

Conformal martingale diffusions and Shilov boundaries

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1. Introduction Let D be a bounded pseudoconvex domain in \mathbb{C}^n . A conformal martingale diffusion (cmd in abbreviation) is by definition a triple $M=(Z_t, \xi, P_Z)$ of a stochastic process $(Z_t)_{0 \leq t}$ with state space D , its life time ξ and probability measures $\{P_Z\}_{Z \in D}$ such that M is a diffusion process on D and Z_t^i and $Z_t^i Z_t^j$, $1 \leq i, j \leq n$, are all local martingales under P_Z , $Z \in D$. In [FO] and [O], it was shown that each symmetrizable cmd is in one-to-one correspondence to a suitable pair (θ, m) of closed positive current θ on D and positive Radon measure m on D . Our aim of this report is to characterize the subset of the boundary ∂D of D where a cmd does not approach in terms of these θ and m . This kind of attempt was essentially made in [DG], where they investigated the Shilov boundary $S(D)$ of D in a probabilistic way. Indeed, they have proved that $\partial D \setminus S(D)$ is the subset of ∂D where a certain Kähler diffusion does not approach if ∂D is nice. In [KT], their argument was extended to the more general domain possessing a suitable family of bounded plurisubharmonic functions. Moreover, it was taken advantage of in the study of the complex Monge-Ampère equations (for details, see [KT, Section

3)]. In this paper, we will see that one can weaken the assumptions in [KT] and will obtain the much simpler expression of the subset of ∂D not approached by a cmd.

The organization of this paper is as follows. In Section 2, we will state our main results. We will also give a brief review on the relationship between cmds and pairs (θ, m) in the same section. Section 3 will be devoted to the proofs of the theorems stated in the preceding section. In Section 4, we will discuss an application of our results to the complex Monge-Ampère equations.

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2. Main results We begin this section with a brief review on the correspondence between symmetrizable cmds and pairs (θ, m) of closed positive currents and Radon measures, following [FO1]. Assume that a cmd M is m -symmetrizable, m being an everywhere-dense positive Radon measure on D : i.e. the transition function $p_t(z, E)$ of M enjoys the property that $\int_F p_t(z, E) m(dz) = \int_E p_t(z, F) m(dz)$ for every Borel subset E and F of D . The Dirichlet form \mathcal{E}^M of M is defined by $\text{Dom}(\mathcal{E}^M) = \text{Dom}(\sqrt{-A})$, $\mathcal{E}^M(u, v) = \langle \sqrt{-A}u, \sqrt{-A}v \rangle$, where A is the infinitesimal generator of the semigroup on $L^2(D; m)$ determined by p_t and $\langle \cdot, \cdot \rangle$ is the inner product in $L^2(D; m)$. If M is C_0^∞ -regular, i.e. $C_0^\infty(D)$ is dense in $\text{Dom}(\mathcal{E}^M)$ with respect to the norm $\| \cdot \| = \{ \mathcal{E}^M(\cdot, \cdot) + \langle \cdot, \cdot \rangle \}^{1/2}$, then there exists a closed positive current θ of bidegree $(n-1, n-1)$

such that $\delta^M = \delta^\theta$ on $C_0^\infty(D) \times C_0^\infty(D)$, where $\text{Dom}(\delta^\theta) = C_0^\infty(D)$, $\delta^\theta(u, v) = \int_D du \wedge d^C v \wedge \theta$ and $d = \partial + \bar{\partial}$, $d^C = \sqrt{-1}(\bar{\partial} - \partial)$. Conversely, given (θ, m) of closed positive current θ on D of bidegree $(n-1, n-1)$ and everywhere-dense positive Radon measure m on D such that δ^θ is closable on $L^2(D; m)$, there exists a C_0^∞ -regular, symmetrizable cmd M related to (θ, m) in the preceding manner (for details, see [FO] and [O]). The pair (θ, m) with the above property is called an admissible pair. Thus we have established the one-to-one correspondence between C_0^∞ -regular, symmetrizable cmds and admissible pairs.

In order to state our results, we introduce some notations. We will use $\text{PSH}(D)$ to denote the sets of all plurisubharmonic functions on D and $\text{PSHB}(D)$ consists of all bounded $\varphi \in \text{PSH}(D)$. We put $E(D) = \{\varphi \in \text{PSHB}(D) \mid \varphi < 0 \text{ on } D \text{ and } \varphi(z) \rightarrow 0 \text{ as } z \rightarrow \partial D\}$. Throughout this and next section, we assume that

$$(2.1) \quad E(D) \neq \emptyset.$$

For $\varphi \in \text{PSH}(D) \cap L_{\text{loc}}^\infty(D)$ and a closed positive current θ of bidegree $(n-1, n-1)$, a positive Radon measure $dd^C \varphi \wedge \theta$ on D is defined by

$$(2.2) \quad \int_D \psi dd^C \varphi \wedge \theta = \int_D \varphi dd^C \psi \wedge \theta \quad \text{for every } \psi \in C_0^\infty(D).$$

We are now ready to state our results.

(2.3) Theorem *Let $M = (Z_t, \xi, P_z)$ be a C_0^∞ -regular, symmetrizable cmd on D and (θ, m) be its corresponding admissible pair. Define*

$$(2.4) \quad \Gamma^\theta = \{ \xi \in \partial D \mid dd^c w \wedge \theta \geq dd^c (-\log(-\varphi)) \wedge \theta \text{ on } D \cap U, \text{ for some } w \in \text{PSHB}(D), \varphi \in E(D) \text{ and open } U \subset \mathbb{C}^n \text{ containing } \xi \}.$$

If M satisfies

$$(2.5) \quad P_z[\lim_{t \uparrow \xi} Z_t \in \partial D] = 1 \quad \text{q.e. } z \in D,$$

then

$$(2.6) \quad P_z[\lim_{t \uparrow \xi} Z_t \in \partial D \setminus \Gamma^\theta] = 1 \quad \text{q.e. } z \in D,$$

where "q.e." means "except on a δ^M -capacity zero set".

(2.7) Remark i) It is obvious that Γ^θ is open in ∂D .

ii) Since D is a bounded domain, Z_t is a uniformly integrable martingale under P_z , $z \in D$. Thus, by the martingale convergence theorem, we see that $\lim_{t \uparrow \xi} Z_t$ exists P_z -a.s. for each $z \in D$.

(2.8) Corollary Let $S(D)$ be the Shilov boundary of D , i.e. $S(D)$ is the smallest closed subset S of ∂D where $\sup_{z \in D} |h(z)| = \sup_{z \in S} |h(z)|$ for every h holomorphic in D and continuous on \bar{D} . Let M and (θ, m) be as in Theorem (2.3). Then $S(D) \subset \partial D \setminus \Gamma^\theta$.

In [KT:Section 2], some cases that the identity $S(D) = \partial D \setminus \Gamma^\theta$ holds were discussed. We now consider sufficient conditions for (2.5) to be satisfied. To see this, we prepare one more notion. A cmd M is said to be irreducible if

$\int_B u(y)P_t(z, dy) = 1_B(z) \int_D u(y)P_t(z, dy)$ for any $u \in L^2(D; m)$ if and only if $m(B) = 0$ or 1 , P_t being its transition function.

(2.9) Theorem Let M be a C_0^∞ -regular, symmetrizable cmf and (θ, m) be an admissible pair associated with it. M enjoys the property (2.5) provided that either of the followings holds:

(a) M is irreducible,

(b) $m \ll d d^c u \wedge \theta$ holds on D for some $u \in \text{PSH}(D) \cap L_{loc}^\infty(D)$.

3. Proofs

Proof of Theorem(2.3)

Let $\xi \in \Gamma^\theta$ and take $w \in \text{PSHB}(D)$, $\varphi \in E(D)$ and open $U \subset \mathbb{C}^n$ containing ξ as stated in the definition (2.4) of Γ^θ .

We first claim that if $q = -\log(-\varphi)$ and $q(Z_t) - q(Z_0) = m_t + A_t$ is Doob-Meyer's decomposition of the continuous semi-martingale $q(Z_t)$ under P_Z , then

$$(3.1) \quad P_Z[\lim_{t \uparrow \xi} A_t = +\infty] = 1 \quad \text{q.e. } z \in D.$$

To see this, note that $q(Z_t)$ is represented as

$$(3.2) \quad q(Z_t) - q(Z_0) = B(\langle m \rangle_t) + A_t,$$

where $B(t)$ is a 1-dimensional Brownian motion starting at 0 and

$\langle m \rangle_t$ is the quadratic variation process for m_t . Since the smooth measures associated with $\langle m \rangle_t$ and A_t are $dq \wedge dd^C q \wedge \theta$ and $dd^C q \wedge \theta$, respectively, and the 1st measure is dominated by the 2nd one (for the proof, see [KT:Lemmas 2.2, 2.3]), we have $\langle m \rangle_t \leq A_t, t \geq 0$. Therefore, if $A_{\xi^-} < +\infty$, then the right hand side of (3.2) remains finite as $t \uparrow \xi$. On the other hand, by virtue of Assumption (2.5), the left hand side of (3.2) tends to infinity as $t \uparrow \xi$. Thus, we obtain (3.1).

Let $w^* = w - (-\varphi)^{1/2}$. It is trivial that $w^* \in \text{PSHB}(D)$. Take an increasing sequence $\{O_k\}$ of relatively compact open subsets of D such that $\overline{O_k} \subset D$ and $D = \bigcup_{k=1}^{\infty} O_k$ and define $D_k = \bigcup O_k$. Note that $\sup\{\varphi(z) \mid z \in O_k\} < 0$ for each k and $dd^C(-(-\varphi)^{1/2}) \geq \frac{1}{4}(-\varphi)^{1/2} dd^C(-\log(-\varphi))$ on D , where for $(1,1)$ -currents $\psi^1 = \frac{\psi^1}{\alpha\bar{\beta}} dz^\alpha \wedge \sqrt{-1} d\bar{z}^\beta$, $i=1,2$, we denote $\psi^1 \geq \psi^2$ if $\sum_{\alpha,\beta=1}^n \eta^{\alpha\bar{\beta}} (\frac{\psi^1}{\alpha\bar{\beta}} - \frac{\psi^2}{\alpha\bar{\beta}})$ is a positive measure for any $\eta \in \mathbb{C}^n$. Thus, by a straightforward computation, it follows from the definition of Γ^θ that there is an $\varepsilon_k > 0$ such that

$$(3.3) \quad dd^C w^* \wedge \theta \geq \varepsilon_k dd^C q \wedge \theta \quad \text{on } D_k \quad \text{for each } k.$$

Due to [FO:Lemma 7], we see that $\varepsilon_k q - w^*$ is locally in $\text{Dom}(\delta^M)$ and δ^M -quasi-continuous. Moreover, the same lemma and (3.3) yields that $\varepsilon_k q - w^*$ is δ^M -superharmonic on D_k . Therefore, by virtue of [FO2:Theorem 9.3], we obtain

$$E_Z [(\varepsilon_k q - w^*)(Z_{\tau_K \wedge T})] \leq (\varepsilon_k q - w^*)(Z) \quad \text{q.e. } Z \in D$$

for every compact $K \subset D$ and $T > 0$, where E_Z stands for the expectation with respect to P_Z and $\tau_K = \inf\{t > 0 \mid Z_t \notin K\}$. Since $E_Z[q(Z_{\tau_K \wedge T}) - q(z)] = E_Z[A_{\tau_K \wedge T}]$ q.e. z , letting $K \uparrow D_k$ and $T \uparrow \infty$, we have

$$(3.4) \quad E_Z[A_{\tau_{D_k}}] \leq (2/\varepsilon_k) \|w^*\|_\infty < +\infty.$$

If we set $B_k = \{Z_t \in D_k \text{ for every } t \in [0, \xi)\}$, then $\tau_{D_k} = \xi$ on B_k . Thus, it follows from (3.1) and (3.4) that $P_Z(B_k) = 0$, q.e. z . Noting that $\{\lim_{t \uparrow \xi} Z_t \in \partial D \cap U\} \subset \bigcup_{k=1}^\infty B_k$, we conclude that $P_Z[\lim_{t \uparrow \xi} Z_t \in \partial D \cap U] = 0$ q.e. z . This completes the proof, because Γ^θ is covered with countable numbers of such U 's.

Proof of Theorem (2.9)

We first assume that M is irreducible. Suppose that M is non-transient. Then M is recurrent and hence, due to [S] (also see [F]), $P_Z[\sigma_G < +\infty] = 1$ q.e. z for every open $G \subset D$, where $\sigma_G = \inf\{t > 0 : Z_t \in G\}$. Take open subsets G_1 and G_2 of D such that $\text{dist}(G_1, G_2) > 0$. Then it follows from the above fact and the strong Markov property that Z_t visits G_1 and G_2 infinitely often P_Z -a.s., q.e. z and which contradicts to the existence of $\lim_{t \uparrow \xi} Z_t$. Thus M is transient and hence (2.5) follows.

We now proceed to the proof of the 2nd assertion. Thus assume that $dm \leq dd^C u \wedge \theta$ on D for some $u \in \text{PSH}(D) \cap L_{loc}^\infty(D)$. Then $t \leq A_t^u$, where $u(Z_t) - u(Z_0) =$ a martingale $+ A_t^u$ is Doob-Meyer's decomposition of the semi-martingale $u(Z_t)$. Therefore, for any

compact $K \subset D$,

$$E_Z[\tau_K] \leq E_Z[A_{\tau_K}] = E_Z[u(Z_{\tau_K}) - u(z)] \leq u(z) + \sup_{y \in K} |u(y)| < +\infty$$

This implies $P_Z[\tau_K < +\infty] = 1$ and hence (2.5) follows.

4. An application to the complex Monge-Ampère equation In this section, we consider an application of our Γ^θ to the complex Monge-Ampère equation. All results we are going to discuss have been already obtained in [KT] and what is new is that our argument in this section is based on Theorem (2.3) which is more general than [KT: Theorem 2.1] that played an essential role in the investigation in [KT]. Suppose that the bounded pseudoconvex domain D possesses a family $\{p_i\}_{i=1}^N \subset \text{PSHB}(D)$ satisfying the following conditions:

$$(4.1) \quad p_i < 0 \quad \text{on } D, \quad i=1, \dots, N,$$

$$(4.2) \quad \prod_{i=1}^N p_i(z) \rightarrow 0 \quad \text{as } z \rightarrow \partial D,$$

$$(4.3) \quad \text{dd}^C(-\sum_{i=1}^N \log(-p_i)) \geq C_D \text{dd}^C |z|^2 \quad \text{for some } C_D > 0 \text{ on each relatively compact } D' \subset D \text{ with } \bar{D}' \subset D.$$

Due to [KT: Lemma 2.1], we see that $p^* = -\prod_{i=1}^N (-p_i)^{1/2N}$ is in $E(D)$.

Moreover, the argument similar to [O: Lemma 3] implies that

$(\theta^*, m^*) \equiv ((\text{dd}^C q^*)^{n-1}, (\text{dd}^C q^*)^n)$ is admissible, where $q^* = -\log(-p^*)$

and for $\varphi \in \text{PSH}(D) \cap L^2_{\text{loc}}(D)$ the closed positive current $(dd^c \varphi)^k$ of bidegree (k, k) , $1 \leq k \leq n$, is defined inductively by

$$(4.4) \quad \int_D \eta \wedge (dd^c \varphi)^k = \int_D \varphi \, dd^c \eta \wedge (dd^c \varphi)^{k-1}$$

for every C_0^∞ $(n-k, n-k)$ -form η on D . We denote by $M^* = (Z_t, \xi, P_Z^*)$ the C_0^∞ -regular, symmetrizable cmd associated with (θ^*, m^*) . By virtue of Theorem (2.9), we notice that M^* enjoys the property (2.5) and hence

$$(4.5) \quad P_Z^*[\lim_{t \uparrow \xi} Z_t \in \partial D \setminus \Gamma^{\theta^*}] = 1 \quad \text{q.e. } z \in D.$$

Define an open subset $\tilde{\Gamma}$ of ∂D by

$$\tilde{\Gamma} = \{ \xi \in \partial D \mid dd^c w \wedge \theta^* \geq dm^* \text{ on } U \cap D \text{ for some } w \in \text{PSH}(D) \text{ and open } U \subset \mathbb{C}^n \text{ containing } \xi \}.$$

Then $\tilde{\Gamma} \subset \Gamma^{\theta^*}$. Combining this with (4.5), we have

$$(4.6) \quad P_Z^*[\lim_{t \uparrow \xi} Z_t \in \partial D \setminus \tilde{\Gamma}] = 1 \quad \text{q.e. } z \in D.$$

By the same argument as in the proof of [KT: Theorem 3.1], we deduce from (4.6) the following:

(4.7) Theorem ([KT: Theorem 3.1]) *Assume that $u, v \in \text{PSH}(D)$ satisfy that*

$$(4.8) \quad (dd^c u)^n \leq (dd^c v)^n \quad \text{on } D,$$

$$(4.9) \quad \liminf_{z \rightarrow \xi, z \in D} (u-v)(z) \geq 0 \quad \text{for every } \xi \in \partial D \setminus \tilde{\Gamma},$$

$$(4.10) \quad dd^c(u+v) \leq C \left\{ \prod_{i=1}^N (-p_i) \right\}^\alpha dd^c q^* \quad \text{on } D \cap V$$

for some $C \geq 0, \alpha > 0$ and open $V \subset \mathbb{C}^n$ with $V \cap \partial D = \tilde{\Gamma}$. Then

$$(4.11) \quad u(z) \geq v(z) \quad \text{for every } z \in D.$$

(4.12) Corollary The complex Monge-Ampère equation:

$$(dd^c u)^n = f dz,$$

(4.13)

$$\lim_{z \rightarrow \xi, z \in D} u(z) = \varphi(\xi) \quad \text{for every } \xi \in \partial D \setminus \tilde{\Gamma},$$

where $f \in L_{loc}^\infty(D)$, $f \geq 0$, dz is the Lebesgue measure on D and $\varphi \in C(\tilde{\Gamma})$, possesses at most one solution $u \in \text{PSHB}(D)$ satisfying

$$(4.14) \quad dd^c u \leq C \left\{ \prod_{i=1}^N (-p_i) \right\}^\alpha dd^c q^* \quad \text{on } V \cap D$$

for some $C \geq 0, \alpha > 0$ and open $V \subset \mathbb{C}^n$ such that $V \cap \partial D = \tilde{\Gamma}$.

Before closing this section, we give a comment on the condition (4.14). We assume that $f \equiv 0$ and $\varphi \equiv a$ (constant) in (4.13). Then $u \equiv a$ is the only one solution to (4.13) satisfying

(4.14). In other words, for every non-trivial solution $u \in \text{PSHB}(D)$ to (4.13) with $f=0$ and $\varphi=a$, $dd^c u$ grows faster than $\{\prod_{i=1}^N (-p_i)\}^\alpha dd^c q^*$, $\alpha > 0$, near $\tilde{\Gamma}$. For example, let $D = \{z = (z_1, z_2) \in \mathbb{C}^2 : |z_1| \vee |z_2| < 1\}$ and $p_i(z) = |z_i|^2 - 1$, $i=1,2$. Then $\tilde{\Gamma} = \{|z_1|=1, |z_2| < 1\} \cup \{|z_1| < 1, |z_2|=1\}$ and the condition (4.14) is equivalent to

$$(4.14)' \quad dd^c u \leq C \left\{ \frac{(1-|z_2|^2)^\alpha}{(1-|z_1|^2)^{\alpha-2}} \sqrt{-1} dz_1 \wedge d\bar{z}_1 + \frac{(1-|z_1|^2)^\alpha}{(1-|z_2|^2)^{\alpha-2}} \sqrt{-1} dz_2 \wedge d\bar{z}_2 \right\}.$$

Let $g(x) = \sum_{j=0}^d c_j x^j$ be a polynomial on \mathbb{R}^1 such that $c_j \geq 0$, $0 \leq j \leq d$, $c_d > 0$ and $\sum_{j=1}^d c_j = 1$. By a straightforward calculation, we see that $u(z) = g(|z_1|)$ satisfies (4.13) with $f=0$ and $\varphi=1$ but does not satisfy (4.14)'.

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