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<th>Integral representations of weighted Beppo Levi functions (Potential Theory and Its Related Fields)</th>
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<td>Author(s)</td>
<td>AIKAWA, Hiroaki</td>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (1987), 610: 1-26</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1987-02</td>
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<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/99752">http://hdl.handle.net/2433/99752</a></td>
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<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
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<td>Textversion</td>
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Kyoto University
Integral representations of weighted Beppo Levi functions

Hiroaki AIKAWA (学習院大学理学部 相川弘明)

§1. Introduction

It is well known that functions in Sobolev spaces can be represented as Bessel potentials ([17; Chapter V, Theorem 3]). In this paper we shall consider a similar problem for weighted Beppo Levi functions.

Let $1 < p < \infty$ and let $w$ be a weight (nonnegative Lebesgue measurable function) satisfying the Muckenhoupt $A_p$ condition:

$$(A_p) \quad \sup_Q \left( \frac{1}{|Q|} \int_Q w(x) dx \right) \left( \frac{1}{|Q|} \int_Q w^{1/(1-p)}(x) dx \right)^{p-1} < \infty,$$

where the supremum is taken over all cubes $Q$ with sides parallel to the axes and $|Q|$ stands for the Lebesgue measure of $Q$ (see [1]).

By $A_p$ we denote the class of weights $w$ satisfying $(A_p)$. We write

$$\|f\|_{L^p,w} = \left( \int_{\mathbb{R}^n} |f(x)|^p w(x) dx \right)^{1/p}, \quad L^p(\mathbb{R}^n, w) = \{f; \|f\|_{L^p,w} < \infty\}.$$

By $BL_m(L^p(\mathbb{R}^n, w))$, $m \geq 1$, we denote the space of distributions whose partial derivatives of $m$-th order all belong to $L^p(\mathbb{R}^n, w)$ (see [2]). Since $w^{1/(1-p)}$ is locally integrable by $(A_p)$, it follows from Hölder's inequality and Kryloff's theorem [16; Chapitre VI, Théorème 15] that a distribution in $BL_m(L^p(\mathbb{R}^n, w))$ is a locally integrable function whenever $w \in A_p$. Therefore we call a locally integrable function in $BL_m(L^p(\mathbb{R}^n, w))$ a Beppo Levi function of order $m$ with
weight \( w \). If \( w = 1 \), then we write simply \( \|f\|_{L^p_w} \), \( L^p(R^n) \) and \( BL_m(L^p(R^n)) \) for \( \|f\|_{L^p_w} \), \( L^p(R^n, w) \) and \( BL_m(L^p(R^n, w)) \), respectively. Hereafter we limit ourselves to the case \( 1 \leq m \leq n - 1 \).

For a multiindex \( \alpha = (\alpha_1, \ldots, \alpha_n) \) we write \( |\alpha| = \alpha_1 + \cdots + \alpha_n \), \( \alpha! = \alpha_1! \cdots \alpha_n! \) and

\[
D^\alpha = \partial^{|\alpha|}/\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}.
\]

By \( h_m \) we denote the Riesz kernel \( \gamma(m)^{-1} |x|^{m-n} \) with \( \gamma(m) = \pi^{n/2} \Gamma(m/2)/\Gamma(n-m/2) \) (cf. [17; p.117]). If \( |\alpha| = m \), then \( D^\alpha h_m \) is not a locally integrable function. It will be stated in Lemma 2 in \$2\) that \( D^\alpha h_m \) is the sum of a principal value distribution \( S_\alpha \) and a multiple \( b_\alpha \delta \) of the Dirac measure \( \delta \) at the origin. It is proved in [1] that this kind of distribution is related to weights \( w \) in \( A_p \) as follows:

\begin{equation}
\|(D^\alpha h_m)_* g\|_{L^p_w} \leq \text{const.} \|g\|_{L^p_w} \quad \text{for } g \in L^p(R^n, w).
\end{equation}

Let \( D \) be the space of indefinitely differentiable functions with compact support and \( P_{m-1} \) the space of all polynomials of degree smaller than or equal to \( m - 1 \). Let \( c_\alpha = (-1)^m \alpha! / \alpha! \). Mizuta proved

Theorem A ([8; Theorem 5.2]). Let \( 2m < n \) and let \( f \in BL_m(L^p(R^n)) \). If

\begin{equation}
\text{there is a sequence } \{\psi_j\}_j \subset D \text{ such that } D^\alpha \psi_j + D^\alpha f \text{ in } L^p(R^n)
\end{equation}

for \( |\alpha| = m \),

and \( g = \sum_{|\alpha| = m} c_\alpha (D^\alpha h_m)_* D^\alpha f \) satisfies

\begin{equation}
\int_{R^n} (1 + |x|)^{m-n} |g(x)| \, dx < \infty,
\end{equation}

then

\begin{equation}
\|(D^\alpha h_m)_* g\|_{L^p_w} \leq \text{const.} \|g\|_{L^p_w} \quad \text{for } g \in L^p(R^n, w).
\end{equation}
then \( f = h_m \ast g + P \) a.e. on \( \mathbb{R}^n \) with some \( P \in P_{m-1} \).

We shall show that assumption (2) is superfluous and the theorem extends to the case when \( 1 \leq m \leq n-1 \) and \( f \in BL_m(L^p(\mathbb{R}^n, w)) \) with general \( w \in A_p \). More precisely, we shall prove

**Theorem 1.** Let \( w \in A_p \). Suppose that \( f \in BL_m(L^p(\mathbb{R}^n, w)) \) and

\[
\sum_{|\alpha|=m} c_\alpha (D^\alpha h_m) \ast D^\alpha f.
\]

If \( g \) satisfies (3), then \( f = h_m \ast g + P \) a.e. on \( \mathbb{R}^n \) with some \( P \in P_{m-1} \). Moreover this representation is unique in the sense that if \( f = h_m \ast \mu + P' \) a.e. on \( \mathbb{R}^n \), where \( P' \in P_{m-1} \) and \( \mu \) is a signed measure such that

\[
\int_{\mathbb{R}^n} (1 + |x|)^{m-n} |\mu|(x) < \infty,
\]

then \( \mu \) is absolutely continuous, \( d\mu = gdx \) and \( P' = P \).

Ohtsuka [13] proved Theorem 1 for \( m = 1 \) and \( w = 1 \) by using extremal length (see also [12] for the definition and the properties of extremal length). In case \( m > 1 \), however, the theory of extremal length is not applicable to \( BL_m(L^p(\mathbb{R}^n, w)) \), so our argument will depend on the general theory of distributions and singular integrals (see [1], [11], [16] and [17]).

Let \( \omega_{n-1} = 2\pi^{n/2}/\Gamma(\frac{n}{2}) \) be the surface area of the unit sphere in \( \mathbb{R}^n \) and let \( a_d = m/(\alpha! \omega_{n-1}) \). Mizuta proved

**Theorem B ([8; Theorem 3.1]).** Let \( f \in BL_m(L^p(\mathbb{R}^n)) \) satisfy (2). If

\[
\int_{\mathbb{R}^n} (1 + |x|)^{m-n} |D^\alpha f(x)| dx < \infty \quad \text{for any} \ \alpha \ \text{with} \ |\alpha| = m,
\]
then

\[ f(x) = \sum_{|\alpha| = m} \alpha x |D^\alpha f(y)| dy + P(x) \quad \text{a.e. on } \mathbb{R}^n, \]

where \( P \in P_{m-1} \).

In case \( m = 1 \) Ohtsuka [13; Theorem 29] proved that (2) can be dropped. We shall extend Ohtsuka's result to higher order Beppo Levi functions with weight \( w \) in \( A_p \).

Theorem 2. Let \( w \in A_p \) and let \( f \in BL_m(L^p(\mathbb{R}^n, w)) \). If \( f \) satisfies (5), then (6) holds.

It is easy to see that \( w(x) = (1 + |x|)^{rp} \) belongs to \( A_p \) if and only if \( -n < rp < n(p - 1) \). Hence this theorem includes Kurokawa [4; Theorem 2.6].

In case \( g = \sum_{|\alpha| = m} c_\alpha (D^\alpha h_m) * D^\alpha f \) does not satisfy (3), the weighted Beppo Levi function \( f \) cannot be represented as the sum of a Riesz potential and a polynomial. However, a certain modification of the Riesz kernel (cf. [3; Chapter IV]) will enable us to represent \( f \) as the sum of a modified Riesz potential and a polynomial, and to show

Theorem 3 (cf. [14], [4; Theorem 3.2]). Let \( w \in A_p \). If \( f \in BL_m(L^p(\mathbb{R}^n, w)) \), then there is a sequence \( \{\psi_j\} \subset D \) such that

\[ \lim_{j \to \infty} \sum_{|\alpha| = m} \| D^\alpha f - D^\alpha \psi_j \|_{L^p(w)} = 0. \]

In the rest of this section we deal with \( w \in A_p \) for which every
$g \in L^p(\mathbb{R}^n, w)$ satisfies (3). In order to simplify the notation we denote by $A_{p,m}$ the class of all weights $w \in A_p$ such that every $g \in L^p(\mathbb{R}^n, w)$ satisfies (3). We shall show that $w \in A_{p,m}$ if and only if

$$
\int_{\mathbb{R}^n} \left(1 + |x|\right)^{(m-n)p/(p-1)}w(x)^{1/(1-p)}dx < \infty.
$$

See Theorem 7 in §5. Since $w(x) = (1 + |x|)^{r_p}$ belongs to $A_{p,m}$ if and only only if $m - n/p < r < n(1 - 1/p)$, it follows that $A_{p,m}$ is a proper subclass of $A_p$. If $w \in A_{p,m}$, then Theorem 1 gives a decomposition

$$
BL_m(L^p(\mathbb{R}^n, w)) = I_m(L^p(\mathbb{R}^n, w)) \oplus P_{m-1},
$$

where $I_m(L^p(\mathbb{R}^n, w)) = \{h \ast g ; g \in L^p(\mathbb{R}^n, w)\}$. We shall consider a condition for $f \in BL_m(L^p(\mathbb{R}^n, w))$ to belong to $I_m(L^p(\mathbb{R}^n, w))$. For this purpose we introduce a notion which describes the behavior at $\infty$ of a weighted Beppo Levi function.

**Definition.** Let $f_j$ and $f \in BL_m(L^p(\mathbb{R}^n, w))$. We say that $f_j$ converges to $f$ in the $BL_m(L^p(\mathbb{R}^n, w))$ sense if

$$
\lim_{j \to \infty} \sum_{|\alpha| = m} \|D^\alpha f_j - D^\alpha f\|_{L^p, w} = 0,
$$

$$
\lim_{j \to \infty} f_j = f \text{ a.e. on } \mathbb{R}^n.
$$

We say that $f$ vanishes at $\infty$ in the $BL_m(L^p(\mathbb{R}^n, w))$ sense if there is a sequence $\{\psi_j\}_j \subset D$ converging to $f$ in the $BL_m(L^p(\mathbb{R}^n, w))$ sense.

We shall show

**Theorem 4.** Let $w \in A_{p,m}$. Then $f \in BL_m(L^p(\mathbb{R}^n, w))$ belongs to
If and only if $f$ vanishes at $\infty$ in the $\text{BL}_m(L^p(\mathbb{R}^n, w))$ sense.

**Corollary 1.** Let $w \in A_{p,m}$. If $f \in \text{BL}_m(L^p(\mathbb{R}^n, w))$ and 
\[ \lim_{|x| \to \infty} f(x) = 0, \]
then $f \in \text{I}_m(L^p(\mathbb{R}^n, w))$, and hence $f$ vanishes at $\infty$ in the $\text{BL}_m(L^p(\mathbb{R}^n, w))$ sense.

We shall give a criterion for $f \in \text{BL}_m(L^p(\mathbb{R}^n, w))$ to vanish at $\infty$ in terms of the integrability of $f$ in case $w = V^p$ is a weight introduced by Muckenhoupt and Wheeden [11].

**Lemma A ([11; Theorem 4]).** Let $1 < p < n/m$ and $1/p^* = 1/p - m/n$. Suppose that $V \geq 0$ satisfies
\begin{equation}
\sup_Q \left( \frac{1}{|Q|} \int_Q V^{p^*} dx \right)^{1/p^*} \left( \frac{1}{|Q|} \int_Q V^{-p'} dx \right)^{1/p'} < \infty,
\end{equation}
where $p' = p/(p-1)$ and the supremum is taken over all cubes $Q$ with sides parallel to the axes. Then
\[ \| (h_{m^*} g) V \|_{L^{p^*}} \leq \text{const.} \| g V \|_{L^p} \] for $g \in L^p(\mathbb{R}^n, V^p)$.

Obviously, Hölder's inequality yields that if $V$ satisfies (8), then $V^p \in A_p$. Hence we can easily deduce from (1) and this lemma that $V^p \in A_{p,m}$. We shall show

**Theorem 5.** Let $1 < p < n/m$, $1/p^* = 1/p - m/n$ and $V$ satisfy (8).

(i) A function $f$ in $\text{BL}_m(L^p(\mathbb{R}^n, V^p))$ vanishes at $\infty$ in the
\( \text{BL}_m(L^p(R^n, \nu^p)) \) sense if and only if \( f \in L^{p^*}(R^n, \nu^{p^*}) \).

(ii) If \( f \in \text{BL}_m(L^p(R^n, \nu^p)) \) satisfies

\[
\int_{R^n} |f(x)|^q \nu(x)^r dx < \infty,
\]

for some \( q > 0 \) and some \( r, 0 < r \leq p^* \), then \( f \) vanishes at \( \infty \) in the \( \text{BL}_m(L^p(R^n, \nu^p)) \) sense.

This theorem yields the implication

\[
\text{BL}_m(L^p(R^n, \nu^p)) \cap \left( \bigcup_{q>0} L^q(R^n, \nu^r) \right)_{0<r<p^*} 
\subseteq \text{BL}_m(L^p(R^n, \nu^p)) \cap L^{p^*}(R^n, \nu^{p^*}) = \text{I}_m(L^p(R^n, \nu^p)).
\]

By virtue of (1), Theorem 1 and Lemma A we readily have an improvement of [11; Theorem 9].

Corollary 2. Let \( m, p, p^* \) and \( \nu \) be as in Theorem 5. Then there is a positive constant \( C \) depending only on \( m, p \) and \( \nu \) such that

\[
\|f \nu\|_{L^{p^*}} \leq C \sum_{|\alpha|=m} \|D^\alpha f \nu\|_{L^p}
\]

for \( f \in \text{BL}_m(L^p(R^n, \nu^p)) \cap \left( \bigcup_{q>0} L^q(R^n, \nu^r) \right)_{0<r<p^*} \).

§2. Preliminaries

We collect some basic results on the theory of distributions. We shall mainly use the notation of [16]. We write
\[ \langle T, \Psi \rangle = T(\Psi) \]

for a distribution \( T \) and a test function \( \Psi \). In order to avoid confusion, we write

\[ \langle T_x, \Psi \rangle \]

if \( \Psi \) involves two variables \( x \) and \( y \), and the distribution \( T \) acts on \( \Psi(\cdot, y) \) for each fixed \( y \). As in [16; Chapitre VII] we define the Fourier transform of \( \Psi \in \mathcal{S} \) and that of \( T \in \mathcal{S}' \) by

\[ \hat{\Psi}(y) = \Psi(y) = \int_{\mathbb{R}^n} e^{-2\pi i x \cdot y} \phi(x) dx, \]

\[ \langle \hat{T}, \Psi \rangle = \langle T, \Psi \rangle = \langle T, \Phi \rangle \quad \text{for} \quad \Psi \in \mathcal{S}, \]

where \( \mathcal{S} \) is the space of indefinitely differentiable functions decreasing rapidly at \( \infty \) and \( \mathcal{S}' \) is the space of tempered distributions. We note that the Fourier transform defined here corresponds to the inverse Fourier transform in [17]. By \( \mathcal{E}' \) and \( \mathcal{D}'_{L^p} \)

we denote the space of distributions of compact support and that of distributions \( T \) of the form

\[ T = \sum_{|a| \leq k} D^a f_a, \quad \text{where} \ k \geq 0 \ \text{and} \ f_a \in L^p(\mathbb{R}^n). \]

Schwartz [16; Chapitres VI and VII] proved

Lemma B. (i) If \( 1 \leq p \leq q \leq \infty \), then

\[ \mathcal{E}' \subset \mathcal{D}' \subset \mathcal{D}'_{L^p} \subset \mathcal{S}' \subset \mathcal{D}'. \]

(ii) If \( 0 \leq 1/r = 1/p + 1/q - 1 \leq 1 \), then the convolution \( S \ast T \)

exists and belongs to \( \mathcal{D}'_{L^r} \) for \( S \in \mathcal{D}'_{L^p} \) and \( T \in \mathcal{D}'_{L^q} \).

(iii) If \( S \) and \( T \) belong to \( \mathcal{D}'_{L^2} \), then \( \hat{\mathcal{F}} S \) and \( \hat{\mathcal{F}} T \) belong to

\( L^2_{\text{loc}}(\mathbb{R}^n) \) and \( \hat{\mathcal{F}}(S \ast T) = \hat{\mathcal{F}} S \ast \hat{\mathcal{F}} T \).

We can easily give another condition for the convolution of a
function and a measure to be defined.

Lemma 1. (i) Let \( \lambda \) be a real number. Suppose that \( f \in L^1_{\text{loc}}(\mathbb{R}^n) \) and \( |f(x)| \leq \text{const.} \ |x|^{\lambda} \) for \( |x| \geq 1 \). If a signed measure \( \mu \) satisfies

\[
\int_{\mathbb{R}^n} (1 + |x|)^{\lambda} d|\mu|(x) < \infty,
\]

then \( f*\mu \) is well-defined and belongs to \( L^1_{\text{loc}}(\mathbb{R}^n) \); moreover

\[
D^\beta(f*\mu) = (D^\beta f)*\mu = f*(D^\beta \mu)
\]

for any multiindex \( \beta \).

(ii) Let \( 0 < m < n \). If a signed measure \( \mu \) satisfies (4), then \( h_m f*\mu \) exists and belongs to \( L^1_{\text{loc}}(\mathbb{R}^n) \). If \( \mu \) does not satisfy (4), then \( h_m f*|\mu| \equiv \infty \) on \( \mathbb{R}^n \).

We need several results from the theory of singular integrals.

Consider the class consisting of all distributions \( T \) of the form

(9) \[ T = c\delta + \text{v.p.} \frac{\Omega(x)}{|x|^n}, \]

i.e., \( \langle T, \psi \rangle = c\psi(0) + \lim_{\varepsilon \to 0} \int_{|x| > \varepsilon} \frac{\Omega(x)}{|x|^n} \psi(x) dx \) for \( \psi \in D \),

where \( c \) is a constant; \( \Omega \) is a homogeneous function of degree 0, which is indefinitely differentiable on the unit sphere and

\[
\int_{|x|=1} \Omega(x) d\sigma(x) = 0.
\]

Lemma C ([17; Chapter III, Theorem 6]). A distribution \( T \) in \( \mathcal{S}' \) is written as (9) if and only if the Fourier transform \( \hat{T} \) is a homogeneous function of degree 0, which is indefinitely differentiable on the unit sphere.
The Muckenhoupt $A_p$ condition is related to distributions of the form (9) as follows:

**Lemma D ([1]).** Let $w \in A_p$ and let $T$ be a distribution of the form (9). Then

\begin{equation}
\|T g\|_{L^p_w} \leq \text{const.} \|g\|_{L^p_w} \quad \text{for } g \in L^p(R^n, w).
\end{equation}

From Lemmas C and D we can derive a generalization of (1).

**Lemma 2 (cf. [8; §3]).** Let $m \geq 1$ and $\ell \geq 0$. If $|\alpha| = m$ and $|\beta| = \ell$, then the distribution

\[ T = D^a(-\frac{x^\beta}{|x|^{n-m+\ell}}), \]

in particular $D^a h_m$, is of the form (9) and satisfies (10).

By $D'_{L^p_w}$ we denote the class of distributions of the form

\[ \sum_{|\alpha| \leq k} D^a f_{a}, \text{ where } k \geq 0 \text{ and } f_{a} \in L^p(R^n, w). \]

We shall have

**Lemma 3.** Let $S$ and $T$ be distributions of the form (9) and $w \in A_p$. Then

(i) $S$ and $T$ belong to $D'_{L^q}$ for any $q > 1$.

(ii) The convolution $S\ast T$ exists and is of the form (9).

(iii) If $f \in L^p(R^n, w)$, then $(S\ast T) \ast f = S \ast (T \ast f) \in L^p(R^n, w)$.

(iv) If $U \in D'_{L^p_w}$, then $(S\ast T) \ast U = S \ast (T \ast U) \in D'_{L^p_w}$.
It is proved in [1] that every \( w \in A_p \) satisfies the Muckenhoupt \( A_\infty \) condition:

There are positive constants \( C, \delta > 0 \) such that given any cube \( Q \) and any measurable subset \( E \) of \( Q \),

\[
(A_\infty) \quad \frac{w(E)}{w(Q)} \leq C \left( \frac{|E|}{|Q|} \right)^\delta, \quad \text{where } w(A) = \int_A w \, dx \text{ for } A \subset \mathbb{R}^n.
\]

We shall denote by \( A_\infty \) the class of weights \( w \) satisfying \( (A_\infty) \). Then \( A_\infty = \bigcup_{p>1} A_p \) (see [1]). We collect some properties of \( A_p \) and \( A_\infty \) weights.

**Lemma 4.** Let \( w_1 \) and \( w_2 \) belong to \( A_p \). Then the weights \( \max\{w_1, w_2\} \), \( \min\{w_1, w_2\} \) and \( w_1 + w_2 \) belong to \( A_p \).

**Lemma 5.** Let \( w \in A_\infty \). If \( L \) is a cone with vertex at the origin, then \( w(L) = \infty \). Moreover, there are no nonnegative functions \( u \) and \( v \) such that

\[
(11) \quad w \leq u + v \text{ on } L, \\
\int_L u^q \, dx + \int_L v^r \, dx < \infty \quad \text{for some } q, r, 0 < q \leq r \leq 1.
\]

If a polynomial \( P \) belongs to \( L^s(\mathbb{R}^n, w) \) for some \( s > 0 \), then \( P \equiv 0 \).

**Lemma 6.** Let \( w \in A_p \). Then

(i) \[
\int_{\mathbb{R}^n} (1 + |x|)^{-n} |g(x)| \, dx < \infty \quad \text{for } g \in L^p(\mathbb{R}^n, w).
\]

(ii) \[
\int_{\mathbb{R}^n} (1 + |x|)^{-np} w(x) \, dx < \infty.
\]
§3. Proof of Theorems 1 and 2

Proof of Theorem 1. In this proof we let $\alpha$, $\beta$ and $\gamma$ be multiindices of length $m$. Take $\Psi \in \mathcal{D}$ such that $\Psi = 1$ on a neighborhood of the origin and let $h'_m = \Psi h_m$ and $h''_m = (1 - \Psi) h_m$. Since $h''_m \in L^q(\mathbb{R}^n)$ for any $q > n/(n-m)$ and $h'_m \in \mathcal{E}'$, it follows from Lemma B (i) that $h'_m \in \mathcal{D}' \subset L^q$ for any $q > n/(n-m)$. Since $D^\alpha h_m$ belongs to $\mathcal{D}' \subset L^q$ for any $q > 1$ by Lemmas 2 and 3, we have from Lemma B (ii) that the convolution $h'_m * D^\alpha h_m$ is well-defined and belongs to $\mathcal{D}' \subset L^q$ for any $q > n/(n-m)$. We observe that

$$D^\beta (h'_m * D^\alpha h_m) = D^\alpha h_m * D^\beta h_m = h'_m * D^\alpha D^\beta h_m.$$  

Noting that $D^\alpha h_m \in \mathcal{D}' \subset L^2$, we obtain from Lemma B (iii) that

$$F(h'_m * \sum_{|\alpha| = m} c_\alpha D^2 \alpha h_m) = F(\sum_{|\alpha| = m} c_\alpha D^\alpha h_m * D^\alpha h_m)$$
$$= \sum_{|\alpha| = m} c_\alpha \{ (2\pi i)^\alpha (2\pi |x|)^{-m} \}^2 = 1,$$

because

$$(2\pi |x|)^2m = F((-\Delta)^m) = F(\sum_{|\alpha| = m} c_\alpha D^{2\alpha}) = \sum_{|\alpha| = m} c_\alpha (2\pi i)^{2\alpha}.$$  

Accordingly

$$(12) \quad h'_m * \sum_{|\alpha| = m} c_\alpha D^2 \alpha h_m = \sum_{|\alpha| = m} c_\alpha D^\alpha h_m * D^\alpha h_m = \delta.$$

By (3) and Lemma 1 we obtain that the convolution $h'_m * g$ is well-defined and belongs to $L^1_{loc}(\mathbb{R}^n)$. We infer from Lemmas 2, 3 and (12) that

$$D^\beta D^\gamma (h'_m * g) = D^\beta h'_m * D^\gamma g = D^\beta h'_m * ((\sum_{|\alpha| = m} c_\alpha D^\alpha h_m) * D^\gamma D^\alpha f)$$
$$= (D^\beta h'_m * \sum_{|\alpha| = m} c_\alpha D^2 \alpha h_m) * D^\gamma f = D^\beta \delta * D^\gamma f = D^\beta D^\gamma f.$$
Since $\beta$ is arbitrary, it follows that
\[ D^\gamma f = D^\gamma (h_m \ast g) + P_\gamma \quad \text{for any } \gamma \text{ with } |\gamma| = m, \]
where $P_\gamma \in P_{m-1}$. However $D^\gamma f \in L^p(\mathbb{R}^n, w)$ and
\[ D^\gamma (h_m \ast g) = \sum |\alpha| = m c_\alpha D^\gamma h_m \ast (D^\alpha h_m \ast D^\alpha f) \in L^p(\mathbb{R}^n, w) \]
by Lemma 3, and hence $P_\gamma$ must be identically 0 by Lemma 5. Since
\[ D^\gamma f = D^\gamma (h_m \ast g) \text{ for any } \gamma \text{ with } |\gamma| = m, \]
it follows that
\[ f = h_m \ast g + P, \]
where $P \in P_{m-1}$.

The uniqueness of the representation readily follows from the following proposition, which may be of some independent interest.

**Proposition 1.** Let $0 < m < n$ and let $\mu$ be a signed measure satisfying (4). If $h_m \ast \mu$ coincides with some polynomial $P$, then $\mu = 0$ and hence $P$ must be 0.

**Proof.** We define a sequence of signed measures $\mu_j$ of compact support by $\mu_j(E) = \mu((x \in E; |x| \leq j))$. Since
\[ \left| \int_{|x| > j} \psi d\mu(x) \right| \leq \text{const.} \int_{|x| > j} (1 + |x|)^{m-n} d|\mu|(x) \quad \text{for } \psi \in \mathcal{S}, \]
it follows that $\mu_j \ast \mu$ in $\mathcal{S}'$. We claim that $h_m \ast \mu_j \ast h_m \ast \mu$ in $\mathcal{S}'$.

Let $\Psi \in \mathcal{S}$. Take $\psi \in \mathcal{D}$ such that $\psi(x) = 1$ for $|x| \leq 1$ and write
\[ \psi = \psi \psi + (1 - \psi) \psi = \psi_1 + \psi_2. \]

It is easy to see that $h_m \ast |\psi_1|(x) = 0(|x|^{m-n})$ as $|x| \to \infty$. Since $\psi$ decreases rapidly, we have $|\psi_2(\gamma)| \leq \text{const.} |\gamma|^{-n-1}$. Let $|x| > 2$. Then
\[ h_m^*|\psi_2|(x) \leq \text{const.} \int_{|y| > 1} |x - y|^{m-n} |y|^{-n-1} dy \]

\[ = \text{const.} |x|^{m-n-1} \left\{ \int_{|x|^{-1} < |z| < 2} - \int_{|z| > 2} \frac{|x|}{|z|} - z|^{m-n} |z|^{-n-1} dz \right\}. \]

We see that the first integral is not greater than

\[ \text{const.} \int_{|x|^{-1}}^{2^{-1}} t^{-2} dt \leq \text{const.} |x|, \]

and that the second integral is a finite value independent of \( x \).

Therefore \( h_m^*|\psi_2|(x) = O(|x|^{m-n}) \) as \( |x| \to \infty \). Accordingly

\[ |\int \psi(h_m^*(u-u_j))dx| \leq \text{const.} \int_{|x| > j} |x|^{m-n} d|u|(x) \to 0 \]

by (4) and Fubini's theorem. Thus \( h_m^* \mu_j = h_m u \) in \( \mathcal{S}' \).

Now we see that \( \hat{\mathcal{F}}(h_m^* \mu_j) = (2\pi|x|)^{-m} \hat{\mu}_j \in L^1_{\text{loc}}(\mathbb{R}^n) \). In fact, since \( \mu_j \in \mathcal{E}' \subset D' \) for any \( q > 1 \), it follows that \( h_{m/2}^* \mu_j \) belong to \( D' \), and from Riesz's composition formula that \( h_m^* \mu_j = h_{m/2}^*(h_{m/2}^* \mu_j) \). We infer from Lemma B (iii) that

\[ \hat{\mathcal{F}}(h_m^* \mu_j) = \hat{\mathcal{F}}(h_{m/2}^*(h_{m/2}^* \mu_j)) = \hat{h}_{m/2}^* \hat{h}_{m/2}^* \hat{\mu}_j = (2\pi|x|)^{-m} \hat{\mu}_j. \]

Since the total variation of \( \mu_j \) is finite, it follows that \( \hat{\mu}_j \) is a bounded function, so that \( (2\pi|x|)^{-m} \hat{\mu}_j \in L^1_{\text{loc}}(\mathbb{R}^n) \).

Noting that \( \mu_j \to \mu \) and \( h_m^* \mu_j \to h_m^* \mu = P \) in \( \mathcal{S}' \), we obtain that

\[ \hat{\mu}_j \to \hat{\mu} \quad \text{and} \quad (2\pi|x|)^{-m} \hat{\mu}_j \to \hat{\mathcal{F}}(P) = \mathcal{F}(\frac{-1}{2\pi^2} \frac{\partial}{\partial x}) \delta \text{ in } \mathcal{S}'. \]

For any \( \psi \in \mathcal{S} \) vanishing on a neighborhood of the origin we have

\[ \psi = (2\pi|x|)^m \psi(x) \in \mathcal{S} \text{ and} \]

\[ \langle \hat{\mu}, \psi \rangle = \lim_{j \to \infty} \langle \hat{\mu}_j, \psi \rangle = \lim_{j \to \infty} \int \hat{\mu}_j(x) \psi(x) dx \]

\[ = \lim_{j \to \infty} \int (2\pi|x|)^{-m} \hat{\mu}_j(x) \psi(x) dx = \langle P(\frac{-1}{2\pi^2} \frac{\partial}{\partial x}) \delta, \psi \rangle = 0. \]

This implies that \( \hat{\mu} \) is supported on \{0\}. Hence we can write
\[ \hat{\Omega} = \mathcal{P}'\left(\frac{n-1}{2\pi i} \frac{2}{\partial x}\right) \delta \]

with some polynomial \( \mathcal{P}' \). By the inverse Fourier transform we have \( \mu = \mathcal{P}' \), i.e., \( \mu \) is absolutely continuous and \( d\mu = \mathcal{P}'dx \). Since \( \mu \) satisfies (4), it follows that \((1 + |x|)^{m-n}\mathcal{P}'(x)\) is integrable, so that \( \mathcal{P}' \) must be identically zero. Hence \( \mu = 0 \) and \( \mathcal{P} = 0 \).

**Remark 1.** The above proof works even if \( m \) is not an integer. In case \( m \) is an integer, \( d\mu = gd\theta \), \( g \in L^p(R^n, w) \) and \( \mathcal{P} \in \mathcal{P}_{m-1} \), it is possible to give a simple proof. In fact by (12) and Lemma 3

\[ g = \sum |\alpha| = m c_\alpha D^\alpha h_m * D^\alpha (h_m * g) = \sum |\alpha| = m c_\alpha D^\alpha h_m * D^\alpha \mathcal{P} = 0. \]

**Proof of Theorem 2.** By using polar coordinates and integration by parts, we can prove

\[ \sum |\alpha| = m a_\alpha D^\alpha \left( \frac{x^\alpha}{|x|^n} \right) = \delta \]

(see [15; Lemma 6.2]). Let \( |\beta| = |\gamma| = m \). Applying Lemma 1 to \( \ell = m - n \), \( f = x^\alpha / |x|^n \) and \( d\mu = D^\gamma f dx \), we obtain that

\[ \left( \frac{x^\alpha}{|x|^n} \right) * D^\gamma f \in L^1_{loc}(R^n) \] for each \( \alpha \) and \( \gamma \).

We infer from Lemma 2 and (13) that

\[ D^\beta D^\gamma \left( \sum |\alpha| = m a_\alpha \left( \frac{x^\alpha}{|x|^n} \right) * D^\alpha f \right) = \sum |\alpha| = m a_\alpha D^\beta \left( \frac{x^\alpha}{|x|^n} \right) * D^\alpha D^\gamma f \]

\[ = \sum |\alpha| = m a_\alpha D^\beta \left( \frac{x^\alpha}{|x|^n} \right) * D^\gamma f = D^\beta \left( \sum |\alpha| = m a_\alpha D^\alpha \left( \frac{x^\alpha}{|x|^n} \right) \right) * D^\gamma f \]

\[ = D^\beta \delta * D^\gamma f = D^\beta D^\gamma f. \]

Now the same argument as in the proof of Theorem 1 completes the proof.
§4. Proof of Theorem 3

Let us begin with modifying the Riesz kernel. The following technique is found in [3; Chapter IV] and [9, 10]. Observe that if \( y \neq 0 \), then \( h_m(x - y) \) has a multiple power series expansion in \( x_1, x_2, \ldots, x_n \), convergent in a neighborhood of the origin. We write

\[
h_m(x - y) = \sum_{\nu=0}^{\infty} a_{\nu}(x, y),
\]

where, for fixed \( \nu \) and \( y \neq 0 \), \( a_{\nu}(x, y) \) is a homogeneous polynomial in \( x_1 \) to \( x_n \) of degree \( \nu \) and continuous in \( x, y \) jointly for \( y \neq 0 \) (cf. [3; Lemma 4.1]). We now set

\[
k_m(x, y) = \begin{cases} 
h_m(x - y) & \text{if } |y| \leq 1 \\
h_m(x - y) - \sum_{\nu=0}^{m-1} a_{\nu}(x, y) & \text{if } |y| > 1.
\end{cases}
\]

Obviously \( D_\alpha^x k_m(x, y) = D_\alpha^x h_m(x - y) \) for \( |\alpha| \geq m \). Since

\[
|k_m(x, y)| \leq \text{const.} |x|^m |y|^{-n} \quad \text{if } 2|x| \leq |y|
\]

(cf. [3; Lemma 4.2]), we can easily prove from Lemma 6 (i)

Lemma 7. Let \( w \in A_p \). If \( g \in L_p^D(\mathbb{R}^n, w) \), then

\[
\int_{\mathbb{R}^n} k_m(x, y)g(y)dy \in L_{1\text{loc}}^1(\mathbb{R}^n),
\]

\[
D_\alpha^x(\int_{\mathbb{R}^n} k_m(x, y)g(y)dy) = (D_\alpha^x h_m) * g \quad \text{for } |\alpha| \geq m.
\]

This lemma and the same argument as in Theorem 1 yield

Theorem 6. Let \( w \in A_p \). If \( f \in B_\text{loc}(L_p^D(\mathbb{R}^n, w)) \), then
\[ f = \int_{\mathbb{R}^n} k_m(x, y)g(y)dy + P, \quad g = \sum_{|\alpha| = m} c_\alpha (D^\alpha h_m) \ast D^\alpha f, \]

where \( P \in P_{m-1} \).

Let \( \mathcal{E} \) be the space of all indefinitely differentiable functions on \( \mathbb{R}^n \). We show

Lemma 8. Let \( f \in \text{Im}(L^p(\mathbb{R}^n, w)) \cap \mathcal{E} \). Then for \( \varepsilon > 0 \) and \( r > 0 \) there is a function \( \psi \in \mathcal{D} \) such that

\[
(14) \quad \sum_{|\alpha|=m} \left\| D^\alpha \psi - D^\alpha f \right\|_{L^p, w} < \varepsilon \quad \text{and} \quad \sup_{|x|<r} |\psi(x) - f(x)| < \varepsilon.
\]

Proof. First we treat the case when \( f = h_m \ast g \) with \( g \in \mathcal{D} \). Let \( R > r \) and \( \text{supp } g \in \{ y; |y| < R \} \). Take \( \psi \in \mathcal{D} \) such that \( 0 \leq \psi \leq 1 \) and \( \psi(x) = 1 \) for \( |x| < 3R \) and put \( \psi_j(x) = \psi(x/j) \). We observe that

\[
0 \leq \psi_j \leq 1, \quad \psi_j(x) = 1 \text{ for } |x| < 3Rj,
\]

\[
(15) \quad \sum_{k=0}^m \sum_{|\alpha|=k} \sup_{|x|<r} \left| x^k D^\alpha \psi_j(x) \right| = \sum_{k=0}^m \sum_{|\alpha|=k} \sup_{|x|<r} \left| x^k D^\alpha \psi(x) \right| < \infty.
\]

Let \( h_{m,j}(x) = \psi_j(x)h_m(x) \). Then \( h_{m,j} \ast g \in \mathcal{D} \) and

\[
\begin{align*}
    h_{m,j} \ast g(x) &= \int_{|y|<R} \psi_j(x-y)h_m(x-y)g(y)dy \\
    &= \int_{|y|<R} h_m(x-y)g(y)dy = h_m \ast g(x) \quad \text{for } |x| < 2Rj
\end{align*}
\]

by (15). Let \( \alpha \) be a multiindex of length \( m \). We have

\[
D^\alpha h_{m,j} \ast g(x) = D^\alpha h_m \ast g(x) \quad \text{for } |x| < 2Rj,
\]

and hence

\[
D^\alpha h_{m,j} \ast g = D^\alpha h_m \ast g \quad \text{on } \mathbb{R}^n.
\]

In view of (15) and Leibniz's formula we have
\[ |D^{\alpha}h_{m,j}(x-y)| \leq \text{const.}|x|^{-n} \quad \text{for } |x| > 2R \text{ and } |y| < R, \]
and hence
\[ |D^{\alpha}h_{m,j} * g(x) - D^{\alpha}h_{m} * g(x)| \leq \text{const.}|x|^{-n} \quad \text{for } |x| > 2R. \]

Now it follows from Lemma 6 (ii) and the dominated convergence theorem that
\[
\int_{R^n} |D^{\alpha}h_{m,j} * g(x) - D^{\alpha}h_{m} * g(x)|^{p_w(x)} dx = \int_{|x| > 2R} |D^{\alpha}h_{m,j} * g(x) - D^{\alpha}h_{m} * g(x)|^{p_w(x)} dx \to 0,
\]
so that \( D^{\alpha}(h_{m,j} * g) \to D^{\alpha}(h_{m} * g) \) in \( L^p(R^n, w) \). Therefore \( \Psi = h_{m,j} * g \) satisfies \( (14) \) if \( j \) is sufficiently large.

Next we consider the general case. From the uniqueness in Theorem 1 \( f \) is written as \( f = h_{m} * g \) with \( g = \sum |\alpha| = m c_\alpha (D^{\alpha}h_{m}) * D^{\alpha}f \in L^p(R^n, w) \cap \mathcal{E} \). It is easy to find \( \psi \in \mathcal{D} \) such that \( 0 \leq \psi \leq 1 \),
\[
\sum |\alpha| = m \| D^{\alpha}h_{m} * (\psi g) - D^{\alpha}f \|_{L^p,w} \leq \text{const.} \| \psi g - g \|_{L^p,w} < \varepsilon/2,
\]
and
\[
\sup_{|x| < r} |h_{m} * (\psi g)(x) - h_{m} * g(x)| < \varepsilon/2.
\]

From the first part there is a function \( \Psi \in \mathcal{D} \) such that
\[
\sum |\alpha| = m \| D^{\alpha}\Psi - D^{\alpha}h_{m} * (\psi g) \|_{L^p,w} < \varepsilon/2,
\]
\[
\sup_{|x| < r} |\Psi(x) - h_{m} * (\psi g)(x)| < \varepsilon/2.
\]
This \( \Psi \) satisfies \( (14) \).

Proof of Theorem 3. Let \( g \) be as in Theorem 6. It is easy to find a sequence \( \{g_j\}_j \subset \mathcal{D} \) such that \( \|g_j - g\|_{L^p,w} \to 0 \). Since \( g_j \) has compact support, \( h_{m} * g_j \) exists and by Lemma 7
\[ \int_{\mathbb{R}^n} k_m(x, y) g_j(y) dy = h_m * g_j + P_j \]

with some \( P_j \in P_{m-1} \). Now Lemma 8 gives a sequence \( \{ \psi_j \}_j \subset D \) such that

\[ \sum_{|\alpha|=m} \| D^\alpha h_m * g_j - D^\alpha \psi_j \|_{L^p, w} < 1/j. \]

We infer from Theorem 6, Lemmas 2 and 7 that

\[ \sum_{|\alpha|=m} \| D^\alpha f - D^\alpha \psi_j \|_{L^p, w} \]

\[ = \sum_{|\alpha|=m} \| D^\alpha (\int_{\mathbb{R}^n} k_m(x, y) g(y) dy) - D^\alpha \psi_j \|_{L^p, w} \]

\[ = \sum_{|\alpha|=m} \| (D^\alpha h_m) * g - D^\alpha \psi_j \|_{L^p, w} \]

\[ \leq \sum_{|\alpha|=m} \| (D^\alpha h_m) * (g - g_j) \|_{L^p, w} + 1/j + 0. \]

The theorem is proved.

§5. Proof of Theorem 4

Lemma 9. If \( \psi \in D \), then \( \psi = h_m * g \in I_m (L^p(R^n, w)) \), where \( g = \)

\[ \sum_{|\alpha|=m} c_\alpha D^\alpha h_m * D^\alpha \psi \in L^p(R^n, w) \cap \mathcal{C}. \]

Proof. Since \( D^\alpha h_m \) is of the form (8) and \( D^\alpha \psi \) has compact support, it follows that \( g(x) = O(|x|^{-n}) \) as \( |x| \to \infty \), so that \( g \)
satisfies (3) and \( h_m * g = o(1) \) as \( |x| \to \infty \). By Theorem 1 we have \( \psi = h_m * g + P \) with some \( P \in P_{m-1} \). However \( P \) must be equal to zero, for \( P(x) = \psi(x) - h_m * g(x) \to 0 \) as \( |x| \to \infty \).

Lemma 10. A function \( f = h_m * g + P \) in \( BL_m(L^p(R^n, w)) \) vanishes at \( \infty \) in the \( BL_m(L^p(R^n, w)) \) sense if and only if there is a sequence
\( \{g_j\}_j \subset L^p(\mathbb{R}^n, w) \cap \mathcal{E} \) such that

\[
\|g_j - g\|_{L^p, w} + 0 \quad \text{and} \quad h_m * g_j + f \text{ a.e. on } \mathbb{R}^n.
\]

(16)

Proof. First suppose that \( \{g_j\}_j \subset L^p(\mathbb{R}^n, w) \cap \mathcal{E} \) satisfies (16). Then by Lemma 8 there is a sequence \( \{\psi_j\}_j \subset \mathcal{D} \) such that

\[
\sum_{|\alpha| = m} \|D^\alpha h_m * g_j - D^\alpha \psi_j\|_{L^p, w} < 1/j,
\]

\[
\sup_{x} \left| \psi_j(x) - \psi_j(x) \right| < 1/j.
\]

We easily see that \( \psi_j \) converges to \( f \) in the \( BL_m(L^p(\mathbb{R}^n, w)) \) sense.

Conversely suppose that \( \{\psi_j\}_j \subset \mathcal{D} \) converges to \( f \) in the \( BL_m(L^p(\mathbb{R}^n, w)) \) sense. We infer from Lemmas 2 and 9 that \( \psi_j = h_m * g_j \), \( g_j \in L^p(\mathbb{R}^n, w) \cap \mathcal{E} \) and

\[
\|g_j - g\|_{L^p, w} = \sum_{|\alpha| = m} \sum_{\alpha} \|D^\alpha h_m * \psi_j - D^\alpha \psi_j\|_{L^p, w}
\]

converges to zero. Thus \( \{g_j\}_j \) satisfies (16).

For \( E \subset \mathbb{R}^n \) we define a capacity \( R_{m, p, w}(E) \) by

\[
R_{m, p, w}(E) = \inf\{\|g\|_{L^p, w}^p \ ; \ g \geq 0, \ h_m * g \geq 1 \text{ on } E\}.
\]

The next theorem combines conditions (3) and (7), the capacity \( R_{m, p, w} \) and the vanishing property of Beppo Levi functions.

Theorem 7. The following statements on \( w \in A_p \) are equivalent:

(a) \( w \in A_{p, m} \).

(b) For every \( g \in L^p(\mathbb{R}^n, w) \) the convolution \( h_m * g \) exists and
belongs to $L^1_{loc}(R^n)$.

(c) If $\|g_j\|_{L^p_w} \to 0$, then $h_m \ast g_j \to 0$ in measure on any ball.

c') If $\|g_j\|_{L^p_w} \to 0$, then $h_m \ast g_j \to 0$ in measure on some ball.

(d) The constant function 1 does not vanish at $\infty$ in the $\mathcal{B}L_m(L^p(R^n, w))$ sense.

(e) There is a set of positive $R_{m, p, w}$ capacity.

(f) If $|E| > 0$, then $R_{m, p, w}(E) > 0$.

(g) $w$ satisfies (7).

Proof. The equivalence between (a) and (b) readily follows from Lemma 1. The implications (c) $\Rightarrow$ (c') and (f) $\Rightarrow$ (e) are obvious. We have (f) $\Rightarrow$ (c) from [6; Theorem 4] and (g) $\Rightarrow$ (a) from Hölder's inequality. We shall complete the proof by showing (b) $\Rightarrow$ (f) $\Rightarrow$ (g), (e) $\Rightarrow$ (b) and (c') $\Rightarrow$ (d) $\Rightarrow$ (a).

(b) $\Rightarrow$ (f): Suppose that there is a measurable set $E$ such that $|E| > 0$ but $R_{m, p, w}(E) = 0$. By [6; Theorem 3] we find a nonnegative function $g$ in $L^p(R^n, w)$ such that $h_m \ast g = \infty$ on $E$. Since $|E| > 0$, it follows that $h_m \ast g$ is not locally integrable, so that (b) does not holds.

(f) $\Rightarrow$ (g): Since the unit ball $B$ has positive capacity, it follows from [6; Theorem 14] that there exists a measure $\mu$ concentrated on $B$ such that $\mu(B) > 0$ and $h_m \ast \mu \in L^{p'}(R^n, w^{1/(1-p)})$. Noting that $h_m \ast \mu(x) \geq \text{const.} \mu(B) h_m(x)$ for $|x| > 1$, we obtain

$$\int_{|x|>1} |x|^{(m-n)p'} w(x)^{1/(1-p)} dx < \infty,$$

which is equivalent to (7).

(e) $\Rightarrow$ (b): If (b) does not hold, then there is a nonnegative
function $g$ in $L_p^p(R^n, w)$ such that $h_m * g \equiv \infty$ on $R^n$. By definition

$$0 \leq R_{m,p,w}(E) \leq R_{m,p,w}(R^n) \leq \inf_{t>0} \|tg\|_{L_p^p,w}^p = 0.$$ 

Thus (e) does not hold.

(c') $\Rightarrow$ (d): If $1$ vanishes at $\infty$ in the $BL_m(L_p^p(R^n, w))$ sense, then there is a sequence $\{g_j\} \subset L_p^p(R^n, w)$ such that

$$\|g_j\|_{L_p^p,w} \to 0 \text{ and } h_m * g_j \to 1 \text{ a.e. on } R^n$$

by Lemma 10. This contradicts (c').

(d) $\Rightarrow$ (a): Suppose that there is a nonnegative function $g$ in $L_p^p(R^n, w)$ such that (3) does not hold. Mollifying $g$, we may assume that $g \in L_p^p(R^n, w) \cap \mathcal{E}$. We shall prove that $1$ vanishes at $\infty$ in the $BL_m(L_p^p(R^n, w))$ sense. By Lemma 10 it is sufficient to show that if $\varepsilon > 0$ and $R > 0$, then there is $g_1 \in L_p^p(R^n, w) \cap \mathcal{E}$ such that

$$|h_m * g_1(x) - 1| < \varepsilon \text{ for } |x| < R,$$

$$\|g_1\|_{L_p^p,w} < \varepsilon.$$ 

Take $R_1 > R$ such that

$$1 - \varepsilon < h_m(x-y)/h_m(y) < 1 + \varepsilon \text{ for } |x| < R \text{ and } |y| > R_1.$$ 

Since (3) does not hold and $g \in L_p^p(R^n, w)$, we find a function $\psi \in \mathcal{D}$ such that $0 \leq \psi \leq 1$, supp $\psi \subset \{y; |y| > R_1\}$ and $g_1 = \psi g$ satisfies $h_m * g_1(0) = 1$ and $\|g_1\|_{L_p^p,w} < \varepsilon$. We observe that

$$1 - \varepsilon < h_m * g_1(x)/h_m * g_1(0) = h_m * g_1(x) < 1 + \varepsilon \text{ for } |x| < R.$$ 

Hence $g_1$ has the desired property. Thus the theorem is completely proved.

Proof of Theorem 4. Suppose that $f = h_m * g \in I_m(L_p^p(R^n, w))$. 

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Take a nonnegative function $\psi$ in $\mathcal{D}$ such that $\int \psi dx = 1$. Letting $\psi_j(x) = j^n \psi(jx)$, we observe that $g_j = g * \psi_j \in L^p(R^n, w) \cap \mathcal{E}$ satisfies (16). Hence $f$ vanishes at $\infty$ in the $BL_m(L^p(R^n, w))$ sense by Lemma 10.

Conversely suppose that $f = h_m * g + P \in BL_m(L^p(R^n, w))$ vanishes at $\infty$ in the $BL_m(L^p(R^n, w))$ sense. Since $h_m * g$ vanishes at $\infty$ in the $BL_m(L^p(R^n, w))$ sense from the only if part of the theorem, it follows that $P = f - h_m * g$ vanishes at $\infty$ in the $BL_m(L^p(R^n, w))$ sense. Hence there is a sequence $\{g_j\}_j \subset L^p(R^n, w)$ such that

$$\|g_j\|_{L^p, w} \to 0 \quad \text{and} \quad h_m * g_j \to P \quad \text{a.e. on } R^n$$

by Lemma 10. On account of (c) of Theorem 7 we have $P = 0$. The proof is complete.

For the proof of Corollary 1 we prepare

Lemma 11. Let $L$ be a cone with vertex at the origin. Then $R_m, p, w(L)$ is equal to 0 or $\infty$; $R_m, p, w(L) = 0$ if and only if $R_m, p, w(R^n) = 0$. The constant 1 vanishes at $\infty$ in the $BL_m(L^p(R^n, w))$ sense if and only if $R_m, p, w(R^n) = 0$.

Proof. If $0 < R_m, p, w(L) < \infty$, then there would exist a nonnegative function $g$ in $L^p(R^n, w)$ satisfying (3) and $h_m * g \geq 1$ on $L$ by Theorem 7. Since $L$ is not m-thin at $\infty$ in the notation of [5], this contradicts

$$\lim \inf_{|x| \to \infty, x \in L} h_m * g(x) = 0$$
([5; Theorem 3.3]). By Theorem 7 we can easily prove the remainder.

Proof of Corollary 1. Suppose that \( f = h_m \ast g + P \) and \( P \neq 0 \). Then we would find \( \varepsilon > 0 \), \( R > 0 \) and a cone \( L \) with vertex at the origin such that

\[
h_m \ast |g|(x) \geq |f(x) - P(x)| \geq \varepsilon \quad \text{if} \quad |x| \geq R \quad \text{and} \quad x \in L.
\]

By definition

\[
R_{m, p, w}(L) \leq R_{m, p, w}(\{x; \ |x| < R\}) + R_{m, p, w}(\{x \in L; \ |x| \geq R\}) < \infty
\]

and hence by Lemma 11 \( R_{m, p, w}(L) = 0 \). This contradicts (f) of Theorem 7.

§6. Proof of Theorem 5

Proof of Theorem 5. First suppose that \( f \) vanishes at \( \infty \) in the \( BL_m(L^p(R^n, V^p)) \) sense. Thus \( f \) is written as \( h_m \ast g \) with \( g \in L^p(R^n, V^p) \). On account of Lemma 10 there is a sequence \( \{g_j\}_j \subset L^p(R^n, V^p) \) satisfying (16) with \( w = V^p \). Since

\[
\|(h_m \ast g_j)V\|_{L^p} \leq \text{const.} \|g_jV\|_{L^p} \leq \text{const.}
\]

by Lemma A, Fatou's lemma leads to

\[
\|fV\|_{L^p} \leq \liminf_{j \to \infty} \|(h_m \ast g_j)V\|_{L^p}, \ < \infty.
\]

The if part of (i) is included in (ii). Now we shall prove (ii) by contradiction. Suppose that \( f = h_m \ast g + P \), \( g \in L^p(R^n, V^p) \), \( P \in P_{m-1} \) and \( P \neq 0 \). Then we would find \( \varepsilon > 0 \), \( R > 0 \) and a cone \( L \) with vertex at the origin such that

\[
|P(x)| \geq 2\varepsilon \quad \text{for} \quad x \in \{x \in L; \ |x| \geq R\}.
\]
We observe that $V_r^{p*} \leq u + v$ on $L$, where

\[
u(x) = \begin{cases} V(x)^{p*} & \text{if } |f(x)| \geq \epsilon \\ 0 & \text{otherwise} \end{cases}, \quad \nu(x) = \begin{cases} V(x)^{p*} & \text{if } |h_m g(x)| \geq \epsilon \text{ or } |x| \leq R \\ 0 & \text{otherwise} \end{cases}.
\]

Since $V_r^{p*} \in A_\infty$ and $0 < r/p* \leq 1$,

\[
\int_L u^{r/p*} dx = \int_{\{x \in L; \ |f(x)| \geq \epsilon\}} V_r^{r} dx \leq \epsilon^{-r} \int_{\mathbb{R}^n} |f|^q V_r^q dx < \infty,
\]

\[
\int_L v^{r} dx = \int_{\{x \in L; \ |h_m g(x)| \geq \epsilon\}} V_r^{p*} dx + \int_{|x| \leq R} V_r^{p*} dx \leq \epsilon^{-p*} \int_{\mathbb{R}^n} |h_m g|^p V_r^{p*} dx + \int_{|x| \leq R} V_r^{p*} dx < \infty,
\]

we have a contradiction by Lemma 5. The theorem is proved.

References


