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Moduli Spaces of Holomorphic Mappings into Hyperbolically Imbedded Complex Spaces and Locally Symmetric Spaces

Junjiro Noguchi
野口潤次郎

Department of Mathematics, Faculty of Science
Tokyo Institute of Technology
Oh-Okayama, Meguroku, Tokyo 152, Japan

Introduction

The aim of this talk is to describe the results of Noguchi [14]. Let X be a connected Zariski open subset of a compact reduced complex space $\bar{X}$ such that X is complete hyperbolic and hyperbolically imbedded into $\bar{X}$ (cf. [7, 16]). Let N be a Zariski open subset of a compact complex manifold $\bar{N}$ such that $\partial N = \bar{N}-N$ is a hypersurface with only normal crossings; in some case, we consider the case of $\partial N = \phi$. Here we study the structure of the moduli space $\mathrm{Hol}(N, X)$ of all holomorphic mappings $f: N \to X$ of N into X. Especially interesting is the case where X is the quotient $\Gamma\backslash D$ of a symmetric bounded domain D by a torsion-free arithmetic subgroup Γ of the identity component $Aut^0(D)$ of the holomorphic transformation group Aut(D) of D. It is known that $\Gamma\backslash D$ is complete hyperbolic and hyperbolically imbedded into the Satake compactification $\overline{\Gamma\backslash D}$ of $\Gamma\backslash D$ (cf. [10, 2, 8, 9]). Besides the interesting results of [20, 21, 18], the present work is motivated by the results on the Parshin-Arakelov theorems for curves [17, 1] and Abelian varieties [5]. Cf. also [13]. Let π: $\overline{Y} \to \overline{N}$ be a fiber space over $\overline{N}$ which is smooth over N, such that the fibers $Y_x = \pi^{-1}(x)$ with $x \in N$ are curves with a given genus g or g-dimensional Abelian varieties with principal polarization. Then, roughly speaking, the fiber space naturally induces a holomorphic mapping f: $N \to \Gamma\backslash S_g$, where S_g denotes Siegel's generalized upper half space. Then the deformation of π: $\overline{Y} \to \overline{N}$ as fiber space over $\overline{N}$ with degeneration at most over ∂N and the total space of such fiber spaces correspond respectively to the deformation of the holomorphic mapping f and the moduli space Hol(N, $\Gamma\backslash S_g$). Thus it is quite natural to deal with the case where N and $\Gamma\backslash D$ are non-compact. In the case where N is compact, there is an earlier work for a fiber space of Abelian varieties by [11].

§1. Holomorphic mappings into hyperbolically imbedded spaces

The natural topology of Hol(N, X) which we endow with is the compact-open topology. We first prove an extension and convergence theorem.

Theorem (1.1). *Let* X *be a hyperbolic complex space and hyperbolically imbedded into* $\overline{X}$. *Let* N *be a complex manifold* $\overline{N}$ *minus a hypersurface with only normal crossings. If a sequence* $\{f_\nu\}_{\nu=1}^{\infty}$ *of* $f_\nu \in \mathrm{Hol}(N, X)$ *converges to a holomorphic mapping* $f: N \to \overline{X}$, *then there are unique holomorphic extensions* $\overline{f}_\nu: \overline{N} \to \overline{X}$ *of* f_ν *and* $\overline{f}: \overline{N} \to \overline{X}$ *of* f, *and* $\{\overline{f}_\nu\}$ *converges uniformly on compact subsets of* $\overline{N}$ *to* $\overline{f}$.

As for the extension theorem, this generalizes the result of [7], but the method of the proof is different. It will play a fundamental role in our arguments. In the proof of Theorem (1.1) we use the following lemma (cf. [14] for the details).

Lemma (1.2) (cf. [19]). *Let* $B(R)$ *be the open ball of the* m-*dimensional complex vector space* $\mathbf{C}^m$ *with radius* R *and center* 0. *Let* S *be an analytic subset of pure dimension* k *of* $B(R)$ *such that* $0 \in S$. *Then we have*

$$\mathrm{Vol}(S \cap B(r)) \geq \frac{\pi^k}{k!} r^{2k}$$

for $0 < r < R$. *Moreover, if the equality holds for some* $r > 0$, *then* S *is a linear subspace of* $\mathbf{C}^m$.

In what follows, we assume that $\overline{N}$ and $\overline{X}$ are compact, and that X is a Zariski open subset of $\overline{X}$, complete hyperbolic and hyperbolically imbedded into $\overline{X}$. Combining Theorem (1.1) with the Douady theory [3], we have

Theorem (1.3). Hol(N, X) _carries a structure of a complex space with universal property, such that its underlying topology coincides with the compact-open topology, and_

$$\Phi: (f, x) \in \mathrm{Hol}(N, X) \times N \to f(x) \in X$$

is a holomorphic mapping, which is proper for every fixed $x \in N$. _Moreover_, Hol(N, X) _is a Zariski open subset of a compact complex space_.

Sketch of the proof. Let $\mathrm{Hol}(\overline{N}, \overline{X})$ be the space of all holomorphic mappings from $\overline{N}$ into $\overline{X}$ with compact-open topology. Then, by Theorem (1.1) the mapping

$$f \in \mathrm{Hol}(N, X) \to \overline{f} \in \mathrm{Hol}(\overline{N}, \overline{X})$$

is an into-homeomorphism. Hence we identify the topological space Hol(N, X) with its image in $\mathrm{Hol}(\overline{N}, \overline{X})$. By making use of the distance decreasing property of hyperbolic distance for holomorphic mappings, we see that Hol(N, X) is relatively compact. The complete hyperbolicity of X implies that Hol(N, X) is open and closed in $\mathrm{Hol}(\overline{N}, \overline{X})$ and then Theorem (1.1) yields that the topological closure of Hol(N, X) in $\mathrm{Hol}(\overline{N}, \overline{X})$ is a compact complex subspace which contains Hol(N, X) as a Zariski open subset. The complete hyperbolicity of X also implies that $\Phi(\cdot, x)$: Hol(N, X) $\to$ X is proper for every fixed $x \in N$. Q.E.D.

In general, let Y_1 and Y_2 be two complex spaces. For a holomorphic mapping $f: Y_1 \to Y_2$, we set

$$\operatorname{rank} f = \sup\left\{\dim_t Y_1 - \dim_t f^{-1}(f(t));\ t \in Y_1\right\}.$$

The following proposition follows from Lemma (1.2). It reveals a special nature of the complex analyticity of holomorphic mappings but is less known.

Proposition (1.4). *Assume that* Y_1 *and* Y_2 *are compact. Let* $\{f_\nu\}_{\nu=1}^{\infty}$ *be a sequence of points of* $\mathrm{Hol}(Y_1, Y_2)$ *converging to* $f \in \mathrm{Hol}(Y_1, Y_2)$. *If* $\operatorname{rank} f_\nu = k$, *then* $\operatorname{rank} f = k$.

We set

$$\mathrm{Hol}(k; N, X) = \{f \in \mathrm{Hol}(N, X);\ \operatorname{rank} f = k\}.$$

Corollary (1.5). $\mathrm{Hol}(k; N, X)$ *is open and closed in* $\mathrm{Hol}(N, X)$.

§2. The moduli $\mathrm{Hol}(N, \Gamma\backslash D)$

In this section we deal with the case where X is the quotient $\Gamma\backslash D$ of a symmetric bounded domain D by a torsion-free discrete subgroup Γ of $\mathrm{Aut}(D)$. We assume that Γ is uniform or an arithmetic subgroup of $\mathrm{Aut}^0(D)$. In the case where Γ is uniform and $N = \overline{N}$, the results of this section were already obtained in [20, 21, 18]. We are mainly interested in the case where $\Gamma\backslash D$ and N are non-compact, while our arguments work in the compact case. Let $\ell(D)$ (resp. $\ell(\Gamma)$) denote the maximum dimension of proper boundary components of D (resp. Γ-rational boundary components). Let

$\mathrm{Hol}(k;\ N,\ \Gamma\backslash D)$ denote the set of all holomorphic mappings $f\colon N \to \Gamma\backslash D$ with rank $f = k$. Applying the results of the previous section, we have

Theorem (2.1). i) $\mathrm{Hol}(N,\ \Gamma\backslash D)$ carries a structure of a complex space compatible with compact-open topology, such that the evaluation mapping

$$\Phi\colon\ (f,\ x)\in \mathrm{Hol}(N,\ \Gamma\backslash D)\times N \to f(x)\in\Gamma\backslash D$$

is holomorphic. Moreover, $\mathrm{Hol}(N,\ \Gamma\backslash D)$ is a Zariski open subset of the compact complex space $\overline{\mathrm{Hol}(N,\ \Gamma\backslash D)}$[1], and satisfies the universality property; i.e., for a complex space T and a holomorphic mapping $\psi\colon\ T\times N \to \Gamma\backslash D$, the natural mapping

$$t\in T \to \psi(t,\ \cdot)\in \mathrm{Hol}(N,\ \Gamma\backslash D)$$

is holomorphic.

ii) Every connected component of $\mathrm{Hol}(N,\ \Gamma\backslash D)$ is complete hyperbolic and the holomorphic mappings

$$\Phi_x\colon\ f\in \mathrm{Hol}(N,\ \Gamma\backslash D) \to f(x)\in\Gamma\backslash D$$

are proper for all $x\in N$.

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1) $\overline{\mathrm{Hol}(N,\ \Gamma\backslash D)}$ is the closure of $\mathrm{Hol}(N,\ \Gamma\backslash D)$ in $\mathrm{Hol}(\overline{N},\ \overline{\Gamma\backslash D})$.

iii) $\mathrm{Hol}(k; N, \Gamma\backslash D)$ _are open and closed in_ $\mathrm{Hol}(N, \Gamma\backslash D)$.

iv) $\mathrm{Hol}(k; N, \Gamma\backslash D)$ _are compact for_ $k > \ell(\Gamma)$.

v) $\mathrm{Hol}(k; N, \Gamma\backslash D)$ _are finite for_ $k > \ell(D)$.

In the lest of this section, we study in details the structure of $\mathrm{Hol}(N, \Gamma\backslash D)$, assuming that $\overline{N}$ is Kähler and ∂N is a hypersurface with only simple normal crossings. We use the following result on harmonic mapppings by [18]:

(2.2) Let $F: N \to \Gamma\backslash D$ and $G: N \to \Gamma\backslash D$ be free homotopic harmonic mappings with finite energy. Then there is a harmonic mapping $\Psi: \mathbf{R}\times N \to \Gamma\backslash D$ with respect to the product metric $dt\oplus h$ on $\mathbf{R}\times N$ such that

i) $\Psi(0, x) = F(x)$, $\Psi(1, x) = G(x)$ and Ψ provides a free homotopy between F and G, equivalent to the given one;

ii) for every $x\in N$, the curve γ_x: $t\in\mathbf{R} \to \Psi(t, x)\in\Gamma\backslash D$ is a parametrization of a geodesic with constant speed, independent of x, and $e(\Psi(t, \cdot))(x)$ is constant in t.

Lemma (2.3). _Let_ F _and_ G _be as in_ (2.2). _If_ F _is holomorphic_, _then so is_ G.

Remark. 1) In case N is compact, this is a theorem due to Lichenerowicz (cf. Theorem (8.6) of [4]).

2) Since (2.2) actually holds for harmonic mappings from a complete Riemannian manifold with finite volume into a complete Riemannian manifold with non-positive sectional curvatures, Lemma (2.3) is also true for harmonic mappings F and G from a complete Kähler manifold with finite volume into a complete Kähler manifold with non-positive sectional curvatures, provided that F and G have finite energies.

The main result is the following:

Theorem (2.4). i) $Hol(N, \Gamma\backslash D)$ _is smooth and quasi-projective_.

ii) _For a connected component_ Z _of_ $Hol(N, \Gamma\backslash D)$ _and a point_ $x \in N$, _the evaluation mapping at_ x

$$\Phi_x : f \in Z \to f(x) \in \Gamma\backslash D$$

is a proper holomorphic immersion onto a totally geodesic complex submanifold, so that Z _is a free quotient of a symmetric bounded domain_.

iii) _For a connected component_ Z _of_ $Hol(N, \Gamma\backslash D)$, _there is a normal complex projective variety_ $\tilde{Z}$ _such that_ Z _is hyperbolically imbedded into_ $\tilde{Z}$ _and_ Φ_x _holomorphically extends to_ $\bar{\Phi}_x : \tilde{Z} \to \overline{\Gamma\backslash D}$.

iv) $\dim Hol(k; N, \Gamma\backslash D) \leq \ell(D)$ _for_ $k > 0$.

v) *For* $f\in \mathrm{Hol}(N, \Gamma\backslash D)$ *with* $\overline{f}^{-1}(\partial\Gamma\backslash D) \neq \phi$, $\dim_f \mathrm{Hol}(N, \Gamma\backslash D) \leq \ell(\Gamma)$.

As a corollary, we have the following.

Corollary (2.5) (Rigidity). *Let* $f\colon N \to \Gamma\backslash D$ *be a holomorphic mapping. Then* f *is a unique holomorphic mapping among the free homotopy class of* f, *if* f *satisfies one of the following conditions*:

a) *The image of* f *is not contained in a totally geodesic complex proper submanifold of* $\Gamma\backslash D$;

b) $\operatorname{rank} f > \ell(D)$;

c) $\overline{f}^{-1}(\partial\Gamma\backslash D) \neq \phi$ *and* $\operatorname{rank} f > \ell(\Gamma)$.

In general, a holomorphic mapping $f\in \mathrm{Hol}(N, \Gamma\backslash D)$ admits a deformation (cf. [5]). But in the special case where D is the n-th product H^n of the upper half plane $H\subset \mathbf{C}$, we see that any $f\in\mathrm{Hol}(N, \Gamma\backslash H^n)$ is rigid. That is, by making use of the rigidity Theorem 6 of [6], we have

Theorem (2.6). *Let* $\Gamma\subset (PSL(2, \mathbf{R}))^n$ *be an irreducible torsion-free discrete subgroup with* $\mathrm{Vol}(\Gamma\backslash H^n) < \infty$. *Then*

i) *if* $f\colon N \to \Gamma\backslash H^n$ *is a non-constant holomorphic mapping*, f *is a unique holomorphic mapping among the free homotopy class of* f,

so that

ii) *there are only finitely many non-constant holomorphic mappings from* N *into* $\Gamma\backslash H^n$.

Remark. 1) It must be noted that if $\Gamma\backslash H^n$ is not compact, then Γ is arithmetic ([12]). Therefore Γ satisfies our requirement for discrete subgroups.

2) In the case of dim $N = 1$, i) was proved in [6].

3) By the same arguments as in [15], we see that the Kähler assumption for $\overline{N}$ is not necessary in ii). The proof is reduced to the present case.

4) For a compact quotient $\Gamma\backslash H^2$ and a compact complex manifold N, ii) was proved in [15]. For an algebraic curve N and compact $\Gamma\backslash H^n$, it was proved in [6].

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