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1. SPINORS, TWISTORS, AND REDUCED HOLONOMY

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We know from the early work of Marcel Berger that there is a small explicit list of holonomy groups possible on a locally irreducible riemannian manifold M. From the years of subsequent work on the subject we now know that if $\dim(M) = n$, then the only possibilities (for the identity component) are: SO_n, $U_{n/2}$, $SU_{n/2}$, $Sp_1 \cdot Sp_{n/4}$, $Sp_{n/4}$ and the two exceptional cases:

$$G_2 \quad \text{when} \quad n = 7 \qquad \text{and} \qquad Spin_7 \quad \text{when} \quad n = 8.$$

We shall show that except for the case $Sp_1 \cdot Sp_m$, each of these holonomy reductions can be related to the existence of a certain parallel spinor field. Furthermore, in each case the set of topological reductions of the structure group is in one-to-one correspondence with the space of all sections of an appropriate spinor bundle.

We begin first with the correspondence between certain projectivized spinors and almost complex structures (which goes back to É. Cartan). Let $(V, \langle\cdot,\cdot\rangle)$ be an oriented real inner product space of dimension $2m$. An <u>orthogonal almost complex structure</u> on V is an orthogonal transformation $J: V \longrightarrow V$ such that $J^2 = -I$. The space $\mathcal{C}$ of such structures falls into two components $\mathcal{C} = \mathcal{C}^+ \amalg \mathcal{C}^-$, distinguished by the canonical orientation. Each $J \in \mathcal{C}$ determines a decomposition $V \otimes \mathbb{C} = W(J) \oplus \overline{W(J)}$ where $W(J) = \{w : Jw = iw\}$. The space $W(J)$ is <u>totally isotropic</u> which means that $\langle w, w \rangle = 0$ for all $w \in W(J)$. It is easily seen that conversely, each totally isotropic complex subspace $W \subset V \otimes \mathbb{C}$ of complex dimension m, determines a unique $J \in \mathcal{C}$ with $W(J) = W$.

Consider now the complex Clifford algebra $\mathbb{C}l_{2m} = Cl(V, \langle\cdot,\cdot\rangle) \otimes_{\mathbb{R}} \mathbb{C}$

It is an elementary fact that $\mathbb{C}l_{2m} \cong \mathrm{Hom}_{\mathbb{C}}(\mathcal{S},\mathcal{S})$ where $\mathcal{S}$ is a complex vector space of dimension 2^m, called the complex spinor space. Given $\sigma \in \mathcal{S}$, consider the subspace $K_\sigma \equiv \{w \in V\otimes\mathbb{C} : w\cdot\sigma = 0\}$. Since $w\cdot w\cdot\sigma = -\langle w,w\rangle\sigma$, we see that K_σ is totally isotropic. A spinor $\sigma \in \mathcal{S}$ is called pure if $\dim_{\mathbb{C}} K_\sigma = m$, i.e., if K_σ is maximal. Such a spinor determines a complex structure $J_\sigma \in \mathcal{C}$.

Proposition 1. Let $\mathbb{P}(P\mathcal{S})$ denote the projectivization of the space of pure spinors. Then the association $\sigma \mapsto J_\sigma$ gives an SO_{2m}-equivariant diffeomorphism $\mathbb{P}(P\mathcal{S}) \longrightarrow \mathcal{C}$.

Note. There is a decomposition $\mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^-$ into eigenspaces for the complex volume form $\omega_{\mathbb{C}} = i^m e_1 \cdots e_{2m}$. Every pure spinor lies in either $\mathcal{S}^+$ or in $\mathcal{S}^-$. The components $\mathbb{P}(P\mathcal{S}^\pm)$ correspond to the components $\mathcal{C}^\pm$ above.

Note. When $2m = 4$ or 6, every non-zero spinor in $\mathcal{S}^+$ is pure.

Theorem 2. Let M be an oriented riemannian manifold of dimension $2m$. Then topological reductions of the structure group of M to U_m are in one-to-one correspondence with sections of the projectivized bundle of pure spinors $\mathbb{P}(P\mathcal{S}^+)$. Furthermore, if σ is such a section, then

1. J_σ is integrable iff σ is (almost) holomorphic.
2. J_σ is Kähler iff $\nabla\sigma \equiv 0$.

Note. The space $\mathbb{P}(P\mathcal{S}^+)$ is called the twistor space of M. It carries a canonical almost complex structure on its tangent bundle.

Theorem 3. Let M be an oriented riemannian manifold of dimension $2m$. Then topological reductions of the structure group of M to SU_m are in one-to one correspondence with the set of sections σ of the spinor bundle $\mathcal{S}^+$ which satisfy $\|\sigma\| \equiv 1$ on M. Furthermore, if σ is such a section, then the manifold (M, J_σ) is Kähler and Ricci-flat if and only if $\nabla\sigma \equiv 0$.

Theorem 4. Let M be an oriented riemannian 7-manifold. Then M has a topological G_2-reduction if and only if it is a spin manifold. If M is spin, then the topological G_2-reductions of M are in one-to-one correspondence with the set of globally defined spinor fields σ on M such that $\|\sigma\| \equiv 1$. Furthermore, M has G_2-holonomy if and only if there exists a non-zero spinor field σ on M with $\nabla\sigma \equiv 0$.

Theorem 5. Let M be an oriented riemannian 8-manifold. Then M has a topological Spin_7^+-reduction if and only if M is spin and $\chi(\$^+) = 0$, i.e., if and only if $w_2(M) = 0$ and

$$p_1(M)^2 - 4p_2(M) + 8\chi(M) = 0.$$

Under this hypothesis, the topological Spin_7^+-reductions are in one-to-one correspondence with the set of sections $\sigma \in \Gamma(\$^+)$ with $\|\sigma\| \equiv 1$. Furthermore, M has Spin_7^+-holonomy if and only if there exists a non-zero section $\sigma \in \Gamma(\$^+)$ with $\nabla\sigma \equiv 0$.

Corollary 6. Let M be a complex manifold of complex dimension 4. Then M carries a topological Spin_7^+-structure if and only if

$$c_1 \cdot \left[c_1^3 - 4c_1c_2 + 8c_3 \right] = 0.$$

Corollary 7. Let M and N be compact spin 4-manifolds. Then the product $M \times N$ carries a topological Spin_7^+-structure if and only if

$$9\,\mathrm{sig}(M)\,\mathrm{sig}(N) = 4\chi(M)\chi(N).$$

In particular, $M \times M$ has such a structure if and only if

$$3\,\mathrm{sig}(M) = \pm 2\chi(M).$$

Results were proved with R. Harvey and B. Lawson.