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AN INTEGRAL REPRESENTATION THEOREM FOR THE HELMHOLTZ EQUATION

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§1. The purpose of this paper is to show the integral representation for positive solutions of the Helmholtz equation \((\Delta - I)f = 0\) on \((0, \infty)^n \times \mathbb{R}^{N-n}\) by a passage to the theory of the heat equation. In the case \(n = 0\), it is well-known (see for example [2] and [3]) that every positive solution has an integral representation

\[
f(X) = \int_{S^{N-1}} \exp(\langle X, A \rangle) \, d\mu(A)
\]

where \(\mu\) is a positive measure on the sphere. We give here a new proof of this fact as an illustration of our method. Let \(f > 0\) be a solution of \(\Delta f = f\) on \(\mathbb{R}^N\). Then the function \(u(X, t) = e^{tf(X)}\) satisfies \(\Delta u = \frac{\partial u}{\partial t}\) on \(\mathbb{R}^N\times\mathbb{R}\). Hence by the integral representation theorem for positive solutions of the heat equation ([1, p.374]) there is a positive measure \(\mu\) on \(\mathbb{R}^N\) such that

\[
u(X, t) = \int_{\mathbb{R}^N} \exp(\langle X, A \rangle + t\|A\|^2) \, d\mu(A).
\]

Since \(0 = (\Delta - I)^2 f(X) = \int (\|A\|^2 - 1)^2 \exp(\langle X, A \rangle + t(\|A\|^2 - 1)) \, d\mu(A)\), we have \(\text{supp}(\mu) \subset S^{N-1}\), so that (1) is obtained.

In section 2 we describe our main theorem for general \(n \geq 1\). After giving the integral representation theorems for the heat equation in...
section 3, we prove the theorem in section 4. Finally we make a remark about the minimal Martin boundary at infinity with respect to the Helmholtz equation.

§2. Given integers $N$ and $n$ with $1 \leq n \leq N$, let $D = (0,\infty)^N \times \mathbb{R}^{N-n} = \{X = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^N; x_i > 0$ for $i = 1, 2, \ldots, n\}$. The Green function of the Helmholtz equation $\Delta - I$ on $D$ is given by

$$G(X,Y) = \int_0^\infty e^{-t} \left( \prod_{i=1}^N \{ w(x_i - y_i, t) - w(x_i + y_i, t) \} \right) dt$$

where $w(x,t) = (4\pi t)^{-1/2} \exp(-x^2/4t)$ if $t > 0$, and $= 0$ if $t \leq 0$.

Now, for each $A = (a_1, \ldots, a_n) \in \partial D$, we define

$$H_1(X,A) = \left( \prod_{i \in \tau(A)} \frac{\partial}{\partial y_i} \right) G(X,Y) \bigg|_{Y=A}$$

where $\tau(A) = \{i; 1 \leq i \leq n$ and $a_i = 0\}$.

For every subset $\Sigma \subset \{1, 2, \ldots, n\}$, we put $\Sigma_1 = \{1, 2, \ldots, n\} - \Sigma$ and $S_\Sigma = \{A \in \mathbb{R}^{N-1}; a_i = 0$ for any $i \in \Sigma$ and $a_i > 0$ for any $i \in \Sigma_1\}$. For each $A \in \partial D \cap \mathbb{R}^{N-1}$, we also define

$$H_2(X,A) = \prod_{i \in \Sigma} x_i \prod_{i \in \Sigma_1} \sinh(a_i x_i) \prod_{i=n+1}^N \exp(a_i x_i) \text{ if } A \in S_\Sigma.$$  

Observe that $H_j(\cdot, A)$, $j = 1, 2$, are positive solutions of the Helmholtz equation on $D$.

We now state the theorem in this paper.
Theorem. For every positive solution \( f \) of the Helmholtz equation on \( D = (0,\infty)^n \times \mathbb{R}^{N-n} \), there are unique Borel measures \( \mu_1 \) on \( \partial D \) and \( \mu_2 \) on \( \overline{D} \cap S^{N-1} \) such that

\[
(2) \quad f(X) = \int H_1(X,A) \, d\mu_1(A) + \int H_2(X,A) \, d\mu_2(A).
\]

Furthermore, if \( f \) is continuous on \( \overline{D} \) then \( d\mu_1(A) = f(A) \, d\sigma(A) \), where \( d\sigma(.) \) is the surface measure on \( \partial D \).

§3. In this section we give integral representation theorems for the heat equation. Following [1], a solution of the heat equation will be said to be parabolic.

For \( x, t \in \mathbb{R} \) and \( a \geq 0 \), we put

\[
k(x,t,a) = \begin{cases} 
w(x-a,t) - w(x+a,t) & \text{if } a > 0 \\
\frac{X}{t} w(x,t) & \text{if } t = 0, a = 0
\end{cases}
\]

and

\[
k^*(x,t,a) = \begin{cases} 
sinh(ax) \exp(ta^2) & \text{if } a > 0 \\
x & \text{if } a = 0.
\end{cases}
\]

Let \( D = (0,\infty)^n \times \mathbb{R}^{N-n} \) as before. For each \( (X,t) \in D \times (-\infty,\infty) \) and \( (A,s) \in D \times [-\infty,\infty) \) we define
The following was proved in [4, Theorems 2.2 and 3.4] in the case $n = 1$ (see also [5]), and a similar proof can be carried out for arbitrary $n \geq 1$ so that we have

**Proposition 1.** For every positive parabolic function $u$ on $D \times (0, \infty)$, there is a unique Borel measure $\mu$ on $\partial (D \times (0, \infty))$ such that

$$u(X,t) = \int K((X,t),(A,s))d\mu(A,s).$$

In particular if $u$ is continuous on $\overline{D} \times [0, \infty)$ then $d\mu(A,s) = u(A,s)d\sigma(A,s)$, where $d\sigma(.,.)$ is the surface measure of $\partial (D \times (0, \infty))$, and if $u$ is continuous on $\overline{D} \times (0, \infty)$ then $d\mu(A,s) = u(A,s)d\sigma(A)ds$ on $\partial D \times (0, \infty)$, where $d\sigma(.)$ is the surface measure on $\partial D$.

By the Appell transform the integral representation on $D \times (-\infty, 0)$ was given in [4, Theorem 4.1] (in the case $n = 1$). Since this method is available for arbitrary $n \geq 1$, we also see

**Proposition 2.** For every positive parabolic function $u$ on $D \times (-\infty, 0)$ there is a unique Borel measure $\mu$ on $\partial D \times (-\infty, 0) \cup \overline{D} \times \{-\infty\}$ such that
(3) \[ u(X,t) = \int K((X,t),(A,s))d\mu(A,s). \]

In particular if \( u \) is continuous on \( \overline{D} \times (-\infty,0) \) then \( d\mu(A,s) = u(A,s)d\sigma(A)ds \) on \( \partial D \times (-\infty,0) \).

We remark here that the second assertion is deduced from the last assertion in Proposition 1 by applying the Appell transform.

Before returning to the Helmholtz equation, we make an observation on the Martin boundary of \( D \times (-\infty,0) \) with respect to the heat equation. (For details, we refer to [1, p.262-383]). Let \( A_1 = (a_1, a_2, \ldots , a_n) \) with \( a_i = 1, 1 \leq i \leq n \) and \( A = 0, n+1 \leq i \leq N \). Then \((A_1,0),(D \times (-\infty,0))\) is a Martin point set pair ([1, p.359]). By the same manner as in [1, p.374-375, in the case \( N = n = 1 \)] we see that the Martin boundary \( \partial^M(D \times (-\infty,0)) \) for this pair is \( \partial D \times (-\infty,0) \cup \overline{D} \times \{-\infty\} \cup \{0_{\infty}\} \) and the Martin kernel is given by

\[ K^*((X,t),(A,s)) = \frac{K((X,t),(A,s))}{K((A_1,0),(A,s))} \]

for \((A,s) \in \partial D \times (-\infty,0) \cup \overline{D} \times \{-\infty\}\) and \( K^*((X,t),0_{\infty}) = 0 \). In the Martin topology, \((Y,r) \in D \times (-\infty,0)\) tends to \((A,s) \in \partial D \times (-\infty,0)\) if and only if \((Y,r) + (A,s), (Y,r)\) tends to \((A,-\infty) \in \overline{D} \times \{-\infty\}\) if and only if \( r \to -\infty\) and \( Y/(-r) \to A\), and \((Y,r)\) tends to \( 0_{\infty}\) if and only if \( r \to 0\) or \( |Y|/(1-r) \to \infty\). Thus, on \( \partial D \times (-\infty,0) \) the Martin topology coincides with the Euclidean topology. Similarly to [1, p.367], we also see that \( 0_{\infty}\) is the only non-minimal Martin boundary point. If \( u \) is positive parabolic on \( D \times (-\infty,0) \) and \( p^fu(A_1,0) < \infty \) (the parabolic fine limit at
(A_1,0), cf. [1, p.359]), then there is a unique Borel measure \( \mu^* \) on 
\( \mathcal{M}(D \times (-\infty,0)) \) with 
\[ \int d\mu^* = pfu(A_1,0) \] 
such that

\[
(4) \quad u(X,t) = \int K^{*}(X(\tau),(A,s))d\mu^*(A,s).
\]

§4. In this section we give a proof of the theorem. Now, let \( f > 0 \) be a solution of \( \Delta f = f \) on \( D = (0,\infty)^{n} \times R^{N-n} \).

We first assume that \( f \) is continuous on \( \overline{D} \). Then the function 
\( u(X,t) = e^{tf}(X) \) is continuous on \( \overline{D} \times (-\infty,0) \) and parabolic on \( D \times (-\infty,0) \).

By Proposition 2, there is a Borel measure \( \mu_2 \) on \( \overline{D} \) (from now on we identify \( \mathcal{M}(D \times \{-\infty\}) \) with \( \overline{D} \)) such that

\[
(5) \quad u(X,t) = \int_{-\infty}^{0} K((X,t),(A,s))e^{sf}(A)d\sigma(A) + \int K((X,t),(A,-\infty))d\mu_2(A).
\]

An elementary calculation shows that for each \( A \in 3D \)

\[
(6) \quad e^{-t} \int_{-\infty}^{0} K((X,t),(A,s))e^{sf}ds = \int_{0}^{\infty} e^{-t}K((X,t),(A,0))dt = H_1(X,A),
\]

which also implies that \( e^{-t}\int K((X,t),(A,-\infty))d\mu_2(A) \) is independent of \( t \) and is a solution of \( \Delta f = f \). It follows that \( \text{supp}(\mu_2) \subset \overline{D} \cap S^{N-1} \), for

\[
0 = (\Delta - I)^2 \int_{\overline{D}} K((X,t),(A,-\infty))d\mu_2(A)
\]

\[
(7) \quad = \int_{\overline{D}} (|A|^2 - 1)^2 K((X,t),(A,-\infty))d\mu_2(A).
\]
Since for each \( A \in \overline{D} \cap S^{n-1} \)

\[
\lim_{t \to 0} K((X,t),(A,-\infty)) = H_2(X,A) \text{ (increasingly)},
\]

we have the second part of the Theorem by letting \( t \to 0 \) in (5).

In the general case, we put

\[
f_m(X) = f(x_1^{1/m}, x_2^{1/m}, \ldots, x_n^{1/m}, x_{n+1}, \ldots, x_N)
\]

and \( u_m(X,t) = e^{tf_m}(X) \) for each \( m \geq 1 \). Then \( f_m \) is continuous on \( \overline{D} \) and satisfies the Helmholtz equation on \( D \). Hence by (4) and the above proof, there exists a Borel measure \( \mu_{2,m} \) on \( \overline{D} \cap S^{n-1} \) such that

\[
e^{tf_m}(X) = \int K^*((X,t),(A,s))d\mu_{1,m}^*(A,s) + \int K^*((X,t),(A,-\infty))d\mu_{2,m}^*(A)
\]

\[
= \int \int K((X,t),(A,s))e^{sf_m}(A)d\sigma(A) + \int K((X,t),(A,-\infty))d\nu_{2,m}(A),
\]

where \( \mu_{1,m}^* = K((A_1,0),(A,s))e^{sf_m}(A)d\sigma(A) \) and \( \mu_{2,m}^* = K((A_1,0),(A,-\infty))d\nu_{2,m}(A) \). Since \( pf_{m}(A_1,0) = \lim_{t \to 0} e^{tf_m}(A_1) = f_m(A_1) \) is bounded in \( m \), \( (\mu_{1,m}^*)^{\infty}_{m=1} (i = 1, 2) \) is a vaguely bounded sequence of positive measures on the Martin boundary \( \partial^M(D^\subset(-\infty,0)) \), so that we may assume that this has a vague limit \( \mu^*_{j} (i = 1, 2) \). Then we see that \( \text{supp}(\mu_{2}^*) \subset \overline{D} \cap S^{n-1} \) and

\[
\lim_{m \to \infty} \int K((X,t),(A,-\infty))d\nu_{2,m}(A) = \int K^*((X,t),(A,-\infty))d\nu_{2}(A).
\]

Now, we denote by \( \mu_{1,1}^{**} \) and \( \mu_{1,2}^{**} \) the restrictions of the measure \( \mu_{1}^{**} \)
to $\mathbb{R}^3 \times (-\infty, 0)$ and to $\overline{D}$, respectively. We shall show that there is a measure $\mu_1$ on $D$ such that $\mu_{1,1}^*(A, s) = K((A_1,0), (A,s))e^s d\mu_1(A)ds$. Let $\psi$ be an arbitrary continuous function on $\mathbb{R}^3 \times (-\infty, 0)$ with compact support and fix $-\infty < s_0 < 0$. We can easily check that the function $\psi(A,s)e^{\psi K((A_1,0), (A,s))/K((A_1,0), (A,s_0))}$ in $(A,s)$ is continuous and has compact support on $\mathbb{R}^3 \times (-\infty, 0)$ and that there is a constant $C = C(\psi, s_0) > 0$ such that $e^{\psi K((A_1,0), (A,s))} \geq CK((A_1,0), (A,s_0))$ on $\text{supp}(\psi)$. Since

$$f_m(A) \geq \int_{\mathbb{R}^3 \times (-\infty, 0)} K((A_1,0), (A,s))e^s f_m(A)d\sigma(A)ds$$

$$\geq C \iint K((A_1,0), (A,s_0))f_m(A)d\sigma(A)ds, \quad \text{supp}(\psi)$$

we may assume that $K((A_1,0), (A,s_0))f_m(A)d\sigma(A)$ converges vaguely to a Borel measure $\tilde{\mu}$ on $\mathbb{R}^3$ as $m \to \infty$. Then

$$\int \psi d\mu_{1,1}^* = \lim_{m \to \infty} \int \psi d\mu_{1,m}^*$$

$$= \lim_{m \to \infty} \iint \psi(A,s)e^{\psi K((A_1,0), (A,s))/K((A_1,0), (A,s_0))}f_m(A)d\sigma(A)ds$$

$$= \iint \psi(A,s)e^{\psi K((A_1,0), (A,s))/K((A_1,0), (A,s_0))}d\tilde{\mu}(A)ds.$$

Therefore $d\mu_{1,1}^* = K((A_1,0), (A,s))e^s d\mu_1(A)ds$, where $d\mu_1(A) = (K((A_1,0), (A,s_0)))^{-1}d\tilde{\mu}(A)$.

Consequently, letting $m \to \infty$ in (9) and remarking (6) and (10), we
have

\[ e^t f(X) = \int \int K((X, t), (A, s)) e^s ds d\mu_1(A) + \int K^*((X, t), (A, -\infty)) d(\mu_{1, 2}^* + \mu_2)(A) \]

\[ = \int e^t H_1(X, A) d\mu_1(A) + \int K((X, t), (A, -\infty)) d\mu_2(A, s), \]

where \( d\mu_2(A) = (K((A, 0), (A, -\infty)))^{-1} d(\mu_{1, 2}^* + \mu_2)(A). \) By the same manner as in (7) we see supp(\( \mu_2 \)) \( \subset \bar{D} \cap S^{N-1}. \) Hence, as a consequence of (8), the desired integral representation (2) follows by letting \( t \to 0. \) Since the uniqueness of the representation measures follows from Proposition 2, we obtain our theorem.

§5. It is easily seen that our method is also available for the operator \( \Delta - cI \) (\( c: \) real constant) on \( D. \) Remark that if \( c < 0 \) there is no positive solution. In the Martin boundary theoretic view point, our result explains that the minimal Martin boundary of \( D \) at infinity with respect to \( \Delta - cI \) (i.e., the set of normalized minimal solutions which vanish at all finite boundary points) is homeomorphic to \( c(S^{N-1} \cap \bar{D}) = \{cA, A \in S^{N-1} \cap \bar{D}\}. \)

On the other hand, Landis & Nadirashvili [6] tells us that

\[ \{f; \Delta f = 0 \text{ and } f > 0 \text{ in } D_E, f = 0 \text{ on } \partial D_E\} \]

is one dimensional, where \( E \subset S^{N-1} \) is a domain with Lipschitz boundary and \( D_E = \{x \in \mathbb{R}^N; x \neq 0, x/|x| \in E\}. \) By these observations it can be conjectured that the minimal Martin boundary of \( D_E \) at infinity with
respect to $\Delta - cI$ would be homeomorphic to $cE$, but we know no other example which reinforces this conjecture.

REFERENCES


