answered trivially, or, sometimes, cannot be posed at all for designs with $\lambda=1$, present a fascinating challenge for twofold designs. And, of course, whether there exist symmetric twofold designs (i.e. biplanes) for infinitely many orders remains one of the foremost unsolved problems in combinatorial design theory.

In this article we are concerned with twofold designs with block size $k=3$ (Chapter 1) and $k=4$ (Chapter 2). In each chapter, we first survey briefly known results, and then discuss in more detail some more recent results concerning twofold designs with additional properties. At the same time, we will try to identify problems that, to best of our knowledge, remain unsolved. Most of the relevant definitions are given in the respective sections.

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1. Twofold designs with block size k=3.

1.1. Definitions and examples. A Steiner triple system, STS (a twofold triple system, TTS) is a pair (V,B) where V is a set consisting of v elements, and B is a collection of 3-element-subsets of V called triples or blocks such that each 2-subset of V is contained in exactly one (exactly two) triples of B. The number v is called the order of the system. A Steiner triple system, and a twofold triple system of order v are abbreviated as STS(v) and TTS(v), respectively. Another notation for STS(v) is BIBD(v,3,1) or S(2,3,v), while for TTS(v) it is BIBD(v,3,2) or $S_2(2,3,v)$; however, the latter notation is sometimes understood to represent a TTS with no repeated triples (see 1.10 below).

Examples. 1. TTS(3): V={1,2,3}, B={{1,2,3}, {1,2,3}} .
2. TTS(4): V={1,2,3,4}, B={{1,2,3}, {1,2,4}, {1,3,4}, {2,3,4} .
3. TTS(6): V={1,2,3,4,5,6}, B={123, 124, 135, 146, 156, 236, 245, 256, 345, 346} (here we omitted braces for simplicity).

1.2. Existence. (1) An STS(v) exists if and only if $v \equiv 1$ or 3 (mod 6). Many proofs of this result have been given, the first one by Rev. T.P.Kirkman [16].
(2) A TTS(v) exists if and only if $v \equiv 0$ or 1 (mod 3). Apparently the first proof of this was supplied by Bhattacharya in 1943 [2]; see also [11,13].

We present here what is perhaps the simplest existence proof for TTSs (sufficiency only; the necessity is trivial). The constructions of the following two lemmas are used quite often for proving existence of TTSs with additional properties.

Lemma 1. If there exists a TTS(v) then there exists a

TTS(2v+1).

Proof. Let (V,B) be a TTS(v). Let X be a set, $|X|=v+1$, $V\cap X = \emptyset$. Let $F = \{F_1,\ldots,F_v\}$ be any 2-factorization of the complete multigraph $2K_{v+1}$ on X, and let $\alpha:V\longrightarrow\{1,2,\ldots,v\}$ be any bijection. Define, for $a\varepsilon V$, $C_a = \{\{a,x,y\}: \{x,y\}\varepsilon F_{\alpha(a)}\}$. Then $(V\cup X,\ B\cup \bigcup_{a\varepsilon V} C_a)$ is a TTS(2v+1). □

Lemma 2. If there exists a TTS(v) then there exists a TTS(2v+4).

Proof. Let (V,B) be a TTS(v). Take X, $|X|=v+4$, $V\cap X = \emptyset$, say, $X = Z_{v+4}$. Let $C' = \{\{i,i+1,i+3\}: i\varepsilon Z_{v+4}\}$. Delete all pairs (considered as edges) that occur in triples of C' from the complete multigraph $2K_{v+4}$ on X. What remains is a regular graph G' of degree 2v. Let $F = \{F_1,\ldots,F_v\}$ be any 2-factorization of G', and let $\alpha: \longrightarrow\{1,2,\ldots,v\}$ be any bijection. If $C_a = \{\{a,x,y\}: \{x,y\}\varepsilon F_{\alpha(a)}\}$ for $a\varepsilon V$ then $(V\cup X, B\cup C'\cup \bigcup_{a\varepsilon V} C_a)$ is a TTS(2v+4). □

Theorem 1. A TTS(v) exists for all $v\equiv 0,1 \pmod 3$.

Proof. We use induction on v. The statement is true for v=3,4,6 (see Examples 1,2,3 in 1.1). Let now $v \equiv 0,1 \pmod 3$, $v\geq 9$, and assume that a TTS(u) exists for all $u<v$ ($u \equiv 0,1 \pmod 3$ and $u\geq 3$). If $v \equiv 1,3 \pmod 6$, let $u=(v-1)/2$; then $u \equiv 0,1 \pmod 3$ and so, by induction hypothesis, there exists a TTS(u). Applying Lemma 1 gives a TTS(v). If $v \equiv 0,4 \pmod 6$, let $u=(v-4)/2$; then $u \equiv 0,1 \pmod 3$, $u\geq 3$, and by induction hypothesis, there exists a TTS(u). But now applying Lemma 2 gives a TTS(v). □

1.3. Enumeration. Let $N_2(v)$ be the number of nonisomorphic TTS(v). The values of $N_2(v)$ are known exactly only for $v\leq 10$. We

have $N_2(3) = N_2(4) = N_2(6) = 1$, $N_2(7) = 4$, $N_2(9) = 36$ [21,24], $N_2(10) = 960$ [6,9,15]. The value of $N_2(12)$ is not known exactly; we know $N_2(12) \geq 574$ [21]. In general, we have $\lim_{v\to\infty} N_2(v) = \frac{1}{3} v^2 \ln v$, a result obtained easily from [26]. The exact asymptotics for $N_2(v)$ is still unknown (as is, incidentally, the exact asymptotics for the number of STSs).

1.4, Resolvability. If (V,B) is a design, a parallel class in (V,B) is a set of blocks that partitions V. A resolution is a set of parallel classes that partitions B. A resolvable design is a design admitting a resolution.

Theorem 2 [12]. A resolvable TTS(v) exists if and only if $v \equiv 0 \pmod 3$, $v \neq 6$.

A near-parallel class is a set of blocks that partitions $V \setminus \{x\}$, x an element. A near-resolution is a set of near-parallel classes that partitions B. A near-resolvable design (sometimes also called an almost resolvable design) is a design admitting a near-resolution.

Theorem 3 [12]. A near-resolvable TTS(v) exists if and only if $v \equiv 1 \pmod 3$.

1.5. Embedding. A TTS(v) (V,B) is a subdesign of a TTS(w) (W,C) if $V \subseteq W$ and $B \subseteq C$. In such a case one also says that (V,B) is embedded in (W,C).

Theorem 4 [32]. A TTS(v) can be embedded in a TTS(w) if and only if $w \geq 2v+1$, $w \equiv 0,1 \pmod 3$.

1.6. Disjoint TTSs. Two TTSs (V,B_1), (V,B_2) on the same set are disjoint if $B_1 \cap B_2 = \emptyset$.

Theorem 5 [33]. The set of all $\binom{v}{3}$ three-element subsets

of a v-set, $v \equiv 0,4 \pmod 6$, can be partitioned into $(v-2)/2$ pairwise disjoint TTS(v)'s.

Such a set of $(v-2)/2$ pairwise disjoint TTS(v)'s is usually called a large set of disjoint TTSs.

When $v \equiv 1,3 \pmod 6$, a large set of disjoint TTSs cannot exist. In this case, there exist at most $(v-3)/2$ pairwise disjoint TTSs (with no repeated blocks). It follows from the known results on large sets of STSs [19] that such a set of $(v-3)/2$ pairwise disjoint TTSs exists for all $v \equiv 1,3 \pmod 6$, $v \geq 7$, except possibly when v=141, 283, 501, 789, 1501, 2365.

1.7. TTS as the underlying design of a Mendelsohn and directed triple system. A Mendelsohn triple system [directed triple system, respectively] is a pair (V,B) where V is a v-set and B is a collection of cyclic triples [of transitively directed triples, respectively] of elements of V such that each ordered pair of distinct elements of V is contained in exactly one cyclic [transitively directed] triple of B. Here, a cyclic triple (a,b,c) contains the ordered pairs (a,b), (b,c) and (c,a) while a transitively directed triple [a,b,c] contains the ordered pairs (a,b), (a,c) and (b,c). Alternatively, a Mendelsohn triple system and a directed triple system, respectively, is a decomposition of the complete symmetric directed graph into cyclic, and into transitive tournaments on three vertices, respectively.

It is known that a Mendelsohn triple system of order v (MTS(v)) exists if and only if $v \equiv 0,1 \pmod 3$, $v \neq 6$ [23], and

a directed triple system of order v (DTS(v)) exists if and only if $v \equiv 0,1 \pmod 3$ [23]. Clearly, disregarding orientation or direction in an MTS or in a DTS results in a TTS. What can one say about the converse?

Theorem 6 [4,14]. Every TTS is directable (i.e., underlies a DTS).

In fact, a TTS is usually the underlying design of several nonisomorphic DTSs. A TTS may be the underlying design of several nonisomorphic MTSs, however, not every TTS can be oriented into an MTS.

Theorem 7 [1]. For every $v \equiv 0,1 \pmod 3$, $v \geq 6$, there exists a TTS(v) which is not orientable into an MTS.

On the other hand, it is quite easy to decide whether a given TTS underlies an MTS (i.e. whether a TTS is orientable).

1.8. Indecomposable and simple TTSs. A TTS(v) is decomposable if its set of blocks can be partitioned into two subsets each of which is a set of blocks of an STS(v). Otherwise, it is indecomposable.

Trivially, any TTS(v) with $v \equiv 0,4 \pmod 6$ is indecomposable.

Theorem 8 17 . Let $v \equiv 1,3 \pmod 6$. An indecomposable TTS(v) exists if and only if $v>7$.

Proof. (i) There is no indecomposable TTS(v) for v=3 (trivial) or v=7 [24]. (ii) Take TTS(4) (trivially indecomposable), and use Theorem 4. □

It is easy to decide whether a TTS is decomposable: form the block intersection graph of the TTS (its vertices are the

triples; two vertices are adjacent if the corresponding triples share a pair of elements). The TTS is decomposable if and only if this graph is bipartite.

A TTS(v) is simple if it contains no nontrivial sub-TTS.

Theorem 9. A simple TTS(v) exists for all $v \equiv 0,1 \pmod 3$.

Proof. There exists a simple STS(v) for all $v \equiv 1,3 \pmod 6$ [8]. Also, every STS(v), $v>3$, has an isomorphic disjoint mate [34]. Take a simple STS(v), and an isomorphic disjoint copy of it; this gives a simple TTS(v).

For $v \equiv 4 \pmod 6$, take a simple STS(v-1), and, on the same set, a disjoint maximal partial triple system whose leave (i.e. the graph whose edges are the pairs not contained in any triple) is a cycle of length v-1. Such a maximal partial triple system always exists [6]. Add a new element ∞, and triples $\{\infty,x,y\}$ where $\{x,y\}$ is an edge of the (v-1)-cycle. This yields a simple TTS(v). For $v \equiv 0 \pmod 6$, the proof is similar. □

For $v \equiv 1,3 \pmod 6$, the simple TTSs obtained in Theorem 9 are decomposable.

Problem 1. Simplify the proof of the existence of simple TTSs.

Problem 2. Determine thespectrum for simple indecomposable TTSs.

Problem 3. Are almost all TTSs simple?

Remark. It was conjectured in [28] that almost all STSs are simple (a simple STS is sometimes called planar).

1.9. Element neighbourhoods in TTSs. Let (V,B) be a TTS(v). For $x \in V$, define the neighbourhood graph N(x) of x as follows:

$N(x) = (V_x, E_x)$ where $V_x = V \setminus \{x\}$, $E_x = \{\{u,v\}: \{u,v,x\} \in B\}$. A given (multi)-graph G is an <u>element neighbourhood</u> (of a TTS) if there exists a TTS and its element x that $N(x) \simeq G$.

An obvious necessary condition for G to be an element neighbourhood of a TTS is that G must be 2-regular and $|V(G)| \equiv 0,2 \pmod 3$.

<u>Theorem 10</u> [7]. Every 2-regular mutligraph on n vertices, $n \equiv 0,2 \pmod 3$, is an element neighbourhood of a TTS, with exactly two exceptions: $C_2 \cup C_3$, $C_3 \cup C_3$.

1.10. <u>Number of repeated blocks in TTSs</u>. A TTS(v) (V,B) may contain two blocks, say, $b=\{x,y,z\}$, $b'=\{x,y,z\}$ identical as subsets of V; then b is a <u>repeated block</u>.

Let $v \equiv 0,1 \pmod 3$. Denote

$R(v) = \{t: \exists \text{ TTS}(v) \text{ having exactly } t \text{ repeated blocks}\}$.

The set R(v) is called the <u>spectrum for repeated blocks</u> in TTSs. We have $R(3)=\{1\}$, $R(4)=\{0\}$ (trivial), $R(6)=\{0\}$ (cf.Example 1 of 1.1), $R(7)=\{0,1,3,7\}$ [24], $R(9)=\{0,1,2,3,4,6,12\}$ [20,24], $R(10)=\{0,1,2,3,4,5,6,7,9\}$[5,9], $R(12)=\{0,1,2,\ldots,12,13,16\}$[31]. Denote $b_v=v(v-1)/6$, $s_v=v(v-4)/6$. The following theorem completely determines the spectrum for repeated blocks in TTSs.

<u>Theorem 11</u> [18,31]. Let $v>12$. Then

$$R(v) = \begin{cases} \{0,1,\ldots,b_v-6,b_v-4,b_v\} & \text{if } v \equiv 1,3 \pmod 6 \\ \{0,1,\ldots,s_v-2,s_v\} & \text{if } v \equiv 0,4 \pmod{12} \\ \{0,1,\ldots,s_v-1\} & \text{if } v \equiv 6,10 \pmod{12} \end{cases}$$

If we require, in addition, that the TTSs be indecomposable, we may define by analogy the spectrum for repeated blocks in indecomposable TTSs to be the set

$RI(v)=\{t: \exists$ indecomposable TTS(v) with exactly t repeated blocks$\}$

We have trivially $RI(v) = R(v)$ for $v \equiv 0,4 \pmod 6$, so we may assume $v \equiv 1,3 \pmod 6$. We have $RI(3) = \emptyset$, $RI(7) = \emptyset$, $RI(9) = \{0,1,2,4\}$ [20]. In general, $RI(v) \subsetneq R(v)$ for all $v \equiv 1,3 \pmod 6$ as b_v is never contained in RI(v).

Denote $L(v) = \{0,1,\ldots,b_v-9,b_v-8,b_v-6\}$.

<u>Theorem 12</u> [30]. Let $v \equiv 1,3 \pmod 6$, $v \geq 15$. Then $RI(v)=L(v)$. (For v=13, there is one additional exception: $17 \notin RI(13)$ so that $RI(13)=\{0,1,\ldots,15,16,18,20\}$.)

<u>Proof</u> (an outline only). (i) Let Q be a 2-factorization of $2K_{v+1}$ and let q be the total number of 2-cycles in Q; then $t \in RI(v)$ implies $t+q \in RI(2v+1)$. <u>Proof</u>. If x is an element and P is a set of pairs, we write $x*P=\{\{x,a,b\}: \{a,b\} \in P\}$. Let $V=\{a_i: i=1,\ldots,v\}$, $|X|=v+1$, $V \cap X=\emptyset$, $W=V \cup X$. Assume (V,B) is an indecomposable TTS(v) with t repeated triples, $Q = \{Q_1,\ldots,Q_v\}$ a 2-factorization of $2K_{v+1}$ on X. Let $C = \bigcup_{i=1}^{v} a_i*Q_i$. Then $(W,B \cup C)$ is an indecomposable TTS(2v+1) with t+q repeated triples.

(ii) If $v \equiv 1,3 \pmod 6$, $s \in \{0,1,\ldots,v-2,v\}$, $t \in RI(v)$ then $t+s(v+1)/2 \in RI(2v+1)$. <u>Proof</u>. Let $F = \{F_1,\ldots,F_v\}$ be any 1-factorization of K_{v+1} on X, α any permutation of $\{1,2,\ldots,v\}$ fixing exactly s letters. Then take in (i) $Q_i=F_i \cup F_{i\alpha}$.

(iii) If $RI(v)=L(v)$ for $v \equiv 1,3 \pmod 6$, $v \geq 13$, then $RI(2v+1) = LI(2v+1)$ (follows directly from (ii)).

(iv) If $v \equiv 1,3 \pmod 6$, $v \geq 9$, $s \in \{0,1,\ldots,v-2,v\}$, $\delta \in \{0,1\}$, $\gamma \in \{0,1,3,7\}$, $t \in RI(v)$ then $t+s(v+7)/2+\delta v+\gamma \in RI(2v+7)$. <u>Proof</u>. Similar to (ii) but use a $v \to 2v+7$ construction for STSs as given, e.g., in [29].

(v) If $v \equiv 1,3 \pmod 6$, $v \geq 15$, $RI(v) = L(v)$ then $RI(2v+7) = L(2v+7)$ (follows directly from (iv)).

(vi) $RI(13)=L(13)\setminus\{17\}$; $RI(v) = L(v)$ for $v=15$, 19, 21, 25, 27, and 33. This is needed to start the induction; the proof of this is tedious, lengthy and ugly. For details, see [30].

(vii) Induction on v, using (iii),(v),(vi). □

Problem 4. For which values of v does there exist a resolvable TTS(v) without repeated blocks? More generally, what is the spectrum for repeated blocks in resolvable TTSs?

Problem 5 (cf.1.6). For which values of v does there exist a set of $(v-3)/2$ pairwise disjoint indecomposable TTS(v)'s without repeated blocks?

2. Twofold designs with block size 4.

2.1. Definitions and examples; existence. A twofold fourtuple system (TFS) is a pair (V,B) where V is a v-element set, B is a collection of 4-element subsets of V called blocks such that each 2-subset of V is contained in exactly two blocks of B. Another notation: BIBD(v,4,2), $S_2(2,4,v)$.

Examples. 1. TFS(4): $V=\{1,2,3,4\}$, $B=\{\{1,2,3,4\},\{1,2,3,4\}\}$.

2. TFS(7): $V=\{1,2,3,4,5,6,7\}$, $B=\{1234, 1256, 1357, 1467, 2367, 2457, 3456\}$.

3. There exist exactly 3 nonisomorphic TFS(10)'s ([25]; cf.also [10]).

Theorem 13. A TFS(v) exists if and only if $v \equiv 1 \pmod 3$.

This was proved by Hanani in 1960 [11]. While the necessity is obvious, we present here a proof of sufficiency which is

quite different from Hanani's. It has the advantage that it enables one to prove quite easily the existence of TFSs with an additional property, provided this property is preserved by embedding, and the existence of designs with this property can be proved for sufficiently many small orders.

A design (V,B) is <u>i-resolvable</u> if its block set B can be partitioned into subsets $P_1,\ldots,P_q$ called <u>i-fold parallel classes</u> such that each element of V occurs in exactly i blocks of each P_j. A <u>partially resolvable</u> $(v;k_1,k_2;\lambda;m)$-<u>design</u> is a pairwise balanced design of index λ on v elements with block sizes k_1, k_2 such that the blocks of size k_1 can be partitioned into m λ-fold parallel classes.

We are interested here in a special case of PRPs: PRP $(v;3,4;2;m)$. We denote such a PRP simply by $P(v,m)$. Thus a $P(v,m)$ is a PBD of index 2, with blocks of size 3 and 4 where the blocks of size 3 are partitioned into m 2-fold parallel classes. We will consider just two values of m: $m_1 = (v-4)/2$, $m_2 = (v-7)/2$.

<u>Lemma 3</u>. $P(v,m_1)$ exists for all $v \equiv 0 \pmod{12}$.

<u>Proof</u>. Let $V = Z_{4s} \times Z_3$, and let the base blocks mod (4s,3) be:

(1) $\{(0,0),(1,0),(2s,0),(2s+1,0)\}$

(2) $\left.\begin{array}{l}\{(0,0),(r,1),(2s-r,1)\}\\ \{(0,0),(r,2),(2s-r,2)\}\end{array}\right\}$ $r=1,2,\ldots,s-1$

(3) $\left.\begin{array}{l}\{(0,0),(r,1),(2s-1-r,1)\}\\ \{(0,0),(r,2),(2s-1-r,2)\}\end{array}\right\}$ $r=1,2,\ldots,s-2$

(4)(a) $\{(0,0),(2s,1),(2s+1,1)\}$, $\{(0,0),(-(s-1),2),(s,2)\}$

(b) $\{(0,0),(s,1),(2s,2)\}$, $\{(0,0),(-(s-1),1),(-(2s-1),2)\}$

(c) $\{(0,0),(0,1),(0,2)\}$ taken twice □

<u>Lemma 4</u>. $P(v,m_1)$ exists for all $v \equiv 6 \pmod{12}$.

<u>Proof</u>. Let $V = Z_{6s+3} \times \{1,2\}$. Let $P_1^i, P_2^i, \ldots, P_{3s+1}^i$ be the parallel classes of a Kirkman triple system of order 6s+3 on $Z_{6s+3} \times \{i\}$, i=1,2. Without loss of generality, let P_1^1 be $\{0_1,(2s+1)_1,(4s+2)_1\}$ mod 6s+3. Let the base blocks of B modulo 6s+3 be:

(1) $\{0_1,(2s+1)_1,s_2,(s+1)_2\}$

(2) $\{0_2,i_1,(6s+3-i)_1\}$ $\quad i=1,2,\ldots,3s+1$

(3) $\{0_1,0_2,(2s+2)_2\}$

$\{0_1,(2s+2-i)_2,(2s+2+i)_2\}$ $\quad i=1,2,\ldots,s$

$\{0_1,(s-i)_2,(3s+2+i)_2\}$ $\quad i=1,2,\ldots,\lfloor(s-1)/2\rfloor$

$\{0_1,(5s+2-i)_2,(5s+3+i)_2\}$ $\quad i=1,2,\ldots,\lfloor 3s/2\rfloor$

Let B also contain

(4) all triples of the parallel classes $P_2^1, P_3^1, \ldots, P_{3s+1}^1$, and $P_1^2, P_2^2, \ldots, P_{3s+1}^2$. □

<u>Lemma 5</u>. $P(v,m_2)$ exists for all $v \equiv 3 \pmod 6$.

<u>Proof</u>. Let $V = Z_{2s+1} \times Z_3$, and let the base blocks modulo (2s+1,3) be:

(1) $\{(0,0),(2,0),(8,0),(12,0)\}$

(2)(a) $\{(0,0),(0,1),(0,2)\}$ taken twice

(b) $\{(0,0),(1,1),(4,2)\}$, $\{(0,0),(2,1),(6,2)\}$

$\{(5,0),(3,1),(8,2)\}$, $\{(6,0),(5,1),(0,2)\}$

(3) $\left.\begin{array}{l}\{(0,1),(4r-2,1),(2r-1,0)\}\\ \{(0,2),(4r,2),(2r,0)\}\end{array}\right\}$ for r=1,2,3 <u>once</u>, and for r=4,5,...,s/2 <u>twice</u>

When s is odd, take also

(4) $\{(0,1),(2s,1),(s,0)\}$, $\{(0,2),(2s,2),(s,0)\}$.
[This works only if $s\geq 6$. The cases $1\leq s\leq 5$ have to be considered separately; see [27] for details.] □

Theorem 14. Let (V,B) be a TFS(v). Then for $w=3v+a$, $a\varepsilon\{1,4,7\}$, there exists a TFS(w) (W,C) such that (V,B) is embedded in (W,C).

Proof. This is well known for $a=1$. Let $u = w - v$. If $a=4$ then $u = 2v+4 \equiv 0 \pmod 6$, and by Lemmas 3 and 4, there exists a $P(u,v)$. If $a=7$ then $u = 2v+7 \equiv 3 \pmod 6$, and then there exists a $P(u,v)$ by Lemma 5. In either case, let $P_1, P_2,\ldots,P_v$ be the 2-fold parallel classes of this PRP. Let Q be the set of blocks of size 4. If $V = \{a_1,\ldots,a_v\}$, let $D = \{\{a_i,x,y,z\}: \{x,y,z\}\varepsilon P_i, i\varepsilon\{1,\ldots,v\}$, $C = B\cup D\cup Q$. Then (W,C) is a TFS(w) containing (V,B). □

Theorem 15. A TFS(v) with no repeated blocks exists if and only if $v \equiv 1 \pmod 3$, $v>4$.

Proof. Necessity is obvious. For sufficiency, use induction. The TFS(7) in Example 2 in 2.1 has no repeated blocks. Similarly, none of the 3 nonisomorphic TFS(10)'s has repeated blocks. Examples of TFS(13) and TFS(16) without repeated blocks can be found in [22] and [3], respectively. An example of a TFS(19) without repeated blocks is the following: $V = Z_{19}$, $B = \{\{0,1,7,11\}, \{0,2,13,14\}, \{0,4,6,9\} \bmod 19\}$. Let $v\geq 22$, $v \equiv 1 \pmod 3$, and assume that a TFS(u) with no repeated blocks exists for all $u<v$ $(u\geq 4)$. Let $u = (v-a)/3$ where $a=1$ if $v \equiv 4 \pmod 9$, $a=4$ if $v \equiv 7 \pmod 9$, and $a=7$ if $v \equiv 1 \pmod 9$. In either case, $u \equiv 1 \pmod 3$, $u\geq 4$, so there exists a TFS(u) with

no repeated blocks. Applying Theorem 14 with appropriate value of a yields a TFS(v) with no repeated blocks. □

Theorem 16. An indecomposable TFS(v) with no repeated blocks exists if and only if $v \equiv 1 \pmod 3$, $v>4$.

Proof. Similar to that of Theorem 15. □

Problem 6. Determine the spectrum for simple TFSs.

Problem 7. Determine the spectrum for repeated blcoks in TFSs.

Acknowledgements. I would like to express my sincere thanks to the organizers of the Kyoto meeting on Designs and Finite Geometries. I would also like to thank Charlie Colbourn for suggesting the proof of Theorem 9.

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