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# ON RESOLVENTS OF FUNCTION-KERNELS

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## §0. Introduction

In potential theory necessary and sufficient conditions have been studied for kernels G satisfying the domination principle to be resolvent kernels. Here we mean that G is a resolvent kernel if there exists a family $\{G_p\}_{p>0}$ of kernels such that

$$G_p - G_q = (q - p)G_pG_q \quad \text{for every } p, q > 0$$

and

$$\lim_{p\to 0} G_p = G.$$

The composition $G_pG_q$ of kernels $G_p$, $G_q$ or the convergence $\lim_{p\to 0} G_p$ of kernels $\{G_p\}$ is a suitable one for respective kernels.

For example, we can refer to [10], [3], [8] for measure kernels, [4], [7] for convolution kernels and [3], [6], [2] for diffusion kernels. Necessary and sufficient conditions obtained in these papers are 'the dominated convergence properties' or 'regularities'.

In this paper we shall consider the same problems with respect to Borel measurable function-kernels G satisfying the domination principle.

Let X be a locally compact Haudorff space with a countable base. Denote by $\mathcal{R}(G)$ be the class of all positive measures $\mu$ with compact support such that $G\mu$ are locally bounded and let $\mathcal{E}$

be a positive Radon measure on X such that $\xi_K \in \mathcal{B}(G) \cap \mathcal{B}(\check{G})$ for every compact set K. Here $\xi_K$ is the restriction of $\xi$ to K. We define the composition $G_1G_2$ relative to $\xi$ of two Borel measurable functions $G_1$, $G_2$ by

$$(G_1G_2)(x, y) = \int G_1(x, z)G_2(z, y)d\xi(Z)$$

and the convergence $\lim_{p\to 0} G_p = G$ by

$$(0.1) \qquad \lim_{p\to 0} \iint G_p(x, y)d\mu(x)d\nu(y) = \iint G(x, y)d\mu(x)d\nu(y)$$

for every $\mu \in \mathcal{B}(\check{G})$ and $\nu \in \mathcal{B}(G)$. We shall say that G is a $\xi$-resolvent kernel if there exists a family $\{G_p\}_{p>0}$ of Borel measurable function-kernels such that $\{G_p\}_{p>0}$ satisfies (0.1) and

$$\iint (G_p(x, y) - G_q(x, y))d\mu(x)d\nu(y)$$

$$= (q - p)\iint (G_pG_q)(x, y)d\mu(x)d\nu(y)$$

for every $\mu \in \mathcal{B}(\check{G})$ and $\nu \in \mathcal{B}(G)$.

Further we shall say that G possesses the $\xi$-dominated convergence property if it has the following property:

Let $u_n$, u be nonnegative Borel measurable functions on $X \times X$ and b be a positive real number. If

$$\iint (Gu_n)(x, y)d\mu(x)d\nu(y) + \iint u_n(x, y)d\mu(x)d\nu(y)$$

$$\leqq b\iint G(x, y)d\mu(x)d\nu(y)$$

and

$$\lim_{n\to\infty} \iint u_n(x, y)d\mu(x)d\nu(y) = \iint u(x, y)d\mu(x)d\nu(y)$$

for every $\mu \in \mathcal{B}(\check{G})$ and every $\nu \in \mathcal{B}(G)$,

then

$$\lim_{n\to\infty} \iint (Gu_n)(x, y)d\mu(x)d\nu(y) = \iint (Gu)(x, y)d\mu(x)d\nu(y)$$

for every $\mu \in \mathcal{B}(\check{G})$ and every $\nu \in \mathcal{B}(G)$.

In §3 and §4 we shall prove that G is a $\xi$-resolvent kernel if only if both G and $\check{G}$ possess the $\xi$-dominated convergence property. To construct a $\xi$-resolvent family $\{G_p\}$, we shall use the methods similar to that in [8].

Further we shall show in §5 that if G and $\check{G}$ possess the dominated convergence property, then do they the $\xi$-dominated convergence property for each measure $\xi$ and G is a $\xi$-resolvent kernel for each measure $\xi$.

Finally, in §6 we shall assume that G is a continuous function-kernel and we shall construct the $\xi$-resolvent of function-kernels with some degree of continuity. For this purpose we define a capacity $\gamma_\mu^E$ with respect to a compact set E and a measure $\mu \in \mathcal{B}(\check{G})$. Under the same assumptions in §4 we shall show that there exists a $\xi$-resolvent family $\{G_p\}_{p>0}$ associated with G having the further property:

There exists a negligible set N such that for each $y \in X\backslash N$ and each compact set E with $E \times \{y\} \subset (X \times X)\backslash\Delta'$, the function $x \mapsto G_p(x, y)$ is $\gamma_\mu^E$-quasi continuous.

Here $\Delta' = \{(x, y);\ G(x, y) = +\infty\}$.

## §1. Borel measurable function-kernels

Let X be a locally compact Hausdorff spacee with a countable base and G be a Borel measurable function-kernel on X, i.e., a Borel measurable mapping from $X \times X$ into $\mathbf{R}^+ \cup \{+\infty\}$ such that G is locally bounded outside the diagonal set. The adjoint kernel $\check{G}$ of G is defined by

$$\check{G}(x, y) = G(x, y).$$

We note that $\check{G}$ is also a Borel measurable function-kernel.

Now we denote by $M^+$ the set of all positive Radon measures on X. For $\mu \in M^+$ the potential $G\mu$ is defined by

$$G\mu(x) = \int G(x, y)d\mu(y)$$

and the classes of measures are defined as follows:

$$M_0^+ = \{\mu \in M^+;\ \mathrm{supp}(\mu) \text{ is compact}\},$$

$$\mathcal{E}(G) = \{\mu \in M_0^+;\ \int G\mu d\mu < +\infty\},$$

$$\mathcal{B}(G) = \{\mu \in M_0^+;\ G\mu \text{ is locally bounded}\},$$

$$\mathcal{F}(G) = \{\mu \in M_0^+;\ G\mu \text{ is finite and continuous}\}.$$

A Borel measurable set B is said to be G-negligible or simply negligible if $\mu(G) = 0$ for every $\mu \in \mathcal{E}(G)$. Obviously a Borel set B is G-negligible if and only if it is $\check{G}$-negligible. When a property holds on a set F except on a G-negligible set, we say that it holds G-n.e. on F (simply n.e. on F).

In this paper we shall assume that a Borel measurable function-kernel G has the following property:

(p) For every compact set K there exist measures $\lambda \in \mathcal{B}(G)$ and $\lambda' \in \mathcal{B}(\check{G})$ such that

$$G\lambda \geqq 1 \text{ on } K, \qquad \check{G}\lambda' \geqq 1 \text{ on } K.$$

Furthermore we shall assume that G satisfies the domination principle, i.e.,

(d) $G\mu \leqq G\nu$ on $\mathrm{supp}(\mu)$ for $\mu \in \mathcal{E}(G)$ and $\nu \in M_0^+ \Rightarrow G\mu \leqq G\nu$ on X.

The following properties on a Borel measurable function-kernel are well-known.

Proposition A ([4, Theorem 2]). A Boren measurable function-kernel G satisfies the domination principle if and only if $\check{G}$ satisfies the domination principle.

Lemma 1.1. A Borel measurable set B is G-negligible if and only if $\mu(B) = 0$ for every $\mu \in \mathcal{B}(G)$. Furthermore B is G-negligible if and only if $\mu(B) = 0$ for every $\mu \in \mathcal{B}(\check{G})$.

Let f be a nonnegative Borel measurasble function on X. If f has the following property (s), f is called G-supermedian.

(s) $\mu \in \mathcal{E}(G)$, $\nu \in M_0^+$, $G\mu \leqq G\nu + f$ on $\mathrm{supp}(\mu) \Rightarrow$

$G\mu \leqq G\nu + f$ on X.

Since G satisfies the domination principle, $G\lambda$ is G-supermedian for each $\lambda \in M_0^+$.

Now we denote by $\varepsilon_K$ the restriction of $\varepsilon \in M^+$ to a subset K of X and define a class of measures by

$$\mathcal{L}(G) = \{\xi \in M^+ : \xi_K \in \mathcal{B}(G) \cap \mathcal{B}(\check{G}) \text{ for every compact set } K\}.$$

Further, we denote by J the class of all nonnegative Borel measurable functions on X. For $f \in J$, set

$$S_f = \{x \in X;\ f(x) > 0\}.$$

Lemma 1.2. Let $\xi$ be a measure in $\mathcal{L}(G)$ and u be G-supermedian. If $f \in J$ is locally bounded on X and $G(f\xi) \leqq G(g\xi) + u$ n.e. on $S_f$ for $g \in J$, then the same inequality holds everywhere.

## §2. The space $L(\check{G}, G)$

Let u be a Borel measurable function on $X \times X$. To simplyfy the notations we write

$$\iint u\,d\mu d\nu$$

instead of

$$\iint u(x, y)d\mu(x)d\nu(y),$$

if it is well-defined.

Now we define $L(\check{G}, G)$ to be the class of all Borel measurable functions u on $X \times X$ satisfying

$$\iint |u|\,d\mu d\nu < +\infty$$

for every $\mu \in \mathcal{B}(\check{G})$ and every $\nu \in \mathcal{B}(G)$. Let $u, v \in L(\check{G}, G)$. If

$$\iint u\,d\mu d\nu = \iint v\,d\mu d\nu \text{ for every } \mu \in \mathcal{B}(\check{G}) \text{ and every } \nu \in \mathcal{B}(G),$$

then we say that u and v are G-equivalent and write

$$u = v \text{ in } L(\check{G}, G).$$

Similarly, if

$$\iint u\,d\mu d\nu \leqq \iint v\,d\mu d\nu$$

for every $\mu \in \mathcal{B}(\check{G})$ and every $\nu \in \mathcal{B}(G)$, we write

$$u \leqq v \quad \text{in } L(\check{G}, G).$$

The class $L(\check{G}, G)$ is an ordered vector space under the G-equivalence.

Using Lemma 1.1, we can easily show the following lemma.

Lemma 2.1. For $u \in L(\check{G}, G)$ the following assertions are equivalent:

(i) $u = 0$ (resp. $u \geqq 0$) in $L(\check{G}, G)$,

(ii) For every $\mu \in \mathcal{B}(\check{G})$ $y \mapsto \int u(x, y)d\mu(x)$ is equal to 0 (resp. nonnegative) n.e. on X.

(iii) For every $\nu \in \mathcal{B}(G)$ $x \mapsto \int u(x, y)d\nu(y)$ is equal to 0 (resp. nonnegative) n.e. on X.

Let $\{u_n\}$ (resp. $\{u_p\}$) be a family of $L(\check{G}, G)$ and $u \in L(\check{G}, G)$. If

$$\lim_{n\to\infty} \iint u_n d\mu d\nu = \iint u\,d\mu d\nu$$

$$\left(\text{resp. } \lim_{p\to 0} \iint u_p d\mu d\nu = \iint u\,d\mu d\nu\right)$$

for every $\mu \in \mathcal{B}(\check{G})$ and every $\nu \in \mathcal{B}(G)$, we say that $\{u_n\}$ (resp. $\{u_p\}$) converges to u in $L(\check{G}, G)$ and write

$$\text{G-}\lim_{n\to\infty} u_n = u \quad \left(\text{resp. } \text{G-}\lim_{p\to 0} u_p = u\right).$$

Similarly we shall define with respect to $\check{G}$. For example the space $L(G, \check{G})$ is the class of all Borel measurable functions u on $X \times X$ satisfying

$$\iint |u| d\mu d\nu < +\infty$$

for every $\mu \in \mathbb{B}(G)$ and every $\nu \in \mathbb{B}(\check{G})$. The equality

$$u = v \text{ in } L(G, \check{G})$$

means that

$$\iint u d\mu d\nu = \iint v d\mu d\nu$$

for every $\mu \in \mathbb{B}(G)$ and every $\nu \in \mathbb{B}(\check{G})$.

## §3. $\xi$-resolvent kernels

In this section we shall fix a measure $\xi \in \mathcal{L}(G)$.

Let u, v be nonnegative Borel measurable functions on $X \times X$. The composition uv of u and v relative to $\xi$ is defined by

$$(uv)(x, y) = \int u(x, z)v(z, y)d\xi(z).$$

Furthermore $u^n$ is inductively defined by

$$u^1(x, y) = u(x, y),$$

$$u^n(x, y) = \int u(x, z)u^{n-1}(z, y)d\xi(z).$$

We note that

$$u^n(x, y) = \int u^{n-1}(x, z)u(z, y)d\xi(z).$$

Moreover we have

$$(uv)^{\vee}(x, y) = (\check{v}\check{u})(x, y).$$

A Borel measurable function-kernel G is said to be a $\xi$-resolvent kernel, when there exists a family of Borel measurable function-kernels $\{G_p\}_{p>0} \subset L(\check{G}, G)$ with the following properties:

$(r_1)$ $G_p - G_q = (q - p)G_pG_q$ in $L(\check{G}, G)$ for every $p, q > 0$,

$(r_2)$ $G\text{-}\lim_{p\to 0} G_p = G$.

The family $\{G_p\}$ is called a $\xi$-resolvent family associated with G.

The requirement $(r_1)$ implies that $G_p \geqq G_q$ in $L(\check{G}, G)$ for $0 < p < q$. It is easy to see that

$$(3.1) \qquad G - G_p = pGG_p = pG_pG \quad \text{in } L(\check{G}, G)$$

for each $p > 0$.

Remark 3.1. If G is a $\xi$-resolvent kernel, so is $\check{G}$. In fact, since $G_pG_q = G_qG_p$ in $L(\check{G}, G)$, it follows that

$$\begin{aligned} \check{G}_p(x, y) - \check{G}_q(x, y) &= G_p(y, x) - G_q(y, x) \\ &= (q - p)(G_qG_p)(y, x) \\ &= (q - p)(\check{G}_p\check{G}_q)(x, y) \end{aligned}$$

in $L(G, \check{G})$.

A $\xi$-resolvent family has the following property.

Lemma 3.1. If $\{G_p\}_{p>0}$ is a $\xi$-resolvent family associated with G, then for each $p > 0$,

$$(3.2) \qquad \iint pG(pG_p)^m d\mu d\nu \to 0 \quad (m \to \infty)$$

for $\mu \in \mathbb{B}(\check{G})$ and $\nu \in \mathbb{B}(G)$ and

$$(3.3) \qquad pG = G\text{-}\lim_{m\to\infty} \sum_{n=1}^{m} p^n G_p^n.$$

Definition 3.1. Let $\xi \in \mathcal{L}(G)$. A Borel measurable function-kernel G is said to possess the $\xi$-dominated convergence property, if they have the following property:

If nonnegative Borel measurable functions $u_n$, $u \in L(\check{G}, G)$ satisfy $Gu_n + u_n \leqq bG$ for some $b \in R^+$ in $L(\check{G}, G)$ and $\{u_n\}$ converges to u in $L(\check{G}, G)$, then $\{Gu_n\}$ converges to Gu in $L(\check{G}, G)$.

Lemma 3.2. If G satisfies (3.1) and possesses the $\xi$-dominated convergence property, then (3.2) and (3.3) also hold.

By the same methods as that in the proof of Proposition 2.7.3 in [8] we can prove the following theorem.

Theorem 3.1. Suppose that G is a $\xi$-resolvent kernel. Then G and $\check{G}$ possess the $\xi$-dominated convergence property.

## §4. The construction of a $\xi$-resolvent

Let F be a Borel measurable subset of $X \times X$. We denote by B(F) the Banach lattice of all bounded Borel measurable functions u on F with the usual order and the norm

$$\|u\| = \sup \{|u(x, y)|;\ (x, y) \in F\}.$$

For $y \in X$ we define

$$F_y = \{x \in X;\ (x, y) \in F\}.$$

Let H be a closed sublattice of B(H) and denote by $H^+$ the class of all nonnegative functions u in H.

The following lemma can be proved by the usual methods. (cf. [9, X.T10]).

Lemmna 4.1. Let w be a function in H such that $w \geqq 1$ on F. Suppose that a positive bounded linear operator V on H has the following property:

(D) Whenever for $u, v \in H^+$, $y \in X$, $b \in \mathbf{R}^+$

$$(Vu)(\cdot, y) \leqq (Vv)(\cdot, y) + bw(\cdot, y) \text{ on } \{x \in F_y;\ u(x, y) > 0\},$$

the same inequality holds on $F_y$.

Then there exists uniquely a family $\{V_p\}_{p>0}$ of positive bounded linear operators on H such that

$$V - V_p = pVV_p = pV_pV. \tag{4.1}$$

In this section we shall fix $\xi \in \mathcal{L}(G)$. The following lemma is important.

Lemma 4.2. Let K be a compact subset of X and F be a Borel measurable subset of $K \times K$. Further, let v be a nonnegative bounded Borel measurable function on F. Then, for $p > 0$, there exists uniquely $u \in B(F)$ such that

$$p\int G(x, z)\chi_F(z, y)u(z, y)d\xi(z) + u(x, y) = v(x,y) \tag{4.2}$$

for each $(x, y) \in F$. Especially, if $x \mapsto v(x, y)$ is G-supermedian on $F_y$ for each $y \in K$, then u is nonnegative on F.

Proof. We define an operator V on B(F) by

$$(Vf)(x, y) = \int G(x, z)\chi_F(z, y)f(z, y)d\xi(z).$$

Then V is a positive bounded linear operator on B(F). Moreover by the assumption (p) there is $\lambda \in \mathcal{B}(G)$ such that $G\lambda \geqq 1$ on K. Put $w(x, y) = G\lambda(x)$ for $(x, y) \in F$. Then w satisfies (D) in Lemma 4.1. Therefore, there exists a family $\{V_p\}_{p>0}$ of positive bounded linear operators on B(F) satisfying (4.1). Set

$$u = v - pV_p v.$$

Then $u \in B(F)$ and with (4.1)

$$\begin{aligned} p\int G(x, z)\chi_F(z, y)u(z, y)d\xi(z) &+ u(x, y) \\ &= p(Vu)(x, y) + u(x, y) \\ &= pV(v - pV_p v)(x, y) + v(x, y) - p(V_p v)(x, y) \\ &= v(x, y) . \end{aligned}$$

Thus we see that u satisfies (4.2). Next, to show the uniqueness of u, let u' be an another function satisfying (4.2) and set

$$u_1(x, y) = \begin{cases} u(x, y) & (x, y) \in F \\ 0 & \text{otherwise,} \end{cases}$$

$$u_2(x, y) = \begin{cases} u'(x, y) & (x, y) \in F \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$pGu_1 + u_1 = pGu_2 + u_2 \quad \text{on } F. \tag{4.3}$$

Put $f = u_1 - u_2$. Noting that $pGf^+ + f^+ = pGf^- + f^-$, we have

$$p(Gf^+)(\cdot, y) \leqq p(Gf^-)(\cdot, y) \quad \text{on } \{x \in F_y;\ f^+(x, y) > 0\}$$

for each $y \in K$. From Lemma 1.2 it follows that

$$p(Gf^+)(\cdot, y) \leqq p(Gf^-)(\cdot, y) \quad \text{on } X,$$

whence

$$pGu_1 \leqq pGu_2 \quad \text{on } F.$$

Consequently, by (4.3)

$$u_1 \geqq u_2 \quad \text{on } F.$$

Similarly we obtain $u_1 \leqq u_2$ on F and hence $u = u'$ on F. Therefore the uniqueness of u has been shown. Finally, assume that $x \mapsto u(x, y)$ is G-supermedian on $F_y$ for each $y \in K$. Then, for a fixed point $y \in K$,

$$p\int G(x, z)\chi_F(z, y)u(z, y)d\xi(z) \leqq v(x, y)$$

on $\{x \in K;\ \chi_F(x, y)u^+(x, y) > 0\}$. From Lemma 1.2 it follows that

$$p\int G(x, z)\chi_F(z, y)u(z, y)d\xi(z) \leqq v(x, y) \text{ for every } x \in F_y.$$

By (4.3) we see that u must be nonnegative on F, which completes the proof.

Next, we shall show our main theorem.

Theorem 4.1. Suppose that G and $\check{G}$ possess the $\xi$-dominated convergence property. Then G is a $\xi$-resolvent kernel.

Proof. The proof is divided into several steps.

(I) Let $\{U_n\}$ be an exhaustion of X i.e., a family of relatively compact open sets satisfying

$$\bar{U}_n \subset U_{n+1} \quad \text{and} \quad \bigcup_{n=1}^{\infty} U_n = X.$$

Set $K_n = \bar{U}_n$ and denote by $\xi_n$ the restriction of $\xi$ to $K_n$. Note that $\xi_n \in \mathcal{B}(G)$. Put

$$F_n = \{(x, y) \in K_n \times K_n;\ G(x, y) \leqq n\}.$$

We shall fix $p > 0$. Since $x \mapsto G(x, y)$ is G-supermedian and $(x, y) \mapsto G(x, y)$ is bounded on $F_n$, by Lemma 4.2 there exists a nonnegative bounded Borel measurable function $u_n$ on $F_n$ such that

$$p\int G(x, z)u_n(z, y)\chi_{F_n}(z, y)d\xi(z) + u_n(x, y) = G(x,y) \text{ on } F_n.$$

Setting $u_n = 0$ outside $F_n$, we extend $u_n$ to a nonnegative bounded Borel measurable function on $X \times X$. By Lemma 1.2 we obtain

$$(4.4) \qquad p\int G(x, z)u_n(z, y)d\xi(z) + u_n(x, y) \leqq G(x, y) \text{ on } X \times X.$$

(II) We shall show that $\{u_n\}_{n\geqq m}$ is decreasing on $F_m$ for a fixed m. Indeed, we fix $y \in X$ and set $v = u_n - u_{n+1}$. From

$$(4.5) \qquad pGu_n + u_n = pGu_{n+1} + u_{n+1} = G \text{ on } F_n$$

it follows that

$$p(Gv^+)(\cdot, y) \leqq p(Gv^-)(\cdot, y) \quad \text{on } \{x \in X;\ v^+(x, y) > 0\}.$$

By Lemma 1.2 we obtain

$$pGv^+(\cdot, y) \leqq pGv^-(\cdot, y) \quad \text{on } X$$

and hence $pGu_n \leqq pGu_{n+1}$ on $X \times X$. Consequently, by (4.5) we have

$$u_n \geqq u_{n+1} \quad \text{on } F_n.$$

Thus we see that $\{u_n\}_{n\geqq m}$ is decreasing on $F_m$.

(III) Now we define

$$G_p(x, y) = \begin{cases} \lim_{n\to\infty} u_n(x, y) & \text{if } (x, y) \in \bigcup_{m=1}^{\infty} F_m \\ 0 & \text{otherwise.} \end{cases}$$

If $x \neq y$, then $(x, y) \in \bigcup_{m=1}^{\infty} F_m$. Further from (4.4) it follows that

(4.6) $\qquad G_p(x, y) \leqq G(x, y)$ for every $x, y \in X$.

Noting that $G\text{-}\lim_{n\to\infty} u_n = G_p$, we deduce from $\xi$-dominated convergence property

$$G\text{-}\lim_{n\to\infty} Gu_n = GG_p$$

whence by (4.6)

$$\iint pGG_p d\mu d\nu + \iint G_p d\mu d\nu = \lim_{n\to\infty} \left(\iint pGu_n d\mu d\nu + \iint u_n d\mu d\nu\right)$$

$$\geqq \lim_{n\to\infty} \iint G(x, y)\chi_{F_n}(x, y) d\mu(x) d\nu(y)$$

$$= \iint G d\mu d\nu.$$

On the other hand from (4.4) the opposite inequality holds. Hence we have

(4.7) $\qquad pGG_p + G_p = G \quad$ in $L(\check{G}, G)$.

(IV) We shall prove

(4.8) $\qquad GG_p = G_pG \quad$ in $L(\check{G}, G)$.

Since $\check{G}$ satisfies the domination principle and has the $\xi$-dominated convergence property, by the above considerations (I)-(III), there exists a family $\{G'_p\}_{p>0}$ of nonnegative Borel measurable functions on $X \times X$ such that

(4.9) $$p\check{G}G'_p + G'_p = \check{G} \quad \text{in } L(G, \check{G})$$

and

(4.10) $$\check{G}_p(x, y) \leq \check{G}(x, y) \quad \text{for every } x, y \in X.$$

Consequently

$$p\iint \check{G}d\nu d\mu = \sum_{n=1}^{\infty} \iint p^n G'^n_p d\nu d\mu \text{ for } \mu \in \mathcal{B}(\check{G}) \text{ and } \nu \in \mathcal{B}(G),$$

whence

$$p\iint G d\mu d\nu = \sum_{n=1}^{\infty} \iint p^n \check{G}'^n_p d\mu d\nu.$$

Therefore, with (4.7) we obtain

$$pGG_p + G_p = G = p\check{G}G'_p + \check{G}'_p \quad \text{in } L(\check{G}, G),$$

(4.11) $$p\int G(x, z)\{\int G_p(z, y)d\nu(y)\}d\xi(z) + \int G_p(x, y)d\nu(y)$$

$$= p\int G(x, z)\{\int \check{G}'_p(z, y)d\nu(y)\}d\xi(z) + \int \check{G}'_p(x, y)d\nu(y)$$

n.e. on X. Denote by N the negligible set of all points x at which (4.11) does not hold and set

$$h(z) = \int G_p(z, y)d\nu(y) - \int \check{G}'_p(z, y)d\nu(y).$$

Then, by (4.6) and (4.10) h is locally bounded and

(4.12) $$p\int G(x, z)h^+(z)d\xi(z) + h^+(x)$$

$$= p\int G(x, z)h^-(z)d\xi(z) + h^-(x)$$

holds for every $x \in X\backslash N$. Here $h^+(x) = \max\{h(x), 0\}$ and $h^-(x) = \max\{-h(x), 0\}$. From Lemma 1.2 we deduce

$$p\int G(x, z)h(z)d\xi(z) = 0 \quad \text{for all } x \in X$$

and hence by (4.12)

$$h(x) = 0 \text{ for all } x \in X\backslash N.$$

Thus we obtain

$$G_p = \check{G}'_p \quad \text{in } L(\check{G}, G).$$

Therefore, from (4.9) it follows that

$$GG_p = G_pG \qquad \text{in } L(\check{G}, G).$$

(V) We shall show that $\{G_p\}$ satisfies $(r_1)$. In fact, for p, q > 0 we have, by (4.7) and (IV)

$$pqG_q(GG_p) = qG_q(G - G_p) = qG_qG - qG_qG_p = G - G_q - qG_qG_p$$

in $L(\check{G}, G)$ and

$$pq(G_qG)G_p = p(G - G_q)G_p = G - G_p - pG_qG_p \qquad \text{in } L(\check{G}, G),$$

whence

$$G_p - G_q = (q - p)G_qG_p \qquad \text{in } L(\check{G}, G).$$

(VI) Finally we shall show that $\{G_p\}$ satisfies $(r_2)$. Let $\mu \in \mathcal{B}(\check{G})$. If p > q, then by $(r_1)$ we have

$$\check{G}_p\mu \leqq \check{G}_q\mu \qquad \text{n.e. on } X.$$

We define

$$h(x) = \begin{cases} \lim\limits_{n\to\infty} (\check{G}_{1/n}\mu)(x) & \text{if it exists} \\ 0 & \text{otherwise.} \end{cases}$$

Then h is Borel measurable on X and $0 \leqq h \leqq \check{G}\mu$ n.e. on X. Let $\nu \in \mathcal{B}(G)$. Noting that

$$\iint(\check{G}_{1/n}\mu)(z)qG_q(z,\ y)d\xi(z)d\nu(y) = \iint G_{1/n}(qG_q)d\mu d\nu$$

$$\leqq \iint G(qG_q)d\mu d\nu \leqq \iint G d\mu d\nu < +\infty,$$

we obtain, by (V)

$$\int(\int h(z)qG_q(z,\ y)d\xi(z))d\nu(y) = \lim_{n\to\infty} q\iint(\check{G}_{1/n}\mu)(z)G_q(z,y)d\xi(z)d\nu(y)$$

$$= \lim_{n\to\infty}(q - 1/n)\iint G_{1/n}G_q d\mu d\nu = \lim_{n\to\infty}\iint(G_{1/n} - G_q)d\mu d\nu$$

$$= \int(h - \check{G}_q\mu)d\nu.$$

Consequently, by (4.7) we have

$$\int(\check{G}\mu - h)d\nu = \iint(\check{G}\mu(z) - h(z))qG_q(z,\ y)d\xi(z)d\nu(y)$$

for all $\nu \in \mathcal{B}(G)$, whence

$$\check{G}\mu(y) - h(y) = \int(\check{G}\mu(z) - h(z))qG_q(z,\ y)d\xi(z) \quad \text{n.e. on } X.$$

Therefore we have, for every $\nu \in \mathcal{B}(G)$

$$0 \leqq \int(\check{G}\mu - h)d\nu = \iint(\check{G}\mu(z) - h(z))qG_q(z,\ y)d\xi(z)d\nu(y)$$

$$= \iint(\check{G}\mu(z) - h(z))(qG_q)^2(z,\ y)d\xi(z)d\nu(y)$$

$$\leqq \iint\check{G}\mu(z)(qG_q)^n(z,\ y)d\xi(z)d\nu(y)$$

Since $\iint G(qG_q)^n d\mu d\nu \to 0$ by Lemma 3.2, it follows that

$$\int(\check{G}\mu - h)d\nu = 0 \qquad \text{for all } \nu \in \mathcal{B}(G).$$

Therefore, we have

$$\lim_{p\to 0}\iint G_p d\mu d\nu = \lim_{n\to\infty}\iint G_{1/n}d\mu d\nu = \int h d\nu = \iint G d\mu d\nu,$$

which proves $(r_2)$. Thus we have completed the proof of Theorem 4.1.

## §5. The dominated convergence property

In this section we shall study the relation of the dominated convergence property and $\xi$-dominated convergence property.

Definition 5.1. We say that a Borel measurable function-kernel G possesses the dominated convergence property, if G has the following property:

$$\mu \in M_0^+,\ \nu_n \in M^+,\ G\nu_n \leqq G\mu,\quad \nu_n \to \nu \text{ vaguely},$$

$\Rightarrow$

$$\int \check{G}\lambda d\nu_n \quad \to \quad \int \check{G}\lambda d\nu \quad \text{for all } \lambda \in \mathcal{B}(\check{G}).$$

Lemma 5.1. Suppose that G possesses the dominated convergence property, then it also possesses the $\xi$-dominated convergence property for any $\xi \in \mathcal{L}(G)$.

Proof. Let $u_n, u$ be nonnegative Borel measurable functions in $L(\check{G}, G)$ such that

(5.1) $\quad Gu_n + u_n \leqq bG$ for some $b \in \mathbf{R}^+$ in $L(\check{G}, G)$

and $\{u_n\}$ converges to u in $L(\check{G}, G)$. Let $\xi \in \mathcal{L}(G)$ and $\nu \in \mathcal{B}(G)$. The inequality

(5.2) $\quad \int G(x, z)u_n(z, y)d\nu(y)d\xi(z) + \int u_n(x, y)d\nu(y) \leqq bG\nu(x)$

holds n.e. on X by (5.1). We denote by N the negligible set of all points x at which (5.2) does not hold and put

$$h_n(z) = \begin{cases} \int u_n(z, y)d\nu(y) & z \in X\setminus N \\ 0 & z \in N. \end{cases}$$

Since $h_n \leqq bG\nu$ on $X$, $h_n$ is locally bounded on $X$ and

(5.3) $$\int G(x, z)h_n(z)d\xi(z) \leqq bG\nu(x)$$

holds on $\{x \in X;\ h_n(x) > 0\}$. By Lemma 1.2 we see that (5.3) holds on $X$. Denote by $\lambda_n$ (resp. $\lambda$) the measure $h_n(\cdot)\xi$ (resp. $\{\int u(\cdot, y)d\nu(y)\}\xi$) with density $h_n(\cdot)$ (resp. $\int u(\cdot, y)d\nu(y)$). Then, by (5.3) we have

$$G\lambda_n(x) = \int G(x, z)h_n(z)d\xi(z) \leqq bG\nu(x) \quad \text{for all } x \in X.$$

Let $f \in C_0(X)^+$. Noting that $f\xi \in \mathcal{B}(\check{G})$ and $\{u_n\}$ converges to $u$ in $L(\check{G}, G)$, we obtain

$$\begin{aligned}\lim_{n\to\infty} \int f d\lambda_n &= \lim_{n\to\infty} \int f(z)h_n(z)d\xi(z) \\ &= \lim_{n\to\infty} \int f(z)\{\int u_n(z, y)d\nu(y)\}d\xi(z) \\ &= \int f(z)\{\int u(z, y)d\nu(y)\}d\xi(z) = \int f d\lambda.\end{aligned}$$

This shows that $\{\lambda_n\}$ converges vaguely to $\lambda$. Since $G$ satisfies the dominated convergence property, it follows that

$$\lim_{n\to\infty} \int \check{G}\mu(z)d\lambda_n(z) = \int \check{G}\mu(z)d\lambda(z)$$

for every $\mu \in \mathcal{B}(\check{G})$ and hence

$$\lim_{n\to\infty} \iint (Gu_n)d\mu d\nu = \iint (Gu)d\mu d\nu$$

for every $\mu \in \mathcal{B}(\check{G})$. Thus we see that $\{Gu_n\}$ converges to $Gu$ in $L(\check{G}, G)$. This completes the proof.

Using Lemma 5.1 and Theorem 4.1 we obtain

Theorem 5.1. Suppose that both $G$ and $\check{G}$ possess the dominated convergence property, for each $\xi \in \mathcal{L}(G)$ $G$ is a $\xi$-resolvent kernel.

## §6. Continuous function-kernels

A Borel measurable function-kernel $G$ is said to be a continuous function-kernel on $X$, if $G$ is continuous in the extended sense and strictly positive on the diagonal set $\Delta$ of $X \times X$.

The following lemma is well-known.

Proposition B (cf. [11, Proposition 2)). Suppose that a continuous function-kernel $G$ satisfies the domination principle and has the following property (n):

(n) No non-empty open set is negligible.

Then for each compact set $K$ there is $\lambda \in \mathcal{B}(G)$ such that

$$G\lambda \geqq 1 \text{ on } K.$$

Remark 6.1. By Propositions A and B we see that, if a continuous function-kernel $G$ satisfies the domination principle and has the property (n), then it has the property (p).

Let $E$ be a compact subset of $X$ and $\mu$ be a measure in $\mathcal{B}(\check{G})$ such that

$$\check{G}\mu \geqq 1 \text{ on } E.$$

Set

$$S(G) = \{\lim_{n\to\infty} G\lambda_n;\ \lambda_n \in \mathcal{B}(G),\ G\lambda_n \leqq G\lambda_{n+1}\}.$$

For a subset $B$ of $E$ we define

$$\gamma_\mu^E(B) = \inf \{\int u d\mu;\ u \in S(G),\ u \geqq \chi_B \text{ n.e. on } E\}.$$

Here we use the convention $\inf \phi = +\infty$.

Then $\gamma_\mu^E(B) = 0$ if and only if B is negligible.

Further a Borel measurable function f on E is said to be $\gamma_\mu^E$-quasi continuous if for each $\varepsilon > 0$ there exists a closed set $E_\varepsilon \subset E$ such that the restriction of f to $E_\varepsilon$ is continuous and $\gamma_\mu^E(E \backslash E_\varepsilon) < \varepsilon$. We note that the $\gamma_\mu^E$-quasi continuity does not depend on the choice of $\mu$.

Lemma 6.1. Suppose that a continuous function-kernel G satisfies the domination principle and has the property (n). Let E be a compact set and $\mu$ be a measure in $\mathcal{B}(G)$ such that $\check{G}\mu \geqq 1$. Further, let $\lambda \in \mathcal{B}(G)$. Then $G\lambda$ is $\gamma_\mu^E$-quasi continuous on E.

Using Lemma 6.1, we can prove the following theorem.

Theorem 6.1. Let G be a continuous function-kernel satisfying (n) and the domination principle. Suppose that both G and $\check{G}$ possess the $\xi$-dominated convergence properrty for a $\xi \in \mathcal{L}(G)$. Then there exists a $\xi$-resolvent family $\{G_p\}_{p>0}$ of G satisfying the following property:

(LQC) There exists a negligible set N such that for each $y \in X \backslash N$ and each compact set E with $E \times \{y\} \subset (X \times X) \backslash \Delta'$, $G_p(\cdot, y)$ is $\gamma_\mu^E$-quasi continuous on E.

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