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FINE BOUNDARY LIMITS AND MAXIMAL SEQUENCES

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Dedicated to my teacher
Professor Otto Haupt
on the occasion of his 100th birthday

Introduction. A classical result of Brelot [4] concerns the limit behavior of bounded harmonic functions at an irregular boundary point. The result states that, given a bounded domain Ω in $\mathbb{R}^n$ and an irregular boundary point z of Ω, the fine limit

$$f - \lim_{\substack{x \to z \\ x \in \Omega}} v(x)$$

exists for every lower bounded, superharmonic function v on Ω. In particular, this result applies to bounded harmonic functions on Ω. Many years later Brelot [6] improved this result by introducing the so-called maximal sequences. These are sequences (x_n) in Ω converging to z (in the ordinary sense) in such a way that the Green function G_{y_o} of Ω with a pole $y_o \in \Omega$ converges along (x_n) to

$$\limsup_{\substack{x \to z \\ x \in U}} G_{y_o}(x).$$

Brelot proved that, for a bounded harmonic function u on Ω, the sequence $(u(x_n))$ converges for every maximal sequence and that one has

$$\lim_{n\to\infty} u(x_n) = f - \lim_{\substack{x\to z\\ x\in\Omega}} u(x).$$

Recently [3] I could prove that these results remain valid in very general situations, in particular for the heat equation. In the case of the heat equation irregular boundary points appear more naturally than for the Laplace equation. Even smooth domains may have irregular boundary points for the heat equation. A typical example is the "center" of the heat ball which we will discuss shortly.

The purpose of this paper is two-fold: We intend to present the main results of [3] in an expository form. At the same time the main hypotheses of [3] will be loosened considerably. We will show that it is possible to replace Doob's convergence axiom, predominant in [3], by the weak convergence axiom which is the standard convergence property for harmonic spaces in the sense of [8]. This improvement also follows from the more sophisticated results of Hansen [11] concerning finely harmonic functions. However, our techniques remain essentially the same as in [3]. Finally we will improve some results of [3] concerning maximal sequences for the heat equation.

Throughout this paper we will work within the following framework: X is a $\mathfrak{P}$-harmonic space (in the sense of Constantinescu-Cornea [8]) with a <u>countable base</u>. We study harmonic (or superharmonic) functions on an open and relatively compact set $U \subset X$ with topological boundary $U^* = \partial U$. U^*_{irr} stands for the set of irregular boundary points. $H_f = H_f^U$ denotes the generalized solution of the Dirichlet problem for a resolutive boundary function f on U*. Limits (including upper and lower ones) with respect to the fine topology on X will be denoted by f-lim, f-lim sup, f-lim inf, respectively. All other undefined notations will be those of [3] and [8].

1. Maximal Sequences

In what follows z will denote a point in U^*_{irr}. When z is also polar, Brelot's classical result holds in the form that

$$\underset{\substack{x\to z\\ x\in U}}{\text{f-lim}}\ v(x)$$

exists for every hyperharmonic function ≥ 0 on U. Brelot's proof given in [7], p.140, remains valid without essential modification. However, we start in a different way and will not need the above result.

<u>Proposition 1.</u> One has

$$\underset{\substack{x\to z\\ x\in U}}{\text{f-lim}}\ \mu_x^U = \varepsilon_z^{\complement U}$$

in the sense of vague convergence, i.e.

$$\underset{\substack{x\to z\\ x\in U}}{\text{f-lim}}\ H_f(x) = \int f d\varepsilon_z^{\complement U}$$

for all $f \in \mathcal{C}(U^*)$.

The proof relies on the fact that

$$H_{u*}(x) = \hat{R}_u^{\complement U}(x) \qquad \text{for all } x \in U$$

whenever $u \geq o$ is a real-valued continuous superharmonic function on X and that $\hat{R}_u^{\complement U}$ is finely continuous as a superharmonic function.

From this we easily obtain the well-known Köhn-Sieveking-Lemma [10], [12]:

Corollary. There exist sequences (x_n) in U such that

(M) $$\lim x_n = z \qquad \text{and} \qquad \lim \mu_{x_n}^U = \varepsilon_z^{\complement U}.$$

It will soon become evident that in the case of classical potential theory these sequences are just those which Brelot [6] had called maximal. Therefore we define:

Definition. A sequence (x_n) is called maximal with respect to the point $z \in U^*_{irr}$ if the above condition (M) is satisfied.

Consequently, the Köhn-Sieveking-Lemma states the existence of maximal sequences for z. The following theorem characterizes these sequences.

Theorem 1. A sequence (x_n) in U converging to $z \in U^*_{irr}$ is maximal with respect to z if and only if there exists a hyperharmonic function $u \geq o$ on X having the following two properties:

(1.1) $\hat{R}_u^{\complement U}(z) < u(z)$;

(1.2) $\lim\limits_{n\to\infty} \hat{R}_u^{\complement U}(x_n) = \hat{R}_u^{\,U}(z)$.

The last condition is equivalent to

(1.2') $\lim\limits_{n\to\infty} \hat{R}_u^{\complement U}(x_n) = \liminf\limits_{\substack{x\to z\\ x\in U}} \hat{R}_u^{\complement U}(x)$.

The proof given in [3] remains essentially unchanged. However, two observations have to be made:

1. The total masses $\|\mu_x^U\|$ of the harmonic measures μ_x^U, $x \in U$, are upper bounded. This can be seen by choosing a strictly positive potential $p \in \mathcal{C}(X)$, e.g. a strict potential $p \in \mathcal{C}(X)$. Then there are constants $\alpha, \beta \in \mathbb{R}$ such that $o < \alpha \le p(x) \le \beta$ for all $x \in \bar{U}$ due to the compactness of $\bar{U}$. Integration then yields

$$\alpha\|\mu_x^U\| \le \int p d\mu_x^U < p(x) \le \beta,$$

and hence

$$\|\mu_x^U\| < \frac{\beta}{\alpha} \qquad (x \in U).$$

(In [3] we gave a proof which uses the existence of a strictly positive harmonic function on an open neighborhood of $\bar{U}$. It may not exist under our weaker assumption.)

2. In [3] we used an additional condition, namely $u|U^* \in \mathcal{C}(U^*)$, which together with (1.1) and (1.2) characterized maximal sequences. However, as W.Hansen kindly has pointed out in a private communication, this additional condition can be avoided by means of the following argument: As in [3], consider a subsequence

(x_{n_k}) of (x_n) such that $(\mu^U_{x_{n_k}})$ converges vaguely to a positive Radon measure λ on U^*. As explained in [3], λ has the form

$$\lambda = \alpha\varepsilon_z + (1-\alpha)\varepsilon_z^{\complement U}$$

for some $\alpha \in [o,1]$, and $\alpha = o$ has to be proved. We have

$$\hat{R}_u^{\complement U}(z) = \lim_{k\to\infty} \hat{R}_u^{\complement U}(x_{n_k}) \geq \lim_{k\to\infty} \hat{R}_p^{\complement U}(x_{n_k}) = \int p d\lambda$$

$$= \alpha p(z) + (1-\alpha)\hat{R}_p^{\complement U}(z)$$

for every potential $p \leq u$ in $\mathcal{C}(X)$. This implies

$$\hat{R}_u^{\complement U}(z) \geq \alpha u(z) + (1-\alpha)\hat{R}_u^{\complement U}(z)$$

since u is the limit of some increasing sequence of such potentials p. The claim $\alpha = o$ then follows from condition (1.1).

As an example, let U be a bounded domain in $\mathbb{R}^p$, $p \geq 3$. (The case $p = 2$ demands some simple modifications.) Consider the $\mathcal{P}$-harmonic space $X = \mathbb{R}^p$ with respect to the sheaf of solutions of the Laplace equation. Choose a pole $y_o \in U$ and choose for u in the above theorem $u = N_{y_o}$ the Newtonian kernel

$$N_{y_o}(x) = \|x-y_o\|^{2-p} \qquad (x \in \mathbb{R}^p).$$

Then it is an immediate consequence of the theorem that a sequence (x_n) in U converging to a point $z \in U^*_{irr}$ is maximal if and only if

$$\lim_{n\to\infty} G_{y_o}(x_n) = \limsup_{\substack{x\to z \\ x\in U}} G_{y_o}(x)$$

where $G_{y_o} = N_{y_o} - H_{N_{y_o}}$ is the Green function of U with pole $y_o \in U$. This, however, is Brelot's original definition of a maximal sequence given in [5].

2. Fine boundary limits of generalized solutions

Proposition 1 tells us that the generalized solution H_f has a fine limit at $z \in U^*_{irr}$ for every $f \in \mathcal{C}(U^*)$. Furthermore, this limit is identified with the integral $\int f d\varepsilon_z^{\complement U}$ with respect to the swept-out measure $\varepsilon_z^{\complement U}$. This behavior of H_f remains unchanged if we replace the continuous boundary function f by an arbitrary bounded resolutive boundary function f as we will see now. In order to obtain this result we have to assume that the point $z \in U^*_{irr}$ is polar.

<u>Theorem 2.</u> Under the assumption that the point $z \in U^*_{irr}$ is polar, every bounded resolutive function f on U* is $\varepsilon_z^{\complement U}$-integrable and H_f has $\int f d\varepsilon_z^{\complement U}$ as a fine limit at z:

$$(2.1) \qquad \text{f-}\lim_{x \to z} H_f(x) = \int f d\varepsilon_z^{\complement U}.$$

The proof of this theorem given in [3] made use of Doob's convergence axiom in an essential way in order to derive the $\varepsilon_z^{\complement U}$-integrability of f. However, only mild modifications of the original proof have to be made in order to prove the theorem under the present assumptions.

P r o o f. As in [3], pp. 347-348, we can see that (2.1) holds when the bounded function f is upper or lower semicontinuous on U* and vanishes in a neighborhood of z.

Suppose now that f (besides being bounded resolutive) vanishes in a neighborhood of z. Then we observe that for every bounded lower semicontinuous function $\varphi \geq f$ on U* there exists a lower semicontinuous function φ' on U* which also vanishes in a neighborhood of z and satisfies $\varphi \geq \varphi' \geq f$. In fact, we choose a closed neighborhood V of z in U* such that f vanishes in a neighborhood V' of V in U*. Then $\varphi' = \varphi 1_{\complement V}$ has all desired properties since $\varphi(y) \geq f(y) = o$ for all $y \in V'$. Consequently, $\int^* f d\varepsilon_z^{\complement U}$ is the infimum of the integrals $\int \varphi d\varepsilon_z^{\complement U}$ where $\varphi \geq f$ is bounded, lower semicontinuous and vanishes in a neighborhood of z. Since (2.1) holds for this type of functions we obtain - by applying this argument to f and -f -

$$\int_* f d\varepsilon_z^{\complement U} \leq f\text{-}\lim\inf H_f(x) \leq f\text{-}\lim\sup H_f(x) \leq \int^* f d\varepsilon_z^{\complement U}.$$

This, however, proves that (2.1) is valid for all bounded Borel functions f on U* which vanish in a neighborhood of z. This observation enables us to treat now also our bounded resolutive function f which vanishes in a neighborhood of z. Resolutivity of f is equivalent to the μ_x^U-integrability of f for all $x \in U$. Hence, given $x \in U$, there exist bounded Borel functions g_x and h_x on U* vanishing in a (fixed) neighborhood of z such that

(2.2) $g_x \leq f \leq h_x$

and

(2.3) $\int g_x d\mu_x^U = \int f d\mu_x^U = \int h d\mu_x^U.$

Now we can use a trick which was also used by Hansen [11]: We choose a countable dense subset $D \subset U$ and choose the functions g_x, h_x for each $x \in D$ as before. Then

$$g := \sup_{x \in D} g_x \qquad \text{and} \qquad h = \inf_{x \in D} h_x$$

are again Borel functions on U* vanishing in a neighborhood of z such that

(2.4) $$g \le f \le h$$

and

(2.5) $$\int g d\mu_x^U = \int f d\mu_x^U = \int h d\mu_x^U \qquad (x \in D).$$

But

$$x \to \int g d\mu_x^U = H_g(x) \quad \text{and} \quad x \to \int h d\mu_x^U = H_h(x)$$

are harmonic, hence continuous functions. Therefore (2.5) implies the equality

$$H_g(x) = H_h(x) \qquad \text{for all } x \in U.$$

Finally we can pass to the fine limit which yields

$$\int g d\varepsilon_z^{\complement U} = \text{f-}\lim_{x \to z} H_g(x) = \text{f-}\lim_{x \to z} H_h(x) = \int h d\varepsilon_z^{\complement U}.$$

This together with (2.4) proves that f is $\varepsilon_z^{\complement U}$-integrable and that (2.1) holds.

The rest of the proof given in [3], for an arbitrary bounded resolutive function f, remains unchanged.

It is not clear what can be said about unbounded resolutive functions (with or without Doob's convergence axiom). Even for classical potential theory this seems to be an open question. We also do not know whether Theorem 2 holds in full generality without assuming the polarity of the point z. However, the following weaker version of the theorem holds without the polarity of z. This observation, based on a preliminary version of [3], is due to Mrs. Ursula Schirmeier:

Remark. Assume that f is a bounded resolutive function on U^* which is continuous at $z \in U^*_{irr}$. Then f is $\varepsilon_z^{\complement U}$-integrable and (2.1) holds.

P r o o f. As in the proof of Theorem 2.4 of [3] one can see that

$$(2.2) \qquad \int f d\varepsilon_z^{\complement U} \le \text{f-}\liminf_{x\to z} H_f(x) \le \text{f-}\limsup_{x\to z} H_f(x) \le \int f d\varepsilon_z^{\complement U\setminus\{z\}}$$

holds for all bounded lower semicontinuous functions $f \geq o$ which vanish in a neighborhood of z. We choose a sequence (φ_n) of functions $\geq o$ in $\mathcal{C}(U^*)$ such that $\varphi_n \uparrow f$ on U^*. Then

$$\text{f-}\limsup_{x\to z} H_f(x) = \text{f-}\limsup_{x\to z} H_{f-\varphi_n}(x) + \text{f-}\lim_{x\to z} H_{\varphi_n}(x)$$

$$= \text{f-}\limsup_{x\to z} H_{f-\varphi_n}(x) + \int \varphi_n d\varepsilon_z^{\complement U}$$

since Proposition 1 holds without the polarity assumption about z. (2.2) can be applied to each function $f - \varphi_n$, $n \in \mathbb{N}$. So we have

$$\text{f-}\limsup_{x\to z} H_f(x) \le \int (f - \varphi_n) d\varepsilon_z^{\complement U\setminus\{z\}} + \int \varphi_n d\varepsilon_z^{\complement U}$$

for every $n \in \mathbb{N}$. For $n \to \infty$ this gives

$$\text{f-}\limsup_{x\to z} H_f(x) \le \int f d\varepsilon_z^{\complement U}.$$

Consequently, (2.1) holds for bounded lower semicontinuous functions $f \geq o$ vanishing in a neighborhood of z.

Then we continue with the argument of step 3 in [3], p.348, and continue as above in the proof of Theorem 2 in order to obtain (2.1) for all bounded resolutive functions f which vanish in a neighborhood of z. From this we can proceed as follows:

We may assume $f(z) = o$ by passing from f to $f-f(z)$. Then we put

$$f_n = \sup(f,\tfrac{1}{n}) + \inf(f, -\tfrac{1}{n}) \qquad (n \in \mathbb{N}).$$

The continuity of f at z (where $f(z) = o$) implies that $\{|f| < \frac{1}{n}\}$ is a neighborhood of z on which f_n vanishes. According to the result obtained so far, we have

$$\text{f-}\lim_{x\to z} H_{f_n}(x) = \int f_n d\varepsilon_z^{\complement U} \qquad (n \in \mathbb{N}).$$

From this the final result follows since

$$|f_n - f| < \frac{1}{n} \qquad \text{on } U^*,$$

so that (f_n) converges uniformly to f on U^*.

Let us also point out again that in classical potential theory Theorem 2 is due to Brelot [4],[5]. However, when Brelot studied the maximal sequences in [6] he did not not recognize the key rôle of his result.

This key rôle becomes evident from the following result:

<u>Lemma 1.</u> Assume that the point $z \in U^*_{irr}$ is polar and that (x_n)

is a maximal sequence in U with respect to z. Then, for every bounded resolutive function f on U*, the generalized solution H_f of the Dirichlet problem has a limit along the sequence (x_n), namely

$$(2.6) \qquad \lim_{n\to\infty} H_f(x_n) = \int f \, d\varepsilon_z^{\complement U}$$

Let us point out immediately that it is crucial to formulate (2.6) in this form and to combine (2.6) only after having proved it with Theorem 1. The proof itself can be copied from [3]. It remains valid in the new framework and is similar in spirit to the proof of Theorem 1.

Next we could make the combination of Lemma 1 and of Theorem 1 and state (2.6) in the more appropriate form:

$$(2.7) \qquad \lim_{n\to\infty} H_f(x_n) = \text{f-}\lim_{x\to z} H_f(x)$$

for all bounded resolutive functions. However much more is true and the proof goes via (2.7):

Theorem 3. Assume that the point $z \in U^*_{irr}$ is polar and that (x_n) is a maximal sequence in U with respect to z. Then for every bounded harmonic function u on U the following two limits exist and are equal:

$$(2.8) \qquad \lim_{n\to\infty} u(x_n) = \text{f-}\lim_{x\to z} u(x).$$

Again the proof can be copied from [3]. In the case of classical potential theory this theorem was proved by Brelot [6] with totally different methods.

3. Applications to the heat equation

We consider the harmonic space given by the solutions of the heat equation

$$(3.1) \qquad \Delta u - \frac{\partial u}{\partial t} = 0$$

on $\mathbb{R}^{p+1} = \mathbb{R}^p \times \mathbb{R}$, $p \geq 1$. In order to avoid confusions, harmonic (superharmonic) functions with respect to this new harmonic structure will be called caloric (supercaloric) functions.

The fundamental solution W of (3.1) with pole at the origin is given by

$$(3.2) \qquad W(x) = \begin{cases} \left(\frac{1}{4\pi\tau}\right)^{p/2} \exp\left(-\frac{\|\xi\|^2}{4\tau}\right) & , \tau > 0 \\ 0 & , \tau \leq 0 \end{cases}$$

where $x = (\xi, \tau) \in \mathbb{R}^p \times \mathbb{R}$ and where $\|\xi\|$ is the euclidean norm of ξ. We denote by ord x the time coordinate τ of x. By translation we obtain the fundamental solution

$$W_{x_0}(x) = W(x - x_0) \qquad (x \in \mathbb{R}^{p+1})$$

with pole $x_0 \in \mathbb{R}^{p+1}$.

In [2] we have studied the relatively compact domain

$$\Omega(z_0, c) = \{ x \in \mathbb{R}^{p+1} : W(z_0 - x) > \left(\frac{1}{4\pi c}\right)^{p/2} \}$$

where $z_0 \in \mathbb{R}^{p+1}$ and $c > 0$. $\Omega(x_0, c)$ is called the heat ball with "center" z_o and "radius" c. The center z_o turns out to be the only irregular boundary point of $\Omega(z_o,c)$. The corresponding swept-out measure $\varepsilon_{z_o}^{\complement\Omega(z_o,c)}$ is explicitely known in this case; it is the co-called Fulks measure as we have shown in [2]. Furthermore all singletons of $\mathbb{R}^{p+1}$ are polar. Consequently, Theorem 2 can be applied to this situation: the fine limit of H_f can be calculated by integration f with respect to the Fulks measure whenener f is a bounded resolutive function for $\Omega(z_o,c)$.

Let us now improve two results of [3] by means of a different approach. We consider a relatively compact open subset U of $\mathbb{R}^{p+1}$, an irregular boundary point z of U and for $y_o \in U$ the corresponding Green function on U:

$$(3.3) \qquad G_{y_o} = W_{y_o} - H_{W_{y_o}} .$$

Here $H_{W_{y_o}}$ is the generalized solution of the Dirichlet problem with $W_{y_o}|U^*$ as boundary function.

<u>Lemma 2.</u> For arbitrary $y_o \in U$ one has

$$(3.4) \qquad \limsup_{x\to z} G_{y_o}(x) = \text{f-}\lim_{x\to z} G_{y_o}(x) = W_{y_o}(z) - \int W_{y_o}\, d\varepsilon_z^U .$$

P r o o f. We choose a relatively compact open neighborhood V_o of z (in $\mathbb{R}^{p+1}$) such that $y_o \notin \overline{V}_o$, put $V = U \cap V_o$ and define a function f on V* as follows:

$$(3.5) \qquad f(y) = \begin{cases} G_{y_o}(y) , & y \in V^* \cap U \\ o , & y \in V^* \cap U^* . \end{cases}$$

Obviously, we have $f \geq o$ and $f(y) = o$ in a neighborhood of z, namely for all $y \in U^* \cap V_o$. Furthermore, f is a bounded Borel function. From [3], Proposition 2.1 we know that we have

$$(3.6) \qquad \text{f-}\lim_{x\to z} H_f^V(x) = \limsup_{x\to z} H_f^V(x)$$

for the corresponding generalized solution H_f^V. [The proof given in [3] remains valid even under the general assumptions of the previous paragraphs.] We denote by u the restriction of W_{y_o} to U^* and consider on V^* the function

$$(3.7) \qquad v(y) = \begin{cases} H_u(y), & y \in V^* \cap U \\ u(y), & y \in V^* \cap U^*. \end{cases}$$

Then by the well-known restriction lemma [1], p.130, we know that v is a resolutive function for V and that the corresponding generalized solution H_v^V equals the restriction of H_u to V. But we have

$$(3.8) \qquad f = W_{y_o} - v \qquad\qquad \text{on } V^*$$

as one can easily check. Consequently, for all $x \in V$

$$H_f^V(x) = H^V_{W_{y_o}}(x) - H_v^V(x) = H^V_{W_{y_o}}(x) - H_u(x)$$

$$= W_{y_o}(x) - H_u(x) = G_{y_o}(x),$$

since W_{y_o} is caloric (in particular continuous) on $\complement\{y_o\}$, hence in a neighborhood of $\overline{V}$. An application of (3.6) now yields

$$\limsup_{x\to z} G_{y_o}(x) = \text{f-}\lim_{x\to z} G_{y_o}(x) = \text{f-}\lim_{x\to z}(W_{y_o}(x) - H_{W_{y_o}}(x))$$

$$= W_{y_o}(z) - \text{f-}\lim H_{W_{y_o}}(x)$$

$$= W_{y_o}(z) - \int W_{y_o} d\varepsilon_z^{\complement U}$$

according to Theorem 2.

We are now able to give a simpler proof of [3], Proposition 4.2. It characterizes maximal sequences for the heat equation in a similar way as Brelot's original definition of maximal sequences for classical potential theory.

Proposition 2. Let z be again an irregular boundary point of U. Assume that the Green function G_{y_o} of U is strictly positive in the trace $W \cap U$ of a neighborhood W of z in $\mathbb{R}^{p+1}$ for some choice of the pole $y_o \in U$. Then a sequence (x_n) in U converging to z is maximal with respect to z if and only if

$$(3.9) \qquad \lim_{n\to\infty} G_{y_o}(x_n) = \limsup_{x\to z} G_{y_o}(x).$$

P r o o f. The positivity assumption about G_{y_o} implies

$$\limsup_{x\to z} G_{y_o}(x) > o$$

because otherwise G_{y_o} restricted to $W \cap U$ would be a barrier for z. Hence z would be a regular boundary point of $W \cap U$ and hence also of U. So the preceding Lemma 2 yields the inequality

$$\hat{R}^{\complement U}_{W_{y_o}}(z) = \int W_{y_o} d\varepsilon_z^{\complement U} < W_{y_o}(z).$$

Consequently, the superharmonic function $u = W_{y_o}$ on $\mathbb{R}^{p+1}$ (W_{y_o} is even a potential) satisfies condition (1.1) of Theorem 1. According to this theorem the sequence (x_n) is maximal if and only if (1.2) holds, i.e. if and only if

$$\lim_{n\to\infty} H_{W_{y_o}}(x_n) = \int W_{y_o}\, d\varepsilon_z^{\complement U}.$$

Due to the continuity of W_{y_o} in $\complement\{y_o\}$ this is equivalent to

$$\lim_{n\to\infty} G_{y_o}(x_n) = W_{y_o}(z) - \int W_{y_o}\, d\varepsilon_z^{\complement U}$$

$$= \limsup_{x\to z} G_{y_o}(x)$$

according to Lemma 2. So the result follows.

Finally we point out that the condition concerning G_{y_o} in the preceding proposition is easy to verify by choosing the pole y_o appropriately. It suffices to observe (cf. Doob [9], p.300) that $G_{y_o}(x) > o$ holds true for $x \in U$ if and only if there exists a continuous path $\gamma : [o,1] \to U$ in U connecting $x = \gamma(o)$ with $y_o = \gamma(1)$ in such a way that the function $t \to \mathrm{ord}(\gamma(t))$ is strictly decreasing.

REFERENCES

[1] BAUER, H.: Harmonische Räume und ihre Potentialtheorie. Lecture Notes in Mathematics, Vol.22. Springer Verlag, Berlin-Heidelberg-New York, 1966.

[2] BAUER, H.: Heat balls and Fulks measures. Ann. Acad. Scient. Fennicae, Ser.A 1, Math., 10, 317 - 332 (1985).

[3] BAUER, H.: Fine boundary limits of harmonic and caloric functions. Bull. Sc. math., 2e série, 109, 337 - 361 (1985).

[4] BRELOT, M.: Sur les ensembles effilés. Bull. Sc. math., 2e série, 68, 12 - 36 (1944).

[5] BRELOT, M.: Sur l'allure des fonctions harmoniques à la frontière. C.R. Acad. Sc., Paris, 220, 676 - 678 (1945).

[6] BRELOT, M.: On the behavior of harmonic functions in the neighborhood of an irregular boundary point. J. Anal. math., 4, 209 - 221 (1954/56).

[7] BRELOT, M.: Lectures on potential theory, 2nd edition. Tata Institute of Fund. Research, Lectures on Math., Vol. 19 (1967).

[8] CONSTANTINESCU, C., and A. CORNEA: Potential theory on harmonic spaces. Grundlehren der math. Wiss., Vol.158, Springer-Verlag, Berlin-Heidelberg-New York, 1972.

[9] DOOB, J.: Classical potential theory and its probabilistic counterpart. Grundlehren der math. Wiss., Vol.262. Springer-Verlag, Berlin-Heidelberg-New York, 1984.

[10] EFFROS, E.G., and J.-L. KAZDAN: Applications of Choquet simplexes to elliptic and parabolic boundary value problems. J. of differential equations, 8, 95 - 134 (1970).

[11] HANSEN, W.: Fine boundary limits of finely harmonic functions. Math. Ann., 274, 643 - 647 (1986).

[12] KÖHN, J., und M. SIEVEKING: Reguläre und extremale Randpunkte in der Potentialtheorie. Revue Roumaine de Math. pures et appl., 12, 1489 - 1502 (1967).

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