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Integral representations of weighted Beppo Levi functions

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§1. Introduction

It is well known that functions in Sobolev spaces can be represented as Bessel potentials ([17; Chapter V, Theorem 3]). In this paper we shall consider a similar problem for weighted Beppo Levi functions.

Let $1 < p < \infty$ and let w be a weight (nonnegative Lebesgue measurable function) satisfying the Muckenhoupt A_p condition:

$$(A_p) \qquad \sup_Q \left(\frac{1}{|Q|}\int_Q w\,dx\right)\left(\frac{1}{|Q|}\int_Q w^{1/(1-p)}dx\right)^{p-1} < \infty,$$

where the supremum is taken over all cubes Q with sides parallel to the axes and $|Q|$ stands for the Lebesgue measure of Q (see [1]). By A_p we denote the class of weights w satisfying (A_p). We write

$$\|f\|_{L^p,w} = \left(\int_{R^n} |f(x)|^p w(x)dx\right)^{1/p}, \quad L^p(R^n, w) = \{f;\ \|f\|_{L^p,w} < \infty\}.$$

By $BL_m(L^p(R^n, w))$, $m \geq 1$, we denote the space of distributions whose partial derivatives of m-th order all belong to $L^p(R^n, w)$ (see [2]). Since $w^{1/(1-p)}$ is locally integrable by (A_p), it follows from Hölder's inequality and Kryloff's theorem [16; Chapitre VI, Théorème 15] that a distribution in $BL_m(L^p(R^n, w))$ is a locally integrable function whenever $w \in A_p$. Therefore we call a locally integrable function in $BL_m(L^p(R^n, w))$ a Beppo Levi function of order m with weight w. If $w = 1$, then we write simply $\|f\|_{L^p}$, $L^p(R^n)$ and $BL_m(L^p(R^n))$ for $\|f\|_{L^p,w}$, $L^p(R^n, w)$ and $BL_m(L^p(R^n, w))$, respectively. Hereafter we limit ourselves to the case $1 \leq m \leq n - 1$.

For a multiindex $\alpha = (\alpha_1, \ldots, \alpha_n)$ we write $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $\alpha! = \alpha_1!\cdots\alpha_n!$ and

$$D^\alpha = \partial^{|\alpha|}/\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}.$$

By h_m we denote the Riesz kernel $\gamma(m)^{-1}|x|^{m-n}$ with $\gamma(m) = \pi^{n/2}\Gamma(\frac{m}{2})/\Gamma(\frac{n-m}{2})$ (cf. [17; p.117]). If $|\alpha| = m$, then $D^\alpha h_m$ is not a locally integrable function. It will be stated in Lemma 2 in §2 that $D^\alpha h_m$ is the sum of a principal value distribution S_α and a multiple $b_\alpha\delta$ of the Dirac measure δ at the origin. It is proved in [1] that this kind of distribution is related to weights w in A_p as follows:

(1) $$\|(D^\alpha h_m)*g\|_{L^p,w} \leq \text{const.}\|g\|_{L^p,w} \quad \text{for } g \in L^p(R^n, w).$$

Let $\mathcal{D}$ be the space of indefinitely differentiable functions with compact support and P_{m-1} the space of all polynomials of degree smaller than or equal to $m - 1$. Let $c_\alpha = (-1)^m m!/\alpha!$. Mizuta proved

Theorem A ([8; Theorem 5.2]). Let $2m < n$ and let $f \in BL_m(L^p(R^n))$. If

(2) there is a sequence $\{\varphi_j\}_j \subset \mathcal{D}$ such that $D^\alpha\varphi_j \to D^\alpha f$ in $L^p(R^n)$ for $|\alpha| = m$,

and $g = \sum_{|\alpha|=m} c_\alpha(D^\alpha h_m)*D^\alpha f$ satisfies

(3) $$\int_{R^n} (1 + |x|)^{m-n}|g(x)|dx < \infty,$$

then $f = h_m * g + P$ a.e. on R^n with some $P \in P_{m-1}$.

We shall show that assumption (2) is superfluos and the theorem extends to the case when $1 \leqq m \leqq n - 1$ and $f \in BL_m(L^p(R^n, w))$ with general $w \in A_p$. More precisely, we shall prove

Theorem 1. Let $w \in A_p$. Suppose that $f \in BL_m(L^p(R^n, w))$ and $g = \sum_{|\alpha|=m} c_\alpha (D^\alpha h_m) * D^\alpha f$. If g satisfies (3), then $f = h_m * g + P$ a.e. on R^n with some $P \in P_{m-1}$. Moreover this representation is unique in the sense that if $f = h_m * \mu + P'$ a.e. on R^n, where $P' \in P_{m-1}$ and μ is a signed measure such that

$$(4) \qquad \int_{R^n} (1 + |x|)^{m-n} d|\mu|(x) < \infty,$$

then μ is absolutely continuous, $d\mu = g dx$ and $P' = P$.

Ohtsuka [13] proved Theorem 1 for $m = 1$ and $w = 1$ by using extremal length (see also [12] for the definition and the properties of extremal length). In case $m > 1$, however, the theory of extremal length is not applicable to $BL_m(L^p(R^n, w))$, so our argument will depend on the general theory of distributions and singular integrals (see [1], [11], [16] and [17]).

Let $\omega_{n-1} = 2\pi^{n/2}/\Gamma(\frac{n}{2})$ be the surface area of the unit sphere in R^n and let $a_\alpha = m/(\alpha! \omega_{n-1})$. Mizuta proved

Theorem B ([8; Theorem 3.1]). Let $f \in BL_m(L^p(R^n))$ satisfy (2). If

$$(5) \qquad \int_{R^n} (1 + |x|)^{m-n} |D^\alpha f(x)| dx < \infty \quad \text{for any } \alpha \text{ with } |\alpha| = m,$$

then

(6) $$f(x) = \sum_{|\alpha|=m} a_\alpha \int_{R^n} \frac{(x-y)^\alpha D^\alpha f(y)}{|x-y|^n} dy + P(x) \quad \text{a.e. on } R^n,$$

where $P \in P_{m-1}$.

In case m = 1 Ohtsuka [13; Theorem 29] proved that (2) can be dropped. We shall extend Ohtsuka's result to higher order Beppo Levi functions with weight w in A_p.

Theorem 2. Let $w \in A_p$ and let $f \in BL_m(L^p(R^n, w))$. If f satisfies (5), then (6) holds.

It is easy to see that $w(x) = (1 + |x|)^{rp}$ belongs to A_p if and only if $- n < rp < n(p - 1)$. Hence this theorem includes Kurokawa [4; Theorem 2.6].

In case $g = \sum_{|\alpha|=m} c_\alpha (D^\alpha h_m)*D^\alpha f$ does not satisfy (3), the weighted Beppo Levi function f cannot be represented as the sum of a Riesz potential and a polynomial. However, a certain modification of the Riesz kernel (cf. [3; Chapter IV]) will enable us to represent f as the sum of a modified Riesz potential and a polynomial, and to show

Theorem 3 (cf. [14], [4; Theorem 3.2]). Let $w \in A_p$. If $f \in BL_m(L^p(R^n, w))$, then there is a sequence $\{\varphi_j\}_j \subset \mathcal{D}$ such that

$$\lim_{j\to\infty} \sum_{|\alpha|=m} \|D^\alpha f - D^\alpha \varphi_j\|_{L^p,w} = 0.$$

In the rest of this section we deal with $w \in A_p$ for which every

$g \in L^p(R^n, w)$ satisfies (3). In order to simplify the notation we denote by $A_{p,m}$ the class of all weights $w \in A_p$ such that every $g \in L^p(R^n, w)$ satisfies (3). We shall show that $w \in A_{p,m}$ if and only if

$$(7) \qquad \int_{R^n} (1 + |x|)^{(m-n)p/(p-1)} w(x)^{1/(1-p)} dx < \infty.$$

See Theorem 7 in §5. Since $w(x) = (1 + |x|)^{rp}$ belongs to $A_{p,m}$ if and only only if $m - n/p < r < n(1 - 1/p)$, it follows that $A_{p,m}$ is a proper subclass of A_p. If $w \in A_{p,m}$, then Theorem 1 gives a decomposition

$$BL_m(L^p(R^n, w)) = I_m(L^p(R^n, w)) \oplus P_{m-1},$$

where $I_m(L^p(R^n, w)) = \{h_m * g;\ g \in L^p(R^n, w)\}$. We shall consider a condition for $f \in BL_m(L^p(R^n, w))$ to belong to $I_m(L^p(R^n, w))$. For this purpose we introduce a notion which describes the behavior at ∞ of a weighted Beppo Levi function.

Definition. Let f_j and $f \in BL_m(L^p(R^n, w))$. We say that f_j converges to f in the $BL_m(L^p(R^n, w))$ sense if

$$\lim_{j\to\infty} \sum_{|\alpha|=m} \|D^\alpha f_j - D^\alpha f\|_{L^p,w} = 0,$$

$$\lim_{j\to\infty} f_j = f \quad \text{a.e. on } R^n.$$

We say that f vanishes at ∞ in the $BL_m(L^p(R^n, w))$ sense if there is a sequence $\{\varphi_j\}_j \subset \mathcal{D}$ converging to f in the $BL_m(L^p(R^n, w))$ sense.

We shall show

Theorem 4. Let $w \in A_{p,m}$. Then $f \in BL_m(L^p(R^n, w))$ belongs to

$I_m(L^p(R^n, w))$ if and only if f vanishes at ∞ in the $BL_m(L^p(R^n, w))$ sense.

Corollary 1. Let $w \in A_{p,m}$. If $f \in BL_m(L^p(R^n, w))$ and $\lim_{|x|\to\infty} f(x) = 0$, then $f \in I_m(L^p(R^n, w))$, and hence f vanishes at ∞ in the $BL_m(L^p(R^n, w))$ sense.

We shall give a criterion for $f \in BL_m(L^p(R^n, w))$ to vanish at ∞ in terms of the integrability of f in case $w = V^p$ is a weight introduced by Muckenhoupt and Wheeden [11].

Lemma A ([11; Theorem 4]). Let $1 < p < n/m$ and $1/p^* = 1/p - m/n$. Suppose that $V \geqq 0$ satisfies

$$\sup_Q\left(\frac{1}{|Q|}\int_Q V^{p^*}dx\right)^{1/p^*}\left(\frac{1}{|Q|}\int_Q V^{-p'}dx\right)^{1/p'} < \infty, \tag{8}$$

where $p' = p/(p-1)$ and the supremum is taken over all cubes Q with sides parallel to the axes. Then

$$\|(h_m * g)V\|_{L^{p^*}} \leqq \text{const.}\|gV\|_{L^p} \quad \text{for } g \in L^p(R^n, V^p).$$

Obviously, Hölder's inequality yields that if V satisfies (8), then $V^p \in A_p$. Hence we can easily deduce from (1) and this lemma that $V^p \in A_{p,m}$. We shall show

Theorem 5. Let $1 < p < n/m$, $1/p^* = 1/p - m/n$ and V satisfy (8).

(i) A function f in $BL_m(L^p(R^n, V^p))$ vanishes at ∞ in the

$BL_m(L^p(R^n, V^p))$ sense if and only if $f \in L^{p*}(R^n, V^{p*})$.

(ii) If $f \in BL_m(L^p(R^n, V^p))$ satisfies

$$\int_{R^n} |f(x)|^q V(x)^r dx < \infty,$$

for some $q > 0$ and some r, $0 < r \leqq p*$, then f vanishes at ∞ in the $BL_m(L^p(R^n, V^p))$ sense.

This theorem yields the implication

$$BL_m(L^p(R^n, V^p)) \cap \Big(\bigcup_{\substack{q>0 \\ 0<r\leqq p*}} L^q(R^n, V^r)\Big)$$

$$\subset BL_m(L^p(R^n, V^p)) \cap L^{p*}(R^n, V^{p*}) = I_m(L^p(R^n, V^p)).$$

By virtue of (1), Theorem 1 and Lemma A we readily have an improvement of [11; Theorem 9].

Corollary 2. Let m, p, $p*$ and V be as in Theorem 5. Then there is a positive constant C depending only on m, p and V such that

$$\|fV\|_{L^{p*}} \leqq C\sum_{|\alpha|=m} \|(D^\alpha f)V\|_{L^p}$$

for $f \in BL_m(L^p(R^n, V^p)) \cap \Big(\bigcup_{\substack{q>0 \\ 0<r\leqq p*}} L^q(R^n, V^r)\Big)$.

§2. Preliminaries

We collect some basic results on the theory of distributions. We shall mainly use the notation of [16]. We write

$$\langle T, \varphi\rangle = T(\varphi)$$

for a distribution T and a test function φ. In order to avoid confusion, we write

$$\langle T_x, \varphi\rangle$$

if φ involves two variables x and y, and the distribution T acts on $\varphi(\cdot, y)$ for each fixed y. As in [16; Chapitre VII] we define the Fourier transform of $\varphi \in \mathcal{S}$ and that of $T \in \mathcal{S}'$ by

$$\mathcal{F}\varphi(y) = \hat{\varphi}(y) = \int_{R^n} e^{-2\pi i x\cdot y}\varphi(x)dx,$$

$$\langle \mathcal{F}T, \varphi\rangle = \langle \hat{T}, \varphi\rangle = \langle T, \hat{\varphi}\rangle \quad \text{for } \varphi \in \mathcal{S},$$

where $\mathcal{S}$ is the space of indefinitely differentiable functions decreasing rapidly at ∞ and $\mathcal{S}'$ is the space of tempered distributions. We note that the Fourier transform defined here corresponds to the inverse Fourier transform in [17]. By $\mathcal{E}'$ and $\mathcal{D}'_{L^p}$ we denote the space of distributions of compact support and that of distributions T of the form

$$T = \sum_{|\alpha|\leqq k} D^\alpha f_\alpha, \quad \text{where } k \geqq 0 \text{ and } f_\alpha \in L^p(R^n).$$

Schwartz [16; Chapitres VI and VII] proved

Lemma B. (i) If $1 \leqq p \leqq q \leqq \infty$, then

$$\mathcal{E}' \subset \mathcal{D}'_{L^p} \subset \mathcal{D}'_{L^q} \subset \mathcal{S}' \subset \mathcal{D}'.$$

(ii) If $0 \leqq 1/r = 1/p + 1/q - 1 \leqq 1$, then the convolution $S*T$ exists and belongs to $\mathcal{D}'_{L^r}$ for $S \in \mathcal{D}'_{L^p}$ and $T \in \mathcal{D}'_{L^q}$.

(iii) If S and T belong to $\mathcal{D}'_{L^2}$, then $\mathcal{F}S$ and $\mathcal{F}T$ belong to $L^2_{loc}(R^n)$ and $\mathcal{F}(S*T) = \mathcal{F}S\cdot\mathcal{F}T$.

We can easily give another condition for the convolution of a function and a measure to be defined.

Lemma 1. (i) Let ℓ be a real number. Suppose that $f \in L^1_{loc}(R^n)$ and $|f(x)| \leqq \text{const.}|x|^\ell$ for $|x| \geqq 1$. If a signed measure μ satisfies

$$\int_{R^n} (1 + |x|)^\ell d|\mu|(x) < \infty,$$

then $f*\mu$ is well-defined and belongs to $L^1_{loc}(R^n)$; moreover

$$D^\beta(f*\mu) = (D^\beta f)*\mu = f*(D^\beta\mu) \quad \text{for any multiindex } \beta.$$

(ii) Let $0 < m < n$. If a signed measure μ satisfies (4), then $h_m*\mu$ exists and belongs to $L^1_{loc}(R^n)$. If μ does not satisfy (4), then $h_m*|\mu| \equiv \infty$ on R^n.

We need several results from the theory of singular integrals. Consider the class consisting of all distributions T of the form

(9) $\quad T = c\delta + \text{v.p.}\ \dfrac{\Omega(x)}{|x|^n},$

i.e., $\langle T, \varphi\rangle = c\varphi(0) + \lim_{\varepsilon\downarrow 0} \int_{|x|>\varepsilon} \dfrac{\Omega(x)}{|x|^n}\,\varphi(x)dx$ for $\varphi \in \mathcal{D}$,

where c is a constant; Ω is a homogeneous function of degree 0, which is indefinitely differentiable on the unit sphere and

$$\int_{|x|=1} \Omega(x)d\sigma(x) = 0.$$

Lemma C ([17; Chapter III, Theorem 6]). A distribution T in $\mathcal{S}'$ is written as (9) if and only if the Fourier transform $\mathcal{F}T$ is a homogeneous function of degree 0, which is indefinitely differentiable on the unit sphere.

The Muckenhoupt A_p condition is related to distributions of the form (9) as follows:

Lemma D ([1]). Let $w \in A_p$ and let T be a distribution of the form (9). Then

(10) $$\|T*g\|_{L^p,w} \leqq \text{const.}\|g\|_{L^p,w} \quad \text{for } g \in L^p(R^n, w).$$

From Lemmas C and D we can derive a generalization of (1).

Lemma 2 (cf. [8; §3]). Let $m \geqq 1$ and $\ell \geqq 0$. If $|\alpha| = m$ and $|\beta| = \ell$, then the distribution

$$T = D^\alpha\left(\frac{x^\beta}{|x|^{n-m+\ell}}\right),$$

in particular $D^\alpha h_m$, is of the form (9) and satisfies (10).

By $\mathcal{D}'_{L^p,w}$ we denote the class of distributions of the form

$$\sum_{|\alpha|\leqq k} D^\alpha f_\alpha, \text{ where } k \geqq 0 \text{ and } f_\alpha \in L^p(R^n, w).$$

We shall have

Lemma 3. Let S and T be distributions of the form (9) and $w \in A_p$. Then

(i) S and T belong to $\mathcal{D}'_{L^q}$ for any $q > 1$.

(ii) The convolution $S*T$ exists and is of the form (9).

(iii) If $f \in L^p(R^n, w)$, then $(S*T)*f = S*(T*f) \in L^p(R^n, w)$.

(iv) If $U \in \mathcal{D}'_{L^p,w}$, then $(S*T)*U = S*(T*U) \in \mathcal{D}'_{L^p,w}$.

It is proved in [1] that every $w \in A_p$ satisfies the Muckenhoupt A_∞ condition:

There are positive constants C, $\delta > 0$ such that given any cube Q and any measurable subset E of Q,

$$(A_\infty) \qquad \frac{w(E)}{w(Q)} \leqq C\left(\frac{|E|}{|Q|}\right)^\delta, \text{ where } w(A) = \int_A w\,dx \text{ for } A \subset R^n.$$

We shall denote by A_∞ the class of weights w satisfying (A_∞). Then $A_\infty = \cup_{p>1}^{\infty} A_p$ (see [1]). We collect some properties of A_p and A_∞ weights.

Lemma 4. Let w_1 and w_2 belong to A_p. Then the weights $\max\{w_1, w_2\}$, $\min\{w_1, w_2\}$ and w_1+w_2 belong to A_p.

Lemma 5. Let $w \in A_\infty$. If L is a cone with vertex at the origin, then $w(L) = \infty$. Moreover, there are no nonnegative functions u and v such that

$$(11) \qquad \begin{aligned} &w \leqq u + v \text{ on } L, \\ &\int_L u^q dx + \int_L v^r dx < \infty \quad \text{for some } q,\ r,\ 0 < q \leqq r \leqq 1. \end{aligned}$$

If a polynomial P belongs to $L^s(R^n, w)$ for some $s > 0$, then $P \equiv 0$.

Lemma 6. Let $w \in A_p$. Then

(i) $\int_{R^n} (1 + |x|)^{-n}|g(x)|dx < \infty$ for $g \in L^p(R^n, w)$.

(ii) $\int_{R^n} (1 + |x|)^{-np}w(x)dx < \infty$.

§3. Proof of Theorems 1 and 2

Proof of Theorem 1. In this proof we let α, β and γ be multiindicies of length m. Take $\varphi \in \mathcal{D}$ such that $\varphi = 1$ on a neighborhood of the origin and let $h'_m = \varphi h_m$ and $h''_m = (1-\varphi)h_m$. Since $h''_m \in L^q(R^n)$ for any $q > n/(n-m)$ and $h'_m \in \mathcal{E}'$, it follows from Lemma B (i) that $h_m \in \mathcal{D}'_{L^q}$ for any $q > n/(n-m)$. Since $D^\alpha h_m$ belongs to $\mathcal{D}'_{L^q}$ for any $q > 1$ by Lemmas 2 and 3, we have from Lemma B (ii) that the convolution $h_m * D^\alpha h_m$ is well-defined and belongs to $\mathcal{D}'_{L^q}$ for any $q > n/(n-m)$. We observe that

$$D^\beta(h_m * D^\alpha h_m) = D^\alpha h_m * D^\beta h_m = h_m * D^\alpha D^\beta h_m.$$

Noting that $D^\alpha h_m \in \mathcal{D}'_{L^2}$, we obtain from Lemma B (iii) that

$$\begin{aligned}\mathcal{F}(h_m * \textstyle\sum_{|\alpha|=m} c_\alpha D^{2\alpha} h_m) &= \mathcal{F}(\textstyle\sum_{|\alpha|=m} c_\alpha D^\alpha h_m * D^\alpha h_m) \\ &= \textstyle\sum_{|\alpha|=m} c_\alpha \{(2\pi i x)^\alpha (2\pi|x|)^{-m}\}^2 = 1,\end{aligned}$$

because

$$(2\pi|x|)^{2m} = \mathcal{F}((-\Delta)^m) = \mathcal{F}(\textstyle\sum_{|\alpha|=m} c_\alpha D^{2\alpha}) = \textstyle\sum_{|\alpha|=m} c_\alpha (2\pi i x)^{2\alpha}.$$

Accordingly

$$h_m * \textstyle\sum_{|\alpha|=m} c_\alpha D^{2\alpha} h_m = \textstyle\sum_{|\alpha|=m} c_\alpha D^\alpha h_m * D^\alpha h_m = \delta. \tag{12}$$

By (3) and Lemma 1 we obtain that the convolution $h_m * g$ is well-defined and belongs to $L^1_{loc}(R^n)$. We infer from Lemmas 2, 3 and (12) that

$$\begin{aligned}D^\beta D^\gamma(h_m * g) &= D^\beta h_m * D^\gamma g = D^\beta h_m * ((\textstyle\sum_{|\alpha|=m} c_\alpha D^\alpha h_m) * D^\gamma D^\alpha f) \\ &= (D^\beta h_m * \textstyle\sum_{|\alpha|=m} c_\alpha D^{2\alpha} h_m) * D^\gamma f = D^\beta \delta * D^\gamma f = D^\beta D^\gamma f.\end{aligned}$$

Since β is arbitrary, it follows that

$$D^\gamma f = D^\gamma(h_m * g) + P_\gamma \quad \text{for any } \gamma \text{ with } |\gamma| = m,$$

where $P_\gamma \in P_{m-1}$. However $D^\gamma f \in L^p(R^n, w)$ and

$$D^\gamma(h_m * g) = \sum_{|\alpha|=m} c_\alpha D^\gamma h_m * (D^\alpha h_m * D^\alpha f) \in L^p(R^n, w)$$

by Lemma 3, and hence P_γ must be identically 0 by Lemma 5. Since $D^\gamma f = D^\gamma(h_m * g)$ for any γ with $|\gamma| = m$, it follows that

$$f = h_m * g + P,$$

where $P \in P_{m-1}$.

The uniqueness of the representation readily follows from the following proposition, which may be of some independent interest.

Proposition 1. Let $0 < m < n$ and let μ be a signed measure satisfying (4). If $h_m * \mu$ coincides with some polynomial P, then $\mu = 0$ and hence P must be 0.

Proof. We define a sequence of signed measures μ_j of compact support by $\mu_j(E) = \mu(\{x \in E;\ |x| \leq j\})$. Since

$$\left|\int_{|x|>j} \varphi d\mu(x)\right| \leq \text{const.} \int_{|x|>j} (1 + |x|)^{m-n} d|\mu|(x) \quad \text{for } \varphi \in \mathcal{S},$$

it follows that $\mu_j \to \mu$ in $\mathcal{S}'$. We claim that $h_m * \mu_j \to h_m * \mu$ in $\mathcal{S}'$. Let $\varphi \in \mathcal{S}$. Take $\psi \in \mathcal{D}$ such that $\psi(x) = 1$ for $|x| \leq 1$ and write

$$\varphi = \psi\varphi + (1-\psi)\varphi = \varphi_1 + \varphi_2.$$

It is easy to see that $h_m * |\varphi_1|(x) = O(|x|^{m-n})$ as $|x| \to \infty$. Since φ decreases rapidly, we have $|\varphi_2(y)| \leq \text{const.}|y|^{-n-1}$. Let $|x| > 2$. Then

$$h_m * |\varphi_2|(x) \leqq \text{const.}\int_{|y|>1} |x - y|^{m-n}|y|^{-n-1}dy$$

$$= \text{const.}|x|^{m-n-1}\{\int_{|x|^{-1}<|z|<2^{-1}} + \int_{|z|>2^{-1}} |\tfrac{x}{|x|} - z|^{m-n}|z|^{-n-1}dz\}.$$

We see that the first integral is not greater than

$$\text{const.}\int_{|x|^{-1}}^{2^{-1}} t^{-2}dt \leqq \text{const.}|x|,$$

and that the second integral is a finite value independent of x. Therefore $h_m*|\varphi_2|(x) = O(|x|^{m-n})$ as $|x| \to \infty$. Accordingly

$$|\int \varphi(h_m*(\mu-\mu_j))dx| \leqq \text{const.}\int_{|x|>j} |x|^{m-n}d|\mu|(x) \to 0$$

by (4) and Fubini's theorem. Thus $h_m*\mu_j \to h_m*\mu$ in $\mathcal{S}'$.

Now we see that $\mathcal{F}(h_m*\mu_j) = (2\pi|x|)^{-m}\hat{\mu}_j \in L^1_{loc}(R^n)$. In fact, since $\mu_j \in \mathcal{E}' \subset \mathcal{D}'_{L^q}$ for any $q > 1$, it follows that $h_{m/2}$ and $h_{m/2}*\mu_j$ belong to $\mathcal{D}'_{L^2}$, and from Riesz's composition formula that $h_m*\mu_j = h_{m/2}*(h_{m/2}*\mu_j)$. We infer from Lemma B (iii) that

$$\mathcal{F}(h_m*\mu_j) = \mathcal{F}(h_{m/2}*(h_{m/2}*\mu_j)) = \hat{h}_{m/2}\cdot\hat{h}_{m/2}\cdot\hat{\mu}_j = (2\pi|x|)^{-m}\hat{\mu}_j.$$

Since the total variation of μ_j is finite, it follows that $\hat{\mu}_j$ is a bounded function, so that $(2\pi|x|)^{-m}\hat{\mu}_j \in L^1_{loc}(R^n)$.

Noting that $\mu_j \to \mu$ and $h_m*\mu_j \to h_m*\mu = P$ in $\mathcal{S}'$, we obtain that

$$\hat{\mu}_j \to \hat{\mu} \quad \text{and} \quad (2\pi|x|)^{-m}\hat{\mu}_j \to \mathcal{F}(P) = P(\tfrac{-1}{2\pi i}\tfrac{\partial}{\partial x})\delta \quad \text{in } \mathcal{S}'.$$

For any $\varphi \in \mathcal{S}$ vanishing on a neighborhood of the origin we have $\psi = (2\pi|x|)^m\varphi(x) \in \mathcal{S}$ and

$$\langle\hat{\mu}, \varphi\rangle = \lim_{j\to\infty} \langle\hat{\mu}_j, \varphi\rangle = \lim_{j\to\infty} \int\hat{\mu}_j(x)\varphi(x)dx$$

$$= \lim_{j\to\infty} \int(2\pi|x|)^{-m}\hat{\mu}_j(x)\psi(x)dx = \langle P(\tfrac{-1}{2\pi i}\tfrac{\partial}{\partial x})\delta, \psi\rangle = 0.$$

This implies that $\hat{\mu}$ is supported on $\{0\}$. Hence we can write

$$\hat{\mu} = P'\left(\frac{-1}{2\pi i}\frac{\partial}{\partial x}\right)\delta$$

with some polynomial P'. By the inverse Fourier transform we have $\mu = P'$, i.e., μ is absolutely continuous and $d\mu = P'dx$. Since μ satisfies (4), it follows that $(1 + |x|)^{m-n}P'(x)$ is integrable, so that P' must be identically zero. Hence $\mu = 0$ and $P = 0$.

Remark 1. The above proof works even if m is not an integer. In case m is an integer, $d\mu = g\,dx$, $g \in L^p(R^n, w)$ and $P \in P_{m-1}$, it is possible to give a simple proof. In fact by (12) and Lemma 3

$$g = \sum_{|\alpha|=m} c_\alpha D^\alpha h_m * D^\alpha(h_m * g) = \sum_{|\alpha|=m} c_\alpha D^\alpha h_m * D^\alpha P = 0.$$

Proof of Theorem 2. By using polar coordinates and integration by parts, we can prove

$$\sum_{|\alpha|=m} a_\alpha D^\alpha\left(\frac{x^\alpha}{|x|^n}\right) = \delta \qquad (13)$$

(see [15; Lemma 6.2]). Let $|\beta| = |\gamma| = m$. Applying Lemma 1 to $\ell = m - n$, $f = x^\alpha/|x|^n$ and $d\mu = D^\gamma f dx$, we obtain that

$$\left(\frac{x^\alpha}{|x|^n}\right) * D^\gamma f \in L^1_{loc}(R^n) \text{ for each } \alpha \text{ and } \gamma.$$

We infer from Lemma 2 and (13) that

$$D^\beta D^\gamma\left(\sum_{|\alpha|=m} a_\alpha\left(\frac{x^\alpha}{|x|^n}\right) * D^\alpha f\right) = \sum_{|\alpha|=m} a_\alpha D^\beta\left(\frac{x^\alpha}{|x|^n}\right) * D^\alpha D^\gamma f$$

$$= \sum_{|\alpha|=m} a_\alpha D^\beta D^\alpha\left(\frac{x^\alpha}{|x|^n}\right) * D^\gamma f = D^\beta\left(\sum_{|\alpha|=m} a_\alpha D^\alpha\left(\frac{x^\alpha}{|x|^n}\right)\right) * D^\gamma f$$

$$= D^\beta \delta * D^\gamma f = D^\beta D^\gamma f.$$

Now the same argument as in the proof of Theorem 1 completes the proof.

§4. Proof of Theorem 3

Let us begin with modifying the Riesz kernel. The following technique is found in [3; Chapter IV] and [9, 10]. Observe that if $y \neq 0$, then $h_m(x - y)$ has a multiple power series expansion in x_1, $x_2, \dots, x_n$, convergent in a neighborhood of the origin. We write

$$h_m(x - y) = \sum_{\nu=0}^{\infty} a_\nu(x, y),$$

where, for fixed ν and $y \neq 0$, $a_\nu(x, y)$ is a homogeneous polynomial in x_1 to x_n of degree ν and continuous in x, y jointly for $y \neq 0$ (cf. [3; Lemma 4.1]). We now set

$$k_m(x, y) = \begin{cases} h_m(x - y) & \text{if } |y| \leqq 1 \\ h_m(x - y) - \sum_{\nu=0}^{m-1} a_\nu(x, y) & \text{if } |y| > 1. \end{cases}$$

Obviously $D_x^\alpha k_m(x, y) = D_x^\alpha h_m(x - y)$ for $|\alpha| \geqq m$. Since

$$|k_m(x, y)| \leqq \text{const.}|x|^m |y|^{-n} \quad \text{if } 2|x| \leqq |y|$$

(cf. [3; Lemma 4.2]), we can easily prove from Lemma 6 (i)

Lemma 7. Let $w \in A_p$. If $g \in L^p(R^n, w)$, then

$$\int_{R^n} k_m(x, y)g(y)dy \in L^1_{loc}(R^n),$$

$$D^\alpha\left(\int_{R^n} k_m(x, y)g(y)dy\right) = (D^\alpha h_m)*g \quad \text{for } |\alpha| \geqq m.$$

This lemma and the same argument as in Theorem 1 yield

Theorem 6. Let $w \in A_p$. If $f \in BL_m(L^p(R^n, w))$, then

$$f = \int_{R^n} k_m(x, y)g(y)dy + P, \quad g = \sum_{|\alpha|=m} c_\alpha (D^\alpha h_m)*D^\alpha f,$$

where $P \in P_{m-1}$.

Let $\mathcal{E}$ be the space of all indefinitely differentiable functions on R^n. We show

Lemma 8. Let $f \in I_m(L^p(R^n, w)) \cap \mathcal{E}$. Then for $\varepsilon > 0$ and $r > 0$ there is a function $\varphi \in \mathcal{D}$ such that

(14) $\sum_{|\alpha|=m} \|D^\alpha\varphi - D^\alpha f\|_{L^p,w} < \varepsilon$ and $\sup_{|x|<r} |\varphi(x) - f(x)| < \varepsilon$.

Proof. First we treat the case when $f = h_m*g$ with $g \in \mathcal{D}$. Let $R > r$ and supp $g \in \{y; |y| < R\}$. Take $\psi \in \mathcal{D}$ such that $0 \leqq \psi \leqq 1$ and $\psi(x) = 1$ for $|x| < 3R$ and put $\psi_j(x) = \psi(x/j)$. We observe that

$$0 \leqq \psi_j \leqq 1, \ \psi_j(x) = 1 \text{ for } |x| < 3Rj,$$

(15)

$$\sum_{k=0}^m \sum_{|\alpha|=k} \sup(|x|^k |D^\alpha\psi_j(x)|) = \sum_{k=0}^m \sum_{|\alpha|=k} \sup(|x|^k |D^\alpha\psi(x)|) < \infty.$$

Let $h_{m,j}(x) = \psi_j(x)h_m(x)$. Then $h_{m,j}*g \in \mathcal{D}$ and

$$h_{m,j}*g(x) = \int_{|y|<R} \psi_j(x-y)h_m(x-y)g(y)dy$$

$$= \int_{|y|<R} h_m(x-y)g(y)dy = h_m*g(x) \quad \text{for } |x| < 2Rj$$

by (15). Let α be a multiindex of length m. We have

$$D^\alpha h_{m,j}*g(x) = D^\alpha h_m*g(x) \quad \text{for } |x| < 2Rj,$$

and hence

$$D^\alpha h_{m,j}*g \to D^\alpha h_m*g \quad \text{on } R^n.$$

In view of (15) and Leibniz's formula we have

$$|D^\alpha h_{m,j}(x-y)| \leqq \text{const.}|x|^{-n} \quad \text{for } |x| > 2R \text{ and } |y| < R,$$

and hence

$$|D^\alpha h_{m,j}*g(x) - D^\alpha h_m*g(x)| \leqq \text{const.}|x|^{-n} \quad \text{for } |x| > 2R.$$

Now it follows from Lemma 6 (ii) and the dominated convergence theorem that

$$\int_{R^n} |D^\alpha h_{m,j}*g(x) - D^\alpha h_m*g(x)|^p w(x)dx$$

$$= \int_{|x|>2R} |D^\alpha h_{m,j}*g(x) - D^\alpha h_m*g(x)|^p w(x)dx \to 0,$$

so that $D^\alpha(h_{m,j}*g) \to D^\alpha(h_m*g)$ in $L^p(R^n, w)$. Therefore $\varphi = h_{m,j}*g$ satisfies (14) if j is sufficiently large.

Next we consider the general case. From the uniqueness in Theorem 1 f is written as $f = h_m*g$ with $g = \sum_{|\alpha|=m} c_\alpha(D^\alpha h_m)*D^\alpha f \in L^p(R^n, w) \cap \mathcal{E}$. It is easy to find $\psi \in \mathcal{D}$ such that $0 \leqq \psi \leqq 1$,

$$\sum_{|\alpha|=m} \|D^\alpha h_m*(\psi g) - D^\alpha f\|_{L^p,w} \leqq \text{const.}\|\psi g - g\|_{L^p,w} < \varepsilon/2,$$

and

$$\sup_{|x|<r} |h_m*(\psi g)(x) - h_m*g(x)| < \varepsilon/2.$$

From the first part there is a function $\varphi \in \mathcal{D}$ such that

$$\sum_{|\alpha|=m} \|D^\alpha\varphi - D^\alpha h_m*(\psi g)\|_{L^p,w} < \varepsilon/2,$$

$$\sup_{|x|<r} |\varphi(x) - h_m*(\psi g)(x)| < \varepsilon/2.$$

This φ satisfies (14).

Proof of Theorem 3. Let g be as in Theorem 6. It is easy to find a sequence $\{g_j\}_j \subset \mathcal{D}$ such that $\|g_j - g\|_{L^p,w} \to 0$. Since g_j has compact support, h_m*g_j exists and by Lemma 7

$$\int_{R^n} k_m(x,\ y)g_j(y)dy = h_m{*}g_j + P_j$$

with some $P_j \in P_{m-1}$. Now Lemma 8 gives a sequence $\{\varphi_j\}_j \subset \mathcal{D}$ such that

$$\sum_{|\alpha|=m} \|D^\alpha h_m{*}g_j - D^\alpha\varphi_j\|_{L^p,w} < 1/j.$$

We infer from Theorem 6, Lemmas 2 and 7 that

$$\begin{aligned}
&\sum_{|\alpha|=m} \|D^\alpha f - D^\alpha\varphi_j\|_{L^p,w} \\
&= \sum_{|\alpha|=m} \|D^\alpha(\int_{R^n} k_m(x,\ y)g(y)dy) - D^\alpha\varphi_j\|_{L^p,w} \\
&= \sum_{|\alpha|=m} \|(D^\alpha h_m){*}g - D^\alpha\varphi_j\|_{L^p,w} \\
&\leqq \sum_{|\alpha|=m} \|(D^\alpha h_m){*}(g-g_j)\|_{L^p,w} + 1/j \to 0.
\end{aligned}$$

The theorem is proved.

§5. Proof of Theorem 4

Lemma 9. If $\varphi \in \mathcal{D}$, then $\varphi = h_m{*}g \in I_m(L^p(R^n,\ w))$, where $g = \sum_{|\alpha|=m} c_\alpha D^\alpha h_m{*}D^\alpha\varphi \in L^p(R^n,\ w) \cap \mathcal{E}$.

Proof. Since $D^\alpha h_m$ is of the form (8) and $D^\alpha\varphi$ has compact support, it follows that $g(x) = O(|x|^{-n})$ as $|x| \to \infty$, so that g satisfies (3) and $h_m{*}g = o(1)$ as $|x| \to \infty$. By Theorem 1 we have $\varphi = h_m{*}g + P$ with some $P \in P_{m-1}$. However P must be equal to zero, for $P(x) = \varphi(x) - h_m{*}g(x) \to 0$ as $|x| \to \infty$.

Lemma 10. A function $f = h_m{*}g + P$ in $BL_m(L^p(R^n,\ w))$ vanishes at ∞ in the $BL_m(L^p(R^n,\ w))$ sense if and only if there is a sequence

$\{g_j\}_j \subset L^p(R^n, w) \cap \mathcal{E}$ such that

(16) $$\|g_j - g\|_{L^p,w} \to 0 \quad \text{and} \quad h_m{*}g_j \to f \text{ a.e. on } R^n.$$

Proof. First suppose that $\{g_j\}_j \subset L^p(R^n, w) \cap \mathcal{E}$ satisfies (16). Then by Lemma 8 there is a sequence $\{\varphi_j\}_j \subset \mathcal{D}$ such that

$$\textstyle\sum_{|\alpha|=m} \|D^\alpha h_m{*}g_j - D^\alpha\varphi_j\|_{L^p,w} < 1/j,$$

$$\sup_{|x|<j} |h_m{*}g_j(x) - \varphi_j(x)| < 1/j.$$

We easily see that φ_j converges to f in the $BL_m(L^p(R^n, w))$ sense.

Conversely suppose that $\{\varphi_j\}_j \subset \mathcal{D}$ converges to f in the $BL_m(L^p(R^n, w))$ sense. We infer from Lemmas 2 and 9 that $\varphi_j = h_m{*}g_j$, $g_j \in L^p(R^n, w) \cap \mathcal{E}$ and

$$\|g_j - g\|_{L^p,w} = \|\textstyle\sum_{|\alpha|=m} c_\alpha (D^\alpha h_m){*}(D^\alpha\varphi_j - D^\alpha f)\|_{L^p,w}$$

converges to zero. Thus $\{g_j\}_j$ satisfies (16).

For $E \subset R^n$ we define a capacity $R_{m,p,w}(E)$ by

$$R_{m,p,w}(E) = \inf\{\|g\|^p_{L^p,w}\ ;\ g \geqq 0,\ h_m{*}g \geqq 1 \text{ on } E\}.$$

The next theorem combines conditions (3) and (7), the capacity $R_{m,p,w}$ and the vanishing property of Beppo Levi functions.

Theorem 7. The following statements on $w \in A_p$ are equivalent:

(a) $w \in A_{p,m}$.

(b) For every $g \in L^p(R^n, w)$ the convolution $h_m{*}g$ exists and belongs to $L^1_{loc}(R^n)$.

(c) If $\|g_j\|_{L^p,w} \to 0$, then $h_m * g_j \to 0$ in measure on any ball.

(c') If $\|g_j\|_{L^p,w} \to 0$, then $h_m * g_j \to 0$ in measure on some ball.

(d) The constant function 1 does not vanish at ∞ in the $BL_m(L^p(R^n, w))$ sense.

(e) There is a set of positive $R_{m,p,w}$ capacity.

(f) If $|E| > 0$, then $R_{m,p,w}(E) > 0$.

(g) w satisfies (7).

Proof. The equivalence between (a) and (b) readily follows from Lemma 1. The implications (c) => (c') and (f) => (e) are obvious. We have (f) => (c) from [6; Theorem 4] and (g) => (a) from Hölder's inequality. We shall complete the proof by showing (b) => (f) => (g), (e) => (b) and (c') => (d) => (a).

(b) => (f): Suppose that there is a measurable set E such that $|E| > 0$ but $R_{m,p,w}(E) = 0$. By [6; Theorem 3] we find a nonnegative function g in $L^p(R^n, w)$ such that $h_m * g = \infty$ on E. Since $|E| > 0$, it follows that $h_m * g$ is not locally integrable, so that (b) does not holds.

(f) => (g): Since the unit ball B has positive capacity, it follows from [6; Theorem 14] that there exists a measure μ concentrated on B such that $\mu(B) > 0$ and $h_m * \mu \in L^{p'}(R^n, w^{1/(1-p)})$. Noting that $h_m * \mu(x) \geq \text{const.}\mu(B)h_m(x)$ for $|x| > 1$, we obtain

$$\int_{|x|>1} |x|^{(m-n)p'} w(x)^{1/(1-p)} dx < \infty,$$

which is equivalent to (7).

(e) => (b): If (b) does not hold, then there is a nonnegative function g in $L^p(R^n, w)$ such that $h_m * g \equiv \infty$ on R^n. By definition

$$0 \leqq R_{m,p,w}(E) \leqq R_{m,p,w}(R^n) \leqq \inf_{t>0} \|tg\|^p_{L^p,w} = 0.$$

Thus (e) does not hold.

(c') => (d): If 1 vanishes at ∞ in the $BL_m(L^p(R^n, w))$ sense, then there is a sequence $\{g_j\}_j \subset L^p(R^n, w)$ such that

$$\|g_j\|_{L^p,w} \to 0 \quad \text{and} \quad h_m * g_j \to 1 \text{ a.e. on } R^n$$

by Lemma 10. This contradicts (c').

(d) => (a): Suppose that there is a nonnegative function g in $L^p(R^n, w)$ such that (3) does not hold. Mollifying g, we may assume that $g \in L^p(R^n, w) \cap \mathcal{E}$. We shall prove that 1 vanishes at ∞ in the $BL_m(L^p(R^n, w))$ sense. By Lemma 10 it is sufficient to show that if $\varepsilon > 0$ and $R > 0$, then there is $g_1 \in L^p(R^n, w) \cap \mathcal{E}$ such that

$$|h_m * g_1(x) - 1| < \varepsilon \text{ for } |x| < R,$$
$$\|g_1\|_{L^p,w} < \varepsilon.$$

Take $R_1 > R$ such that

$$1 - \varepsilon < h_m(x-y)/h_m(y) < 1 + \varepsilon \quad \text{for } |x| < R \text{ and } |y| > R_1.$$

Since (3) does not hold and $g \in L^p(R^n, w)$, we find a function $\varphi \in \mathcal{D}$ such that $0 \leqq \varphi \leqq 1$, supp $\varphi \subset \{y;\ |y| > R_1\}$ and $g_1 = \varphi g$ satisfies $h_m * g_1(0) = 1$ and $\|g_1\|_{L^p,w} < \varepsilon$. We observe that

$$1 - \varepsilon < h_m * g_1(x)/h_m * g_1(0) = h_m * g_1(x) < 1 + \varepsilon \quad \text{for } |x| < R.$$

Hence g_1 has the desired property. Thus the theorem is completely proved.

Proof of Theorem 4. Suppose that $f = h_m * g \in I_m(L^p(R^n, w))$.

Take a nonnegative function ψ in $\mathcal{D}$ such that $\int \psi dx = 1$. Letting $\psi_j(x) = j^n \psi(jx)$, we observe that $g_j = g * \psi_j \in L^p(R^n, w) \cap \mathcal{E}$ satisfies (16). Hence f vanishes at ∞ in the $BL_m(L^p(R^n, w))$ sense by Lemma 10.

Conversely suppose that $f = h_m * g + P \in BL_m(L^p(R^n, w))$ vanishes at ∞ in the $BL_m(L^p(R^n, w))$ sense. Since $h_m * g$ vanishes at ∞ in the $BL_m(L^p(R^n, w))$ sense from the only if part of the theorem, it follows that $P = f - h_m * g$ vanishes at ∞ in the $BL_m(L^p(R^n, w))$ sense. Hence there is a sequence $\{g_j\}_j \subset L^p(R^n, w)$ such that

$$\|g_j\|_{L^p, w} \to 0 \quad \text{and} \quad h_m * g_j \to P \text{ a.e. on } R^n$$

by Lemma 10. On account of (c) of Theorem 7 we have $P = 0$. The proof is complete.

For the proof of Corollary 1 we prepare

Lemma 11. Let L be a cone with vertex at the origin. Then $R_{m,p,w}(L)$ is equal to 0 or ∞; $R_{m,p,w}(L) = 0$ if and only if $R_{m,p,w}(R^n) = 0$. The constant 1 vanishes at ∞ in the $BL_m(L^p(R^n, w))$ sense if and only if $R_{m,p,w}(R^n) = 0$.

Proof. If $0 < R_{m,p,w}(L) < \infty$, then there would exist a nonnegative function g in $L^p(R^n, w)$ satisfying (3) and $h_m * g \geqq 1$ on L by Theorem 7. Since L is not m-thin at ∞ in the notation of [5], this contradicts

$$\liminf_{|x| \to \infty,\ x \in L} h_m * g(x) = 0$$

([5; Theorem 3.3]). By Theorem 7 we can easily prove the remainder.

Proof of Corollary 1. Suppose that $f = h_m * g + P$ and $P \not\equiv 0$. Then we would find $\varepsilon > 0$, $R > 0$ and a cone L with vertex at the origin such that

$$h_m * |g|(x) \geqq |f(x) - P(x)| \geqq \varepsilon \quad \text{if } |x| \geqq R \text{ and } x \in L.$$

By definition

$$R_{m,p,w}(L) \leqq R_{m,p,w}(\{x;\ |x|<R\}) + R_{m,p,w}(\{x \in L;\ |x| \geqq R\}) < \infty$$

and hence by Lemma 11 $R_{m,p,w}(L) = 0$. This contradicts (f) of Theorem 7.

§6. Proof of Theorem 5

Proof of Theorem 5. First suppose that f vanishes at ∞ in the $BL_m(L^p(R^n, V^p))$ sense. Thus f is written as $h_m * g$ with $g \in L^p(R^n, V^p)$. On account of Lemma 10 there is a sequence $\{g_j\}_j \subset L^p(R^n, V^p)$ satisfying (16) with $w = V^p$. Since

$$\|(h_m * g_j)V\|_{L^{p*}} \leqq \text{const.} \|g_j V\|_{L^p} \leqq \text{const.}$$

by Lemma A, Fatou's lemma leads to

$$\|fV\|_{L^{p*}} \leqq \liminf_{j \to \infty} \|(h_m * g_j)V\|_{L^{p*}} < \infty.$$

The if part of (i) is included in (ii). Now we shall prove (ii) by contradiction. Suppose that $f = h_m * g + P$, $g \in L^p(R^n, V^p)$, $P \in P_{m-1}$ and $P \not\equiv 0$. Then we would find $\varepsilon > 0$, $R > 0$ and a cone L with vertex at the origin such that

$$|P(x)| \geqq 2\varepsilon \quad \text{for } x \in \{x \in L;\ |x| \geqq R\}.$$

We observe that $V^{p*} \leqq u + v$ on L, where

$$u(x) = \begin{cases} V(x)^{p*} & \text{if } |f(x)| \geqq \varepsilon \\ 0 & \text{otherwise} \end{cases}, \quad v(x) = \begin{cases} V(x)^{p*} & \text{if } |h_m * g(x)| \geqq \varepsilon \text{ or } |x| \leqq R \\ 0 & \text{otherwise} \end{cases}.$$

Since $V^{p*} \in A_\infty$ and $0 < r/p^* \leqq 1$,

$$\int_L u^{r/p*} dx = \int_{\{x \in L;\ |f(x)| \geqq \varepsilon\}} V^r dx \leqq \varepsilon^{-q} \int_{R^n} |f|^q V^r dx < \infty,$$

$$\int_L v dx = \int_{\{x \in L;\ |h_m * g(x)| \geqq \varepsilon\}} V^{p*} dx + \int_{|x| \leqq R} V^{p*} dx$$

$$\leqq \varepsilon^{-p*} \int_{R^n} |h_m * g|^{p*} V^{p*} dx + \int_{|x| \leqq R} V^{p*} dx < \infty,$$

we have a contradiction by Lemma 5. The theorem is proved.

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