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<td>KUGA, Ken'ichi</td>
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Kyoto University
4. Remarks on Smoothings of Four-Spaces

Ken'ichi Kuga

One of the striking consequences of Michael Freedman's topological theory of 4-manifolds and Simon Donaldson's non-existence results of certain smooth 4-manifolds is the existence of an exotic smoothing of the euclidean 4-space $\mathbb{R}^4$. Then an example of a manifold with finitely generated homology groups admitting infinitely many smooth structures was found along the same lines $[G][K]$. More recently Taubes showed the existence of uncountably many smooth structures on $\mathbb{R}^4[T]$. Actually Taubes' argument applies to a fairly large class of open 4-manifolds, and it may now be possible to expect uncountably many smooth structures on every non-compact 4-manifold.

In this informal note, we give some remarks and observations concerning smooth structures on non-compact 4-manifolds which seem to indicate the complicated nature of the problem: In §1 we provide a natural construction which possibly produces uncountably many smoothings and discuss some problems on the construction; In §2 we give some observations which shows some difficulties to reasonably topologize the set of smooth structures on a non-compact 4-manifold.

§1 Given a non-compact 4-manifold $V^4$, consider the following construction which is an immediate generalization of Taubes' construction to arbitrary non-compact manifold and actually produces uncountably
many smoothings in many cases (e.g. when an end of $V$ is diffeomorphic to $S^3 \times \mathbb{R}$).

1.1 Construction: Fix an exotic smoothing of $\mathbb{R}^4$, denoted $R$, which is standard on $(-\infty, 0) \times \mathbb{R}^3$, and a smooth properly embedded half-open ray $A$ in $V^4$. Identifying an open tubular neighborhood of $A$ with $(-\infty, 0) \times \mathbb{R}^3 \times \mathbb{R}$ so that the open end of $A$ goes to $0 \times \mathbb{R}^3$, we can form an end connected sum of $V^4$ with $R$, denoted $V^4 \# R$, which is homeomorphic to $V^4$. Then we can define a continuously parametrized smooth submanifolds $V(r)$ of $V^4 \# R$ for $0 < r \leq \infty$ by setting $V(r) = V^4 \cup (\text{open ball of radius } r \text{ centered at the origin in } R)$, which are homeomorphic to $V^4$.

1.2 Remark: When the end of $V$ is diffeomorphic to an end of a punctured definite 1-connected 4-manifold with non-standard intersection form, uncountably many distinct smoothings of $V^4$ can be obtained by taking continuously parametrized parallel ends as in [T]. It is not clear, however, that the above construction (where $R$ is connected to $V$ along the standard structure) yields uncountably many smoothings in these cases.

1.3 Remark: If the smooth structure on the end of $V^4$ is sufficiently complicated, the above construction fails. For example, set $V^4 = \#_{n=1}^{\infty} S^2 \times S^2$ a universal smoothing of $\mathbb{R}^4$ in [FT]. Then, for any choice of $R$, $V(r)'s$ are all diffeomorphic to $V^4$, i.e., the above construction cannot produce any new smoothing.

1.4 $P = \#_{n=1}^{\infty} S^2 \times S^2$: Also, if the end of $V^4$ is topologically complicated, the above construction might fail. A candidate to this is an infinite connected sum of $S^2 \times S^2$. More specifically, consider a countable sequence of small disjoint 4-balls $D^4_n$ in the standard $\mathbb{R}^4$. 


centered at points \((n,0,0,0)\), \(n = 1, 2, \ldots\), and take connected sums with countably many copies of \(S^2 \times S^2\), denoted \((S^2 \times S^2)_n\), at \(D^4_n\)'s. The resulting manifold \(P^4 = \#_{n=1}^{\infty}(S^2 \times S^2)_n\) is an open smooth 4-manifold with one end whose homology groups are infinitely generated. The following observation is an easy consequence of techniques in [FT] which shows the complicatedness of the smooth manifold \(P\).

**1.5 Proposition:** If \(Q^4\) is a smooth 4-manifold topologically homeomorphic to \(P\) (i.e. a possibly different smoothing of \(P\)). Then \(Q\) can be smoothly embedded into \(P\) in such a way that \(\text{int}(P - \text{Image}(Q))\) is topologically an open 4-ball.

**Proof** First observe that any smoothing of \(\mathbb{R}^4\), say \(U\), can be smoothly embedded into \(P\). In fact, one can construct a proper \(h\)-cobordism consisting of (small) 2- and 3-handles between \(P\) and \(U \# (\#_{n=1}^{\infty}(S^2 \times S^2)_n)\) which is topologically a product and smoothly a product near \(\bigcup_{n=1}^{\infty}((S^2 \times S^2)_n - D^4_n)\). The smooth Whitney tricks may be performed after removing self-intersections of Whitney disks by Norman tricks in \((S^2 \times S^2)_n - D^4_n\) in the middle level and we get a diffeomorphism \(U \# (\#_{n=1}^{\infty}(S^2 \times S^2)_n) \cong P\). Then we get an embedding of \(U \subseteq U - \text{(tubular neighborhood of a smooth proper half-open arc which joins } D^4_n\text{'s in } U\text{)}\) into \(P\).

Next consider the universal smoothing \(H\) of \([0, \infty) \times \mathbb{R}^3\) constructed in [FT]. As above we can smoothly embed \(H\) into \(P\), call the image \(H_+\), so that the end of \(H_+\) goes to the end of \(P\). (i.e. a proper embedding). Consider a smooth proper \(h\)-cobordism \(W^5\) between \(P\) and \(Q\) consisting of (small) 2- and 3-handles which is topologically a product (Note that we can extend the smooth structure on \(P\) and \(Q\) to \(W\), since the obstruction \(H^4(\text{relative}; \pi_3(\text{TOP/PL}) = 0)\). Then as in [FT], resmooth-
ing a neighborhood \( N \) of properly embedded half-open ray \( A \) in \( H_+ \) in \( P \) crossed with \([0,1]\) in \( W^5 \) by plugging \( Hx[0,1] \) into \( Nx[0,1] \) in \( W^5 \), smooth Whitney tricks may be performed after smoothing the cores of suitably chosen Casson Handles in the middle level using the resmoothing \( Hx(1/2) \). Then we get a diffeomorphism \((Q - (Nx(1))) \cup (Hx(1)) \cong (P - (Nx(0))) \cup (Hx(0))\), and hence a smooth embedding of \( Q \cong (Q - Nx(1)) \) into the latter manifold (\( P \) with \( Nx0 \) replaced by \( Hx0 \)) so that the compliment of the closure of the image is topologically homeomorphic to the open 4-ball. Finally this manifold, \( P \) with \( Nx0 \) replaced by \( Hx0 \), is actually diffeomorphic to \( P \), since \( Hx0 \) is absorbed into the universal smoothing \( H_+ \) originally embedded in \( P \).

§2 Since there are uncountably many smoothings of a non-compact 4-manifold (at least for many cases), the following seems to be a natural question: Can we find a reasonable (Hausdorff, etc.) topology on the set of smooth structures on a non-compact 4-manifold (\( \mathbb{R}^4 \), for example)?

Let \( S_4 \) be the set of smooth structures on \( \mathbb{R}^4 \). One can define a natural distance (admitting \( \infty \)), call Lipschitz-Shikata distance, on \( S_4 \) as follows: For \( U, V \in S_4 \), define \( d(U,V) \in [0, \infty] \) by \( d(U,V) = \inf \left( \inf \left( \log \max(|h|, |h^{-1}|) \right) \right) \), where \( d \) and \( \bar{d} \) run through all complete riemannian metrics compatible with the smooth structures \( U \) and \( V \) respectively, and \( h \) runs through all onto homeomorphism \( U \to V \), and \( |h| = \sup_{x \neq y, \in U} (\bar{d}(h(x), h(y))/d(x,y)) \in [0, \infty] \). (It is non-trivial that \( d(U,V) = 0 \) implies \( U = V \), which is a consequence of the following proposition.) One can readily generalize a result of Shikata [S] to non-compact manifolds and get:
2.1 Proposition: \( S^4 \) is a discrete space with respect to the Lipschitz-Shikata distance. In other words, if two exotic \( \mathbb{R}^n \)'s admit a sufficiently small lipeomorphism with respect to some riemannian metrics, then they are diffeomorphic.

Proof Fix an isometric imbedding of \( V \) into the euclidean space \( \mathbb{R}^m \) for sufficiently large \( m \) (Nash imbedding theorem is valid for non-compact manifolds), and take a tubular neighborhood \( N \) of \( V \) in \( \mathbb{R}^m \) so that the fibers of \( \pi : N \rightarrow V \) are orthogonal to \( V \) in \( \mathbb{R}^m \). Then, as usual, the continuous \( \mathbb{R}^m \)-valued function \( f = i \circ h : U \rightarrow V \subseteq \mathbb{R}^m \) (where \( h : U \rightarrow V \) is a small lipeomorphism), is approximated by a \( C^\infty \) \( \mathbb{R}^m \)-valued function \( f_t(x) = \int_U f(y)g_t(x,y)dy, x \in U \), for \( t > 0 \), a coordinatewise integration over the riemannian manifold \((U,d)\), where \( g_t(x,y) \) is given by \( g_t(x,y) = g(d(x,y)/t) / \left( \int_U g(d(x,y)/t)dx \right) \) and \( g \) is a non-negative \( C^\infty \) function: \( \mathbb{R} \rightarrow [0,\infty) \) with compact support and \( g = \text{constant near } 0 \in \mathbb{R} \).

Consider the composition \( F(x,t) = \pi \circ f_t(x) \) for \( x \in U \), \( t > 0 \), and \( F(x,0) = h(x) \), for \( x \in U \), where \( \pi \) is the orthogonal projection of \( N \) onto \( V \). Although this composition is not defined everywhere, there is a positive number \( t(L) \) for each compact subset \( L \) of \( U \) such that \( F(x,t) \) is well-defined on \( U[x,0,t(L)] \), i.e., \( f_t(x) \in N \) for \( x \in L \), \( 0 \leq t \leq t(L) \). Furthermore, Shikata's proof in [S] shows that there is a positive constant \( c \) (independent of the manifolds \( U, V \) or the choice of isometric imbedding of \( V \)) such that if \( |h| \) and \( |h^{-1}| \) are both \( < c \), and if \( t(L) > 0 \) is sufficiently small (depending on \( L \)), then \( F \) defines a \( C^\infty \) embedding of (neighborhood of \( L \))\( x(t) \) for \( 0 < t \leq t(L) \).

Hence we can find codimension 0 compact submanifolds \( M_n, L_n \) of \( U \) for \( n = 1, 2, \cdots \), and a deceasing sequence \( t_1 > t_2 > \cdots \) of positive numbers \( 0 < t_n < t(L_n) \) such that: (i) \( \bigcup_{n=1}^\infty M_n = \bigcup_{n=1}^\infty L_n = U \),
\( M_0 \subseteq L_n \subseteq M_{n+1} \); (ii) \( F|_{L_n}(x(t)) \) is a \( C^\infty \) embedding of \( L_n \cong \mathbb{R}^n \) into \( \mathbb{R}^n \) for \( 0 < t \leq t_n \); (iii) \( F(M_n(x[0,t_n])) \subseteq F(L_n(x(t_n))) \subseteq F(M_{n+1}(x(t_{n+1}))) \).

Consider the function \( \overline{F}(x,t) = (F(x,t),t) \in \mathbb{R} \times (0,\infty) \) for \((x,t)\) in a neighborhood of \( X = \bigcup_{n=1}^{\infty} (L_n(x[0,t_n])) \) in \( \text{Im}(\overline{F}) \). Then \( \overline{F} \) \( C^\infty \) embeds \( X - \text{Im}(\overline{F}) \) into \( \mathbb{R} \times (0,\infty) \) (differentiability of \( \overline{F} \) is assured by the compactness of support of \( g \) in the definition of \( f(x) \)). One can construct a \( C^\infty \) vector field \( \mathcal{X} \) on \( \mathbb{R} \times (0,\infty) \) by partition of unity argument with the following properties: (i) \( dp_2(\mathcal{X}) = d/dt \) (where \( p_2 \) is the projection \( \mathbb{R} \times (0,\infty) \rightarrow (0,\infty) \)); (ii) \( \mathcal{X} = d\overline{F}(\partial/\partial t) \) on \( \mathbb{R} \times (0,\infty) \); (iii) \( \mathcal{X} = \partial/\partial t \) outside a neighborhood of \( \bigcup_{n=1}^{\infty} (F(L_n(x(t_n)))) \times (0,t_n) \).

Fixing a number \( t_0 > t_1 \), let \( \Pi : V \times (0,\infty) \rightarrow V \) be the projection along the \( C^\infty \) flow \( \mathcal{X} \) onto \( V = \text{Im}(\overline{F}(x_0)) \), i.e. \( \Pi(y,t) \) is the unique intersection of \( V \) with the trajectory of \( \mathcal{X} \) through \((y,t)\). Then the desired diffeomorphism \( \tilde{h} : U \rightarrow V \) may be written as the union \( \tilde{h}(x) = \bigcup_{n=1}^{\infty} \Pi_n \mathcal{F}(L_n(x(t_n)),x(t_n)) \), which is well-defined from the construction. \( \square \)

2.2 Remark: The above argument is valid for any non-compact manifolds of any dimension (the constant \( c \) depends only on the dimension).

2.3 Remark: The Lipschitz–Shikata distance is, hence, too strong.

There are some candidates for defining weaker topologies on \( S_4 \).

Consider the following spaces of embeddings: (i) \( E_1 \) = topological embeddings of the unit open 4-ball into a universal smoothing \( U \) of \( \mathbb{R}^4 \) in \([FT]\); (ii) \( E_2 \) = topological embeddings of the unit open 4-ball into \( P = \bigoplus_{n=1}^{\infty} S_n^2 \times S_n^2 \) described in 1.4; (iii) \( E_3 \) = \( C^\infty \)-proper embeddings of the universal smoothing \( H \) of the half-space \([0,\infty) \times \mathbb{R}^3 \) in \([FT]\) into \( H, f : H \rightarrow H \). Then we have projections \( p_1 : E_1 \rightarrow S_4 \), defined by \( p_1(f) = \text{Image of } f \) with the induced smooth structure, for
i = 1, 2, and \( p_3(f) = \text{Int}(H - \text{Image}(f)) \) (\( p_1 \) and \( p_3 \) are surjective by [FT], and \( p_2 \) is surjective from the argument in Proof of 1.5). Hence any topology on \( E_1 \) induces a quotient topology on \( S_4 \). It seems, however, not easy to make this topology Hausdorff. For example, if we put compact-open topology on \( E_1 \), the only open sets of \( S_4 \) will be the whole set and the empty set.

2.4 Remark: It would be nice if one could define a reasonable topology on \( S_4 \) with possibly accessible homotopy groups. Related to this is the following naive question: Is there a reasonable topology on \( S_4 \) such that the singular complex \( S(S_4) \) is identifiable with the Kan complex \( \text{DIFF}(\mathbb{R}^4) \) of sliced families of smooth structures on \( \mathbb{R}^4 \) \( (S_4 \text{ is the set of vertices } \text{DIFF}(\mathbb{R}^4)^0) \)? Again, this topology cannot be Hausdorff, since a universal smoothing \( U \) is contained in any neighborhood of any element and the only neighborhood of the standard structure is the whole set.

References


