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Exactly solvable lattice models and character formulas

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§0. Introduction

The subject of this article concerns with critical phenomena in 2 dimensional statistical systems. Apart from approximate methods, there are presently two different approaches to this problem:

(i) Exactly solvable model (ESM)

(ii) Conformal field theory (CFT).

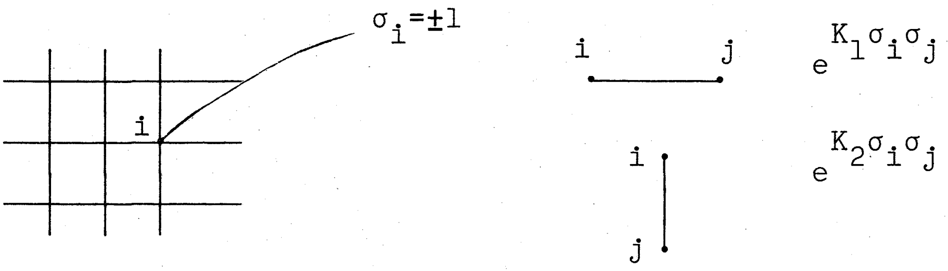
In the first approach, one attempts to construct lattice models for which physical quantities of interest (such as the free energy) can be obtained in a closed form. Representative examples are collected in Baxter's monograph [1]. The second one has been developed rather recently. Instead of working directly on the lattice, one starts with the (Euclidean) field theory that should correspond to the continuum limit of lattice models at criticality. Making full use of the conformal symmetry, along with consistency requirements, the authors of [2] were able to enumerate series of possible continuum models with various types of critical behavior (such a list can be compared to the "periodical table" in atomic chemistry [3]).

The methodology and applicability of these two appear to be quite different. The first covers non-critical models on the lattice, but it is very difficult as a rule to calculate general correlation functions (except for the Ising model discussed in §1). The second, on the other hand, deals with continuous and critical models (the conformal symmetry is achieved only under these restrictions); at this cost the general correlation functions are governed by linear differential equations. Since no reference is made to lattice models, this second approach applies equally well regardless of whether the "corresponding" lattice model is exactly solvable or not. (In other words, there is no direct way to identify a lattice model that tends to a given CFT in the "periodical table".)

In this article we shall mainly deal with ESM. After giving a brief review (§§1-2), we shall present a class of ESM whose one point correlation functions are given in terms of modular forms that arise in the representation theory of affine Lie algebras. We then observe that there is a mysterious correspondence between ESM, CFT and affine Lie algebras.

§1. Ising Model

Among known solvable models, the best understood case is the two dimensional Ising model with nearest neighbor interaction. Consider a planer square lattice. At each site i associate a random variable σ_i that assumes two values $+1$ or -1. (This is a simple model of a magnet consisting of molecules that can point two directions: up-down, or north-south.) The interaction is introduced by assigning a weight $e^{K_\nu \sigma_i \sigma_j}$ (ν=1,2) to each bond joining neighboring sites (i,j), where the coupling constant $K_\nu > 0$ is proportional to the inverse temperature T^{-1} and ν=1,2 refers to whether the bond is horizontal or vertical.



The probability of finding a particular configuration $\boldsymbol{\sigma} = \{\sigma_i\}$ is then given by

$$p(\boldsymbol{\sigma}) = Z^{-1} \prod_{\text{bond}} e^{K_\nu \sigma_i \sigma_j} , \qquad Z = \sum_{\text{config.}} \prod_{\text{bond}} e^{K_\nu \sigma_i \sigma_j} .$$

The quantities of interest are the free energy

$$f = \frac{1}{\text{vol}} \log Z$$

and various correlation functions

$$\langle \sigma_{i_1} \rangle = \sum_{\boldsymbol{\sigma}} \sigma_{i_1} p(\boldsymbol{\sigma}), \quad \langle \sigma_{i_1} \sigma_{i_2} \rangle = \sum_{\boldsymbol{\sigma}} \sigma_{i_1} \sigma_{i_2} p(\boldsymbol{\sigma}), \cdots$$

in the large lattice limit vol(=#(sites)) $\to \infty$. (To be precise the boundary spins are kept fixed to one of the ground states in passing to the limit.)

At zero temperature only two configurations are allowed —— all spins up, or all spins down (extreme order) whereas at infinite temperature, all configurations are given equal weight (extreme disorder). In between there is a critical temperature $T = T_c$ where the quantities above become singular:

$$f_{sing} \sim |T-T_c|^2 \log|T-T_c| \qquad (T \to T_c \pm 0)$$

(1)

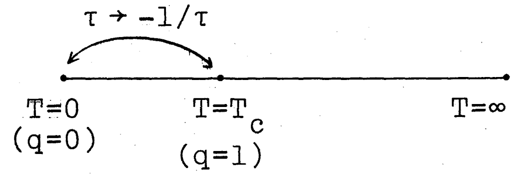
$$\langle \sigma_{i_1} \rangle \sim |T-T_c|^{1/8} \qquad (T \to T_c - 0).$$

The powers 2, 1/8 are the critical exponents which characterize the nature of the phase transition. This much is the classical result of Onsager [4] who found exact expressions for f and $\langle \sigma_{i_1} \rangle$. For example, the formula for $\langle \sigma_{i_1} \rangle$ reads

(2) $$\langle \sigma_{i_1} \rangle = \frac{1-q}{1+q} \frac{1-q^3}{1+q^3} \frac{1-q^5}{1+q^5} \cdots$$

Here we have used Onsager's parametrization of $K_\nu = K_\nu(u,q)$ in terms of elliptic functions; roughly speaking u plays the role of anisotropy $K_1 : K_2$ and q of the temperature. In particular $q \to 0$ corresponds to the extreme order, while $q \to 1$ corresponds to the criticality. The critical behavior (1) can be extracted thanks to the automorphic property of (2) under

(3) $$\tau \to -1/\tau \qquad (q = e^{2\pi i \tau}).$$



Note, however, that there is <u>no a priori reason</u> why $<\sigma_{i_1}>$ should be a modular form.

Higher correlation functions are governed by nonlinear differential/difference equations. When $T \neq T_c$, the two point function at large distance decays exponentially as

$$<\sigma_{i_1}\sigma_{i_2}> - <\sigma_{i_1}><\sigma_{i_2}> \sim e^{-|i_1-i_2|/\xi} .$$

As T approaches T_c, the correlation length ξ diverges like $|T-T_c|^{-1}$. Retaining the ratio $|i_1-i_2|/\xi = mr$ fixed (assuming $K_1=K_2$ for simplicity) and letting $T \to T_c \pm 0$, one gets the scaled two point function $\tau_{\pm}(r)$. As it turns out, they are expressible in terms of $\psi = -\log\left(\frac{\tau_- - \tau_+}{\tau_- + \tau_+}\right)$ that satisfies the nonlinear ODE [5]

$$\frac{d^2}{dr^2}\psi + \frac{1}{r}\frac{d}{dr}\psi = \frac{1}{2}\sinh(2\psi).$$

This is an equivalent form of what is known as a Painlevé equation of the 3rd kind. General correlations obey nonlinear PDEs related to monodromy preserving deformations of linear differential equations. [6] The corresponding difference equations on the lattice are also known [7].

	lattice	continuum
$T \neq T_c$	NL difference eq.	NLDE (Painlevé type)
$T = T_c$		LDE (hypergeometric type)

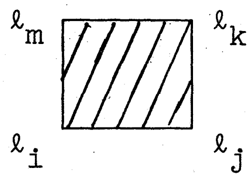
If one stays on criticality, these equations simplify drastically: these nonlinear equations reduce to linear ones. For instance the scaled 2 point function is a simple power function $r^{-1/4}$. This is precisely the regime covered by CFT.*)

Unfortunately, the Ising model is to date an isolated example for which all the lattice correlations are determined.

§2. Exactly Solvable Models

Apart from the Ising model, there exists a class of exactly solvable models in the sense that their free energy and 1 point function can be computed exactly. One of the formulations goes as follows [1].

Consider again a square lattice, and to each site attach ℓ_i (to be called a height variable), this time allowed to take multi-states, say $\ell_i = 1,2,\cdots,L-1$. The interaction is introduced by giving a weight to each configuration round a face (rather than to a bond):



$$\longleftrightarrow \quad W\begin{pmatrix} \ell_m & \ell_k \\ \ell_i & \ell_j \end{pmatrix} .$$

(In the case when $W\begin{pmatrix} d & c \\ a & b \end{pmatrix}$ takes the form $F_{ac}G_{bd}$, the model splits into 2 mutually non-interacting ones on diagonal sublattices. The Ising model can be regarded as this special case with $L-1 = 2$.)

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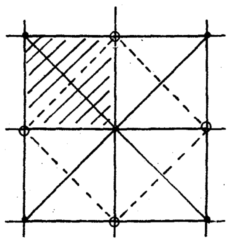
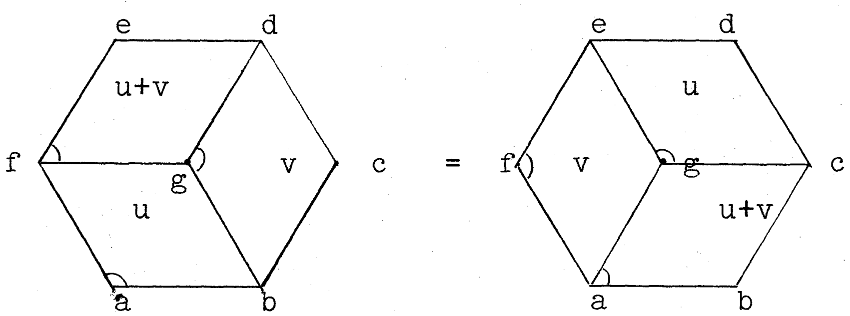
*) We remark that on the lattice the NL difference equations do <u>not</u> reduce to linear ones even at $T=T_c$.



Suppose the weights depend on an additional complex parameter $u \in \mathbb{C}$ so that the following functional equations are satisfied



for all $a, b, \cdots, f = 1, \cdots, L-1$

$$\sum_g W\left(\begin{smallmatrix} f & g \\ a & b \end{smallmatrix}\middle| u\right) W\left(\begin{smallmatrix} d & c \\ g & b \end{smallmatrix}\middle| v\right) W\left(\begin{smallmatrix} e & d \\ f & g \end{smallmatrix}\middle| u+v\right) = \sum_g W\left(\begin{smallmatrix} e & d \\ g & c \end{smallmatrix}\middle| u\right) W\left(\begin{smallmatrix} e & g \\ f & a \end{smallmatrix}\middle| v\right) W\left(\begin{smallmatrix} g & c \\ a & b \end{smallmatrix}\middle| u+v\right).$$

In this circumstance there is a known method to compute the free energy and the one point function.

The functional equations above (the Yang-Baxter equation, or the Star-Triangle relation) involve $(L-1)^4$ unknowns $W\left(\begin{smallmatrix} d & c \\ a & b \end{smallmatrix}\middle| u\right)$, whereas the number of equations is $(L-1)^6$. Each time a solution is found, we get a solvable lattice model. In all known cases the solutions are expressed in terms of elliptic, trigonometric or rational functions (recall the elliptic parametrization in the Ising model). The theory of the Yang-Baxter equation itself has an intriguing algebraic aspect related to Lie algebras and the Braid group [8], but we do not discuss it here.

§3. One point function and character formulas.

We shall now describe a series of models whose one point functions are related to the character formula of the affine Lie algebra $A_1^{(1)} \cong s\ell_2(\mathbb{C}[t,t^{-1}]) \ltimes \mathbb{C}c$.

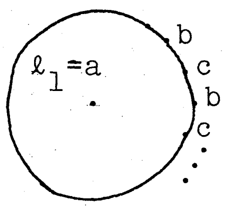
Given positive integers L,N $(L \geq N+3)$, consider an $(L-1)$-state lattice model as in §2 subject to the following restrictions:

(i) $\ell_i = 1,2,\cdots,L-1$

(ii) $\ell_i-\ell_j = N,N-2,\cdots,-N$

(iii) $N < \ell_i+\ell_j < 2L-N$

where (i,j) signify neighboring sites. Any configuration violating these is given zero weight. A system of weights $W\left(\begin{smallmatrix}\ell_m & \ell_k \\ \ell_i & \ell_j\end{smallmatrix}\middle| u\right)$ satisfying the star triangle relation and (i)-(iii) have been found in terms of elliptic functions [9]. The one point function, to be called the local height probability (LHP), is the probability of finding a height ℓ_1 to be a particular value, say, a. The boundary heights are fixed to be one of the ground states. Hereafter we consider the simplest region of the parameter space (called Regime III). In a ground state configuration, all height variables are the same along the southwest-northeast diagonal, and in the horizontal direction have the form $\cdots$ bc bc $\cdots$.



$P(a|bc)=\mathrm{Prob}(\ell_1=a|\mathrm{bdry}=b,c)$

In order to state the result, we need to prepare notations. Consider a pair of Lie algebras

$$\mathfrak{g} = A_1^{(1)} \oplus A_1^{(1)} \supset \mathfrak{k} = \Delta(A_1^{(1)})$$

where Δ signifies the diagonal embedding. Let $\chi_{k,m}(z,q)$ $(0 \le k \le m)$ denote the characters of irreducible highest weight representations of level m (see [10] for the terminology). Decomposing irreducible $\mathfrak{g}$-modules into $\mathfrak{k}$-irreducible pieces, we are led to the identity of the form

$$(4) \qquad \chi_{k_1 m_1}(z,q)\chi_{k_2 m_2}(z,q) = \sum_{0 \le k_3 \le m_3} B_{k_1 k_2 k_3}(q)\chi_{k_3 m_3}(z,q)$$

where $m_3 = m_1 + m_2$. Actually $\chi_{k,m}(z,q)$ is a ratio of theta functions [10], and consequently the B's have automorphic property under the transformation (3). After a lengthy and complicated computation, we find:

<u>Theorem</u> [11].

$$(5) \qquad P(a|b,c) = \frac{B_{k_1 k_2 k_3}(q)\chi_{k_3\ L-2}(\sqrt{q},q)}{\chi_{k_1 N}(\sqrt{q},q)\chi_{k_2\ L-N-2}(\sqrt{q},q)}$$

$$\left(k_1 = \frac{b-c+N}{2},\ k_2 = \frac{b+c-N}{2} - 1,\ \ k_3 = a-1\right).$$

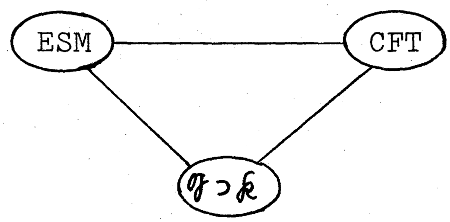
This formula may seem unwieldy, but the structure is quite simple: specialize the identity (4) to $z = \sqrt{q}$ and divide by the LHS so the total probability $\sum_a P(a|bc)$ is 1. Notice that there is the following correspondence:

(6) center height a $\longleftrightarrow$ irreducible representation of $\mathfrak{k}$.

boundary heights b,c $\longleftrightarrow$ irreducible representation of $\mathfrak{g}$.

So, here again the one point function is a modular form; this enables us to know the critical behavior $q \to 1$ of the LHP. However, we know of no mechanism that accounts for this phenomenum. It can be shown in fact that the B's are characters of the Virasoro algebra (= the Lie algebra of infinitesimal conformal transformations). For small N(=1,2,4), the critical exponents of (5) are in agreement with known CFT's in the "periodical table" [3], so it is natural to suspect that these CFT's are the critical, continuum limit of our model. We remark also that the pair of affine Lie algebras $\mathfrak{g} \supset \mathfrak{k}$ played a crucial role in constructing discrete series representations of the Virasoro algebra [12].

To summarize, we have encountered an empirical correspondence among the three objects:



Similar structure has been observed in a different series of ESM [13] ($\mathfrak{g} = A_1^{(1)}$, $\mathfrak{k}$ = (homogeneous) Heisenberg subalgebra), and also in another region of the parameter space (Regime II) of the present model ($\mathfrak{g} = A_{2L-1}^{(1)}$, $\mathfrak{k} = C_L^{(1)}$)[14]. The correspondence (6) is valid in all cases. The true nature of this picture is yet to be explained.

References

[1] R. J. Baxter, Exactly Solved Models in Statistical Mechanics (Academic, London, 1982).

[2] A. A. Belavin, A. M. Polyakov and A. B. Zamolodchikov, Nucl. Phys. B 241, 333 (1984).
D. Friedan, Z. Qiu and S. Shenker, MSRI Publ. 3, 419 (1985).
V. A. Fateev and A. B. Zamolodchikov, Soviet Physics JETP 62, 215 (1986).
See also the Lecture Notes by Vl. S. Dotsenko, RIMS preprint 559, Kyoto University (1986).

[3] V. A. Fateev and A. B. Zamolodchikov, Landau Institute Preprint 1986-3.

[4] L. Onsager, Phys. Rev. 65, 117 (1944).

[5] T. T. Wu, B. M. McCoy, C. A. Tracy and E. Barouch, Phys. Rev. B 13, 316 (1976).

[6] M. Sato, T. Miwa and M. Jimbo, Publ. RIMS, Kyoto University, 14, 223 (1978); 15, 201, 577, 871 (1979); 16, 531 (1980); 17, 137 (1981).

[7] B. M. McCoy, J. H. H. Perk and T. T. Wu, Phys. Rev. Lett. 46, 757 (1981).

[8] A. A. Belavin and V. G. Drinfel'd, Funct. Anal. and Appl. 16, 159 (1982)
V. G. Drinfel'd, Soviet Math. Dokl. 32, 254 (1985).
M. Jimbo, Lett. Math. Phys. 11, 247 (1986).

[9] G. E. Andrews, R. J. Baxter and P. J. Forrester, J. Stat. Phys. 35, 193 (1984).
E. Date, M. Jimbo, T. Miwa and M. Okado, Lett. Math. Phys. 12, 209 (1986).

[10] V. G. Kac, Infinite Dimensional Lie algebras, Cambridge University Press (2^{nd} ed.), 1985.

[11] E. Date, M. Jimbo, T. Miwa and M. Okado, to appear in Phys. Rev. B, Rapid Communication.

[12] P. Goddard, A. Kent and D. Olive, Commun. Math. Phys. 103, 105 (1986).

[13] M. Jimbo, T. Miwa and M. Okado, Nucl. Phys. B275 [FS17], 517 (1986).

[14] E. Date, M. Jimbo, A. Kuniba, T. Miwa and M. Okado, in preparation.