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Kyoto University
Moduli Spaces of Holomorphic Mappings into Hyperbolically Imbedded Complex Spaces and Locally Symmetric Spaces

Junjiro Noguchi

Department of Mathematics, Faculty of Science
Tokyo Institute of Technology
Oh-Okayama, Meguroku, Tokyo 152, Japan

Introduction

The aim of this talk is to describe the results of Noguchi [14]. Let X be a connected Zariski open subset of a compact reduced complex space \( \overline{X} \) such that \( X \) is complete hyperbolic and hyperbolically imbedded into \( \overline{X} \) (cf. [7, 16]). Let \( N \) be a Zariski open subset of a compact complex manifold \( \overline{N} \) such that \( \partial N = \overline{N} - N \) is a hypersurface with only normal crossings; in some case, we consider the case of \( \partial N = \emptyset \). Here we study the structure of the moduli space \( \text{Hol}(N, X) \) of all holomorphic mappings \( f: N \to X \) of \( N \) into \( X \). Especially interesting is the case where \( X \) is the quotient \( \Gamma \backslash D \) of a
symmetric bounded domain $D$ by a torsion-free arithmetic subgroup $\Gamma$ of the identity component $\text{Aut}_0^0(D)$ of the holomorphic transformation group $\text{Aut}(D)$ of $D$. It is known that $\Gamma \backslash D$ is complete hyperbolic and hyperbolically imbedded into the Satake compactification $\overline{\Gamma \backslash D}$ of $\Gamma \backslash D$ (cf. [10, 2, 8, 9]). Besides the interesting results of [20, 21, 18], the present work is motivated by the results on the Parshin-Arakelov theorems for curves [17, 1] and Abelian varieties [5]. Cf. also [13]. Let $\pi: \overline{Y} \to \overline{N}$ be a fiber space over $\overline{N}$ which is smooth over $N$, such that the fibers $Y_x = \pi^{-1}(x)$ with $x \in N$ are curves with a given genus $g$ or $g$-dimensional Abelian varieties with principal polarization. Then, roughly speaking, the fiber space naturally induces a holomorphic mapping $f: N \to \Gamma \backslash S_g$, where $S_g$ denotes Siegel's generalized upper half space. Then the deformation of $\pi: \overline{Y} \to \overline{N}$ as fiber space over $\overline{N}$ with degeneration at most over $\partial N$ and the total space of such fiber spaces correspond respectively to the deformation of the holomorphic mapping $f$ and the moduli space $\text{Hol}(N, \Gamma \backslash S_g)$. Thus it is quite natural to deal with the case where $N$ and $\Gamma \backslash D$ are non-compact. In the case where $N$ is compact, there is an earlier work for a fiber space of Abelian varieties by [11].

§1. Holomorphic mappings into hyperbolically imbedded spaces

The natural topology of $\text{Hol}(N, X)$ which we endow with is the compact-open topology. We first prove an extension
and convergence theorem.

**Theorem (1.1).** Let $X$ be a hyperbolic complex space and hyperbolically imbedded into $\overline{X}$. Let $N$ be a complex manifold $\overline{N}$ minus a hypersurface with only normal crossings. If a sequence $\{f_\nu\}_{\nu=1}^\infty$ of $f_\nu \in \text{Hol}(N, X)$ converges to a holomorphic mapping $f : N \to \overline{X}$, then there are unique holomorphic extensions $\overline{f}_\nu : \overline{N} \to \overline{X}$ of $f_\nu$ and $\overline{f} : \overline{N} \to \overline{X}$ of $f$, and $(\overline{f}_\nu)$ converges uniformly on compact subsets of $\overline{N}$ to $\overline{f}$.

As for the extension theorem, this generalizes the result of [7], but the method of the proof is different. It will play a fundamental role in our arguments. In the proof of Theorem (1.1) we use the following lemma (cf. [14] for the details).

**Lemma (1.2) (cf. [19]).** Let $B(R)$ be the open ball of the $m$-dimensional complex vector space $\mathbb{C}^m$ with radius $R$ and center 0. Let $S$ be an analytic subset of pure dimension $k$ of $B(R)$ such that $0 \in S$. Then we have

$$\text{Vol}(SB(r)) \leq \frac{e^{2k}}{k!} r^{2k}$$

for $0 < r < R$. Moreover, if the equality holds for some $r > 0$, then $S$ is a linear subspace of $\mathbb{C}^m$.

In what follows, we assume that $\overline{N}$ and $\overline{X}$ are compact, and that $X$ is a Zariski open subset of $\overline{X}$, complete hyperbolic and hyperbolically imbedded into $\overline{X}$. Combining Theorem (1.1) with the Douady theory [3], we have
Theorem (1.3). $\text{Hol}(N, X)$ carries a structure of a complex space with universal property, such that its underlying topology coincides with the compact-open topology, and

$$\Phi: (f, x) \in \text{Hol}(N, X) \times N \rightarrow f(x) \in X$$

is a holomorphic mapping, which is proper for every fixed $x \in N$. Moreover, $\text{Hol}(N, X)$ is a Zariski open subset of a compact complex space.

Sketch of the proof. Let $\text{Hol}(\overline{N}, \overline{X})$ be the space of all holomorphic mappings from $\overline{N}$ into $\overline{X}$ with compact-open topology. Then, by Theorem (1.1) the mapping

$$f \in \text{Hol}(N, X) \mapsto \overline{f} \in \text{Hol}(\overline{N}, \overline{X})$$

is an into-homeomorphism. Hence we identify the topological space $\text{Hol}(N, X)$ with its image in $\text{Hol}(\overline{N}, \overline{X})$. By making use of the distance decreasing property of hyperbolic distance for holomorphic mappings, we see that $\text{Hol}(N, X)$ is relatively compact. The complete hyperbolicity of $X$ implies that $\text{Hol}(N, X)$ is open and closed in $\text{Hol}(\overline{N}, \overline{X})$ and then Theorem (1.1) yields that the topological closure of $\text{Hol}(N, X)$ in $\text{Hol}(\overline{N}, \overline{X})$ is a compact complex subspace which contains $\text{Hol}(N, X)$ as a Zariski open subset. The complete hyperbolicity of $X$ also implies that $\Phi(\cdot, x): \text{Hol}(N, X) \rightarrow X$ is proper for every fixed $x \in N$. Q.E.D.

In general, let $Y_1$ and $Y_2$ be two complex spaces. For a holomorphic mapping $f: Y_1 \rightarrow Y_2$, we set
rank \( f = \sup \left\{ \dim_t Y_1 - \dim_t f^{-1}(f(t)) ; t \in Y_1 \right\} \).

The following proposition follows from Lemma (1.2). It reveals a special nature of the complex analyticity of holomorphic mappings but is less known.

**Proposition (1.4). Assume that \( Y_1 \) and \( Y_2 \) are compact. Let \( \{f_\nu\}_{\nu=1}^\infty \) be a sequence of points of \( \text{Hol}(Y_1, Y_2) \) converging to \( f \in \text{Hol}(Y_1, Y_2) \). If \( \text{rank } f_\nu = k \), then \( \text{rank } f = k \).

We set

\[
\text{Hol}(k; N, X) = \{ f \in \text{Hol}(N, X) ; \text{rank } f = k \}.
\]

**Corollary (1.5).** \( \text{Hol}(k; N, X) \) is open and closed in \( \text{Hol}(N, X) \).

§2. The moduli \( \text{Hol}(N, \Gamma \backslash D) \)

In this section we deal with the case where \( X \) is the quotient \( \Gamma \backslash D \) of a symmetric bounded domain \( D \) by a torsion-free discrete subgroup \( \Gamma \) of \( \text{Aut}(D) \). We assume that \( \Gamma \) is uniform or an arithmetic subgroup of \( \text{Aut}^0(D) \). In the case where \( \Gamma \) is uniform and \( N = \overline{\mathbb{N}} \), the results of this section were already obtained in [20, 21, 18]. We are mainly interested in the case where \( \Gamma \backslash D \) and \( N \) are non-compact, while our arguments work in the compact case. Let \( \ell(D) \) (resp. \( \ell(\Gamma) \)) denote the maximum dimension of proper boundary components of \( D \) (resp. \( \Gamma \)-rational boundary components). Let
\( \text{Hol}(k; N, \Gamma \backslash D) \) denote the set of all holomorphic mappings \( f: N \to \Gamma \backslash D \) with rank \( f = k \). Applying the results of the previous section, we have

**Theorem (2.1).** i) \( \text{Hol}(N, \Gamma \backslash D) \) carries a structure of a complex space compatible with compact-open topology, such that the evaluation mapping

\[
\Phi: (f, x) \in \text{Hol}(N, \Gamma \backslash D) \times N \to f(x) \in \Gamma \backslash D
\]

is holomorphic. Moreover, \( \text{Hol}(N, \Gamma \backslash D) \) is a Zariski open subset of the compact complex space \( \overline{\text{Hol}(N, \Gamma \backslash D)} \), and satisfies the universality property: i.e., for a complex space \( T \) and a holomorphic mapping \( \psi: T \times N \to \Gamma \backslash D \), the natural mapping

\[
t \in T \to \psi(t, \cdot) \in \text{Hol}(N, \Gamma \backslash D)
\]

is holomorphic.

ii) Every connected component of \( \text{Hol}(N, \Gamma \backslash D) \) is complete hyperbolic and the holomorphic mappings

\[
\Phi_x: f \in \text{Hol}(N, \Gamma \backslash D) \to f(x) \in \Gamma \backslash D
\]

are proper for all \( x \in N \).

---

1) \( \overline{\text{Hol}(N, \Gamma \backslash D)} \) is the closure of \( \text{Hol}(N, \Gamma \backslash D) \) in \( \text{Hol}(\overline{N}, \overline{\Gamma \backslash D}) \).
iii) $\text{Hol}(k; N, \Gamma \setminus D)$ are open and closed in $\text{Hol}(N, \Gamma \setminus D)$.

iv) $\text{Hol}(k; N, \Gamma \setminus D)$ are compact for $k > \ell(\Gamma)$.

v) $\text{Hol}(k; N, \Gamma \setminus D)$ are finite for $k > \ell(D)$.

In the rest of this section, we study in details the structure of $\text{Hol}(N, \Gamma \setminus D)$, assuming that $\bar{N}$ is Kähler and $\partial N$ is a hypersurface with only simple normal crossings. We use the following result on harmonic mappings by [18]:

(2.2) Let $F: N \to \Gamma \setminus D$ and $G: N \to \Gamma \setminus D$ be free homotopic harmonic mappings with finite energy. Then there is a harmonic mapping $\Psi: RXN \to \Gamma \setminus D$ with respect to the product metric $dt \otimes h$ on $RXN$ such that

i) $\Psi(0, x) = F(x), \Psi(1, x) = G(x)$ and $\Psi$ provides a free homotopy between $F$ and $G$, equivalent to the given one;

ii) for every $x \in N$, the curve $\gamma_x: t \in \mathbb{R} \to \Psi(t, x) \in \Gamma \setminus D$ is a parametrization of a geodesic with constant speed, independent of $x$, and $e(\Psi(t, \cdot))(x)$ is constant in $t$.

Lemma (2.3). Let $F$ and $G$ be as in (2.2). If $F$ is holomorphic, then so is $G$.

Remark. 1) In case $N$ is compact, this is a theorem due to Lichnerowicz (cf. Theorem (8.6) of [4]).
2) Since (2.2) actually holds for harmonic mappings from a complete Riemannian manifold with finite volume into a complete Riemannian manifold with non-positive sectional curvatures, Lemma (2.3) is also true for harmonic mappings $F$ and $G$ from a complete Kähler manifold with finite volume into a complete Kähler manifold with non-positive sectional curvatures, provided that $F$ and $G$ have finite energies.

The main result is the following:

**Theorem (2.4).** i) $\text{Hol}(N, \Gamma \backslash D)$ is smooth and quasi-projective.

ii) For a connected component $Z$ of $\text{Hol}(N, \Gamma \backslash D)$ and a point $x \in N$, the evaluation mapping at $x$

$$\phi_x: f \in Z \rightarrow f(x) \in \Gamma \backslash D$$

is a proper holomorphic immersion onto a totally geodesic complex submanifold, so that $Z$ is a free quotient of a symmetric bounded domain.

iii) For a connected component $Z$ of $\text{Hol}(N, \Gamma \backslash D)$, there is a normal complex projective variety $\tilde{Z}$ such that $Z$ is hyperbolically imbedded into $\tilde{Z}$ and $\phi_x$ holomorphically extends to $\overline{\phi_x}: \tilde{Z} \rightarrow \Gamma \backslash D$.

iv) $\dim \text{Hol}(k; N, \Gamma \backslash D) \leq \ell(D)$ for $k > 0$. 

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v) For $f \in \text{Hol}(N, \Gamma \backslash D)$ with $f^{-1}(\partial \Gamma \backslash D) \neq \emptyset$, 
$\dim_{\mathbb{C}} \text{Hol}(N, \Gamma \backslash D) \leq \ell(\Gamma)$.

As a corollary, we have the following.

**Corollary (2.5) (Rigidity).** Let $f : N \to \Gamma \backslash D$ be a holomorphic mapping. Then $f$ is a unique holomorphic mapping among the free homotopy class of $f$, if $f$ satisfies one of the following conditions:

a) The image of $f$ is not contained in a totally geodesic complex proper submanifold of $\Gamma \backslash D$;

b) $\text{rank } f > \ell(D)$;

c) $f^{-1}(\partial \Gamma \backslash D) \neq \emptyset$ and $\text{rank } f > \ell(\Gamma)$.

In general, a holomorphic mapping $f \in \text{Hol}(N, \Gamma \backslash D)$ admits a deformation (cf. [5]). But in the special case where $D$ is the $n$-th product $H^n$ of the upper half plane $\mathbb{H} \times \mathbb{C}$, we see that any $f \in \text{Hol}(N, \Gamma \backslash H^n)$ is rigid. That is, by making use of the rigidity Theorem 6 of [6], we have

**Theorem (2.6).** Let $\Gamma \subset (\text{PSL}(2, \mathbb{R}))^n$ be an irreducible torsion-free discrete subgroup with $\text{Vol}(\Gamma \backslash H^n) < \infty$. Then

i) if $f : N \to \Gamma \backslash H^n$ is a non-constant holomorphic mapping, $f$ is a unique holomorphic mapping among the free homotopy class of $f$, so that
ii) there are only finitely many non-constant holomorphic mappings from $N$ into $\Gamma \backslash H^n$.

Remark. 1) It must be noted that if $\Gamma \backslash H^n$ is not compact, then $\Gamma$ is arithmetic ([12]). Therefore $\Gamma$ satisfies our requirement for discrete subgroups.

2) In the case of dim $N = 1$, i) was proved in [6].

3) By the same arguments as in [15], we see that the Kähler assumption for $\bar{N}$ is not necessary in ii). The proof is reduced to the present case.

4) For a compact quotient $\Gamma \backslash H^2$ and a compact complex manifold $N$, ii) was proved in [15]. For an algebraic curve $N$ and compact $\Gamma \backslash H^n$, it was proved in [6].

References


