4. Remarks on Smoothings of Four-Spaces

Ken'ichi KUGA

One of the striking consequences of Michael Freedman's topological theory of 4-manifolds and Simon Donaldson's non-existence results of certain smooth 4-manifolds is the existence of an exotic smoothing of the euclidean 4-space $\mathbb{R}^4$. Then an example of a manifold with finitely generated homology groups admitting infinitely many smooth structures was found along the same lines [G] [K]. More recently Taubes showed the existence of uncountably many smooth structures on $\mathbb{R}^4$ [T]. Actually Taubes' argument applies to a fairly large class of open 4-manifolds, and it may now be possible to expect uncountably many smooth structures on every non-compact 4-manifold.

In this informal note, we give some remarks and observations concerning smooth structures on non-compact 4-manifolds which seem to indicate the complicated nature of the problem: In §1 we provide a natural construction which possibly produces uncountably many smoothings and discuss some problems on the construction; In §2 we give some observations which shows some difficulties to reasonably topologize the set of smooth structures on a non-compact 4-manifold.

§1 Given a non-compact 4-manifold $V^4$, consider the following construction which is an immediate generalization of Taubes' construction to arbitrary non-compact manifold and actually produces uncountably
many smoothings in many cases (e.g. when an end of \( V \) is diffeomorphic to \( S^3 \times \mathbb{R} \)).

1.1 Construction: Fix an exotic smoothing of \( \mathbb{R}^4 \), denoted \( R \), which is standard on \( (-\infty,0) \times \mathbb{R}^3 \), and a smooth properly embedded half-open ray \( A \) in \( V^4 \). Identifying an open tubular neighborhood of \( A \) with \( (-\infty,0) \times \mathbb{R}^3 \times \mathbb{R}^3 \) \( R \) so that the open end of \( A \) goes to \( 0 \times \mathbb{R}^3 \), we can form an end connected sum of \( V^4 \) with \( R \), denoted \( V^4 \# R \), which is homeomorphic to \( V^4 \). Then we can define a continuously parametrized smooth submanifolds \( V(r) \) of \( V^4 \# R \) for \( 0 < r < \infty \) by setting \( V(r) = V^4 \cup \) (open ball of radius \( r \) centered at the origin in \( R \)), which are homeomorphic to \( V^4 \).

1.2 Remark: When the end of \( V \) is diffeomorphic to an end of a punctured definite 1-connected 4-manifold with non-standard intersection form, uncountably many distinct smoothings of \( V^4 \) can be obtained by taking continuously parametrized parallel ends as in [T]. It is not clear, however, that the above construction (where \( R \) is connected to \( V \) along the standard structure) yields uncountably many smoothings in these cases.

1.3 Remark: If the smooth structure on the end of \( V^4 \) is sufficiently complicated, the above construction fails. For example, set \( V^4 = \) a universal smoothing of \( \mathbb{R}^4 \) in [FT]. Then, for any choice of \( R \), \( V(r) \)'s are all diffeomorphic to \( V^4 \), i.e., the above construction cannot produce any new smoothing.

1.4 \( P = \#_{n=1}^\infty S^2 \times S^2 \): Also, if the end of \( V^4 \) is topologically complicated, the above construction might fail. A candidate to this is an infinite connected sum of \( S^2 \times S^2 \). More specifically, consider a countable sequence of small disjoint 4-balls \( D^4_n \) in the standard \( \mathbb{R}^4 \).
centered at points \((n,0,0,0)\), \(n = 1,2, \ldots\), and take connected sums with countably many copies of \(S^2 \times S^2\), denoted \(S^2 \times S^2\), at \(D^4\)'s. The resulting manifold \(P^4 = \#_{n=1}^{\infty} (S^2 \times S^2)\) is an open smooth 4-manifold with one end whose homology groups are infinitely generated. The following observation is an easy consequence of techniques in [FT] which shows the complicatedness of the smooth manifold \(P\).

1.5 Proposition: If \(Q^4\) is a smooth 4-manifold topologically homeomorphic to \(P\) (i.e. a possibly different smoothing of \(P\)). Then \(Q\) can be smoothly embedded into \(P\) in such a way that \(\text{int}(P - \text{Image}(Q))\) is topologically an open 4-ball.

Proof First observe that any smoothing of \(\mathbb{R}^4\), say \(U\), can be smoothly embedded into \(P\). In fact, one can construct a proper h-cobordism consisting of (small) 2- and 3-handles between \(P\) and \(U = \#_{n=1}^{\infty} (S^2 \times S^2)\) which is topologically a product and smoothly a product near

\[\bigcup_{n=1}^{\infty} ((S^2 \times S^2) - D^4)\].

The smooth Whitney tricks may be performed after removing self-intersections of Whitney disks by Norman tricks in \((S^2 \times S^2) - D^4\) in the middle level and we get a diffeomorphism \(U = \#_{n=1}^{\infty} (S^2 \times S^2)\) \(\cong P\). Then we get an embedding of \(U \subseteq U - \text{tubular neighborhood of a smooth proper half-open arc which joins } D^4\)'s in \(U\) into \(P\).

Next consider the universal smoothing \(H\) of \([0, \infty) \times \mathbb{R}^3\) constructed in [FT]. As above we can smoothly embed \(H\) into \(P\), call the image \(H_+\), so that the end of \(H_+\) goes to the end of \(P\) (i.e. a proper embedding). Consider a smooth proper h-cobordism \(W^5\) between \(P\) and \(Q\) consisting of (small) 2- and 3-handles which is topologically a product (Note that we can extend the smooth structure on \(P\) and \(Q\) to \(W\), since the obstruction \(H^4(\text{relative}; \pi_3(\text{TOP/PL})) = 0\)). Then as in [FT], resmooth-
ing a neighborhood $N$ of properly embedded half-open ray $A$ in $H_+$ in $P$ crossed with $[0,1]$ in $W^5$ by plugging $H(x[0,1])$ into $N(x[0,1])$ in $W^5$, smooth Whitney tricks may be performed after smoothing the cores of suitably chosen Casson Handles in the middle level using the resmoothing $H(x[1/2])$. Then we get a diffeomorphism $(Q - (N(x(1))))_y(H(x(1)))
abla (P - (N(x(0))))_y(H(x(0)))$, and hence a smooth embedding of $Q \cong (Q - N(x(1)))$ into the latter manifold ( $P$ with $N(x)0$ replaced by $H(x)0$) so that the compliment of the closure of the image is topologically homeomorphic to the open 4-ball. Finally this manifold, $P$ with $N(x)0$ replaced by $H(x)0$, is actually diffeomorphic to $P$, since $H(x)0$ is absorbed into the universal smoothing $H_+$ originally embedded in $P$ [1].

§2 Since there are uncountably many smoothings of a non-compact 4-manifold (at least for many cases), the following seems to be a natural question: Can we find a reasonable (Hausdorff, etc.) topology on the set of smooth structures on a non-compact 4-manifold ($\mathbb{R}^4$, for example)?

Let $S_4$ be the set of smooth structures on $\mathbb{R}^4$. One can define a natural distance (admitting $\infty$), call Lipschitz-Shikata distance, on $S_4$ as follows: For $U, V \in S_4$, define $d(U,V) \in [0, \infty]$ by $d(U,V) = \inf \left( \inf \left( \log \max(|h|, |h^{-1}|) \right) \right)$, where $d$ and $\varrho$ run through all complete riemannian metrics compatible with the smooth structures $U$ and $V$ respectively, and $h$ runs through all onto homeomorphism $U \to V$, and $|h| = \sup_{x\neq y, \in U} (\varrho(h(x), h(y))/d(x,y)) \in [0, \infty]$. (It is non-trivial that $d(U,V) = 0$ implies $U = V$, which is a consequence of the following proposition.) One can readily generalize a result of Shikata [S] to non-compact manifolds and get:
2.1 Proposition: \( S_4 \) is a discrete space with respect to the Lipschitz-Shikata distance. In other words, if two exotic \( \mathbb{R}^n \)'s admit a sufficiently small liceomorphism with respect to some riemannian metrics, then they are diffeomorphic.

Proof Fix an isometric imbedding of \( V \) into the euclidean space \( \mathbb{R}^m \) for sufficiently large \( m \) (Nash imbedding theorem is valid for non-compact manifolds), and take a tubular neighborhood \( N \) of \( V \) in \( \mathbb{R}^m \) so that the fibers of \( \pi: N \to V \) are orthogonal to \( V \) in \( \mathbb{R}^m \). Then, as usual, the continuous \( \mathbb{R}^m \)-valued function \( f = i \circ h: U \to V \subseteq \mathbb{R}^m \) (where \( h: U \to V \) is a small liceomorphism), is approximated by a \( C^\infty \mathbb{R}^m \)-valued function \( f_t(x) = \int_U f(y) g_t(x,y) dy, \ x \in U, \) for \( t > 0 \), a coordinatewise integration over the riemannian manifold \( (U, d) \), where \( g_t(x,y) \) is given by \( g_t(x,y) = g(d(x,y)/t)/(\int_U g(d(x,y)/t) dx) \) and \( g \) is a non-negative \( C^\infty \)-function: \( \mathbb{R} \to [0, \infty) \) with compact support and \( g = \) constant near \( 0 \in \mathbb{R} \).

Consider the composition \( F(x,t) = \pi \circ f_t(x) \) for \( x \in U, \ t > 0, \) and \( F(x,0) = h(x) \), for \( x \in U \), where \( \pi \) is the orthogonal projection of \( N \) onto \( V \). Although this composition is not defined everywhere, there is a positive number \( t(L) \) for each compact subset \( L \) of \( U \) such that \( F(x,t) \) is well-defined on \( L \times [0, t(L)] \), i.e., \( f_t(x) \in N \) for \( x \in L, \ 0 \leq t \leq t(L) \). Furthermore, Shikata's proof in [S] shows that there is a positive constant \( c \) (independent of the manifolds \( U, V \) or the choice of isometric imbedding of \( V \)) such that if \( |h| \) and \( |h^{-1}| \) are both < \( c \), and if \( t(L) > 0 \) is sufficiently small (depending on \( L \)), then \( F \) defines a \( C^\infty \) embedding of (neighborhood of \( L \))\( x(t) \) for \( 0 < t \leq t(L) \).

Hence we can find codimension 0 compact submanifolds \( M_n, L_n \) of \( U \) for \( n = 1, 2, \cdots \), and a deceasing sequence \( t_1 > t_2 > \cdots \) of positive numbers \( 0 < t_n < t(L_n) \) such that: (i) \( \bigcup_{n=1}^\infty M_n = \bigcup_{n=1}^\infty L_n = U \),
\( M_n \subseteq L_n \subseteq M_{n+1} \); (ii) \( F|_{L_n \backslash x(t)} \) is a \( C^\infty \) embedding of \( L_n \cong L_n x(t) \) into \( V \) for \( 0 < t \leq t_n \); (iii) \( F(M_n x[0,t_n]) \subset F(L_n x(t_n)) \subset F(M_{n+1} x(t_{n+1})) \).

Consider the function \( \tilde{F}(x,t) = (F(x,t), t) \in V_{x}(0,\infty) \) for \( (x,t) \) in a neighborhood of \( X = \bigcup_{n=1}^{\infty} (L_n x[0,t_n]) \) in \( V x(0,\infty) \). Then \( \tilde{F} \) \( C^\infty \) embeds \( X - U_{0}(0) \) into \( V_{x}(0,\infty) \) (differentiability of \( \tilde{F} \) is assured by the compactness of support of \( g \) in the definition of \( f_c(x) \)). One can construct a \( C^\infty \) vector field \( \xi \) on \( V x(0,\infty) \) by partition of unity argument with the following properties: (i) \( dp_2(\xi) = d/dt \) (where \( p_2 \) is the projection \( V x(0,\infty) \rightarrow (0,\infty) \)); (ii) \( \xi = d\tilde{F}(\partial/\partial t) \) on \( F(U_{n=1}^{\infty} M_n x[0,t_n]) \); (iii) \( \xi = \partial/\partial t \) outside a neighborhood of \( \bigcup_{n=1}^{\infty} (F(L_n x(t_n)) \times (0,t_n)) \).

Fixing a number \( t_0 > t_1 \), let \( \Pi : V x (0,\infty) \rightarrow V \) be the projection along the \( C^\infty \) flow \( \xi \) onto \( V = V x(t_0) \), i.e. \( \Pi(y,t) \) is the unique intersection of \( V x(t_0) \) with the trajectory of \( \xi \) through \( (y,t) \). Then the desired diffeomorphism \( \bar{h} : U \rightarrow V \) may be written as the union
\[
\bar{h}(x) = \bigcup_{n=1}^{\infty} \Pi \circ F|_{M_n x(t_n)}(x,t_n),
\]
which is well-defined from the construction. \( \square \)

2.2 Remark: The above argument is valid for any non-compact manifolds of any dimension (the constant \( c \) depends only on the dimension).

2.3 Remark: The Lipschitz-Shikata distance is, hence, too strong.

There are some candidates for defining weaker topologies on \( S_4 \).

Consider the following spaces of embeddings: (i) \( E_1 \) = topological embeddings of the unit open 4-ball into a universal smoothing \( U \) of \( R^4 \) in [FT]; (ii) \( E_2 \) = topological embeddings of the unit open 4-ball into \( P = \#_{n=1}^{\infty}(S^2_x \times S^2_n) \) described in 1.4; (iii) \( E_3 \) = \( C^\infty \) proper embeddings of the universal smoothing \( H \) of the half-space \( [0,\infty) \times R^3 \) in [FT] into \( H, f : H \rightarrow H \). Then we have projections \( p_1 : E_1 \rightarrow S_4 \), defined by \( p_1(f) = \text{Image of } f \) with the induced smooth structure, for
i = 1, 2, and \( p_3(f) = \text{Int}(H - \text{Image}(f)) \) (\( p_1 \) and \( p_3 \) are surjective by [FT], and \( p_2 \) is surjective from the argument in Proof of 1.5). Hence any topology on \( E_1 \) induces a quotient topology on \( S_4 \). It seems, however, not easy to make this topology Hausdorff. For example, if we put compact-open topology on \( E_1 \), the only open sets of \( S_4 \) will be the whole set and the empty set.

2.4 Remark: It would be nice if one could define a reasonable topology on \( S_4 \) with possibly accessible homotopy groups. Related to this is the following naive question: Is there a reasonable topology on \( S_4 \) such that the singular complex \( S(S_4) \) is identifiable with the Kan complex \( \text{DIFF}(\mathbb{R}^4) \) of sliced families of smooth structures on \( \mathbb{R}^4 \) (\( S_4 \) is the set of vertices \( \text{DIFF}(\mathbb{R}^4)^0 \))? Again, this topology cannot be Hausdorff, since a universal smoothing \( U \) is contained in any neighborhood of any element and the only neighborhood of the standard structure is the whole set.

References


[K] K. Kuga, On immersed 2-spheres in \( S^2 \times S^2 \), to appear
