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# On Derivations and Congruences of Siegel Modular Forms of Degree Two

東工大・理　佐藤　孝和　(Takakazu Satoh)

## 1. Introduction

Let $M_k(\Gamma_n)$ be the space of Siegel modular forms of degree $n$ and of weight $k$ and $S_k(\Gamma_n)$ its subspace consisting of cuspforms. Denote Fourier expansion of $f\in M_k(\Gamma_n)$ by

$$f(Z) = \sum_{N\geq 0} a(N,f)\exp(2\pi i\mathrm{Tr}(NZ)),$$

where $N$ runs over all symmetric half-integral matrices of size $n$. Put

$$M_k(\Gamma_n;\mathbf{Z}) = \{ f\in M_k(\Gamma_n) \mid a(N,f)\in\mathbf{Z} \text{ for all } N\geq 0 \},$$

$$S_k(\Gamma_n;\mathbf{Z}) = M_k(\Gamma_n;\mathbf{Z})\cap S_k(\Gamma_n),$$

and

$$M_k(\Gamma_n;R) = M_k(\Gamma_n;\mathbf{Z})\otimes_{\mathbf{Z}} R,$$

$$S_k(\Gamma_n;R) = S_k(\Gamma_n;\mathbf{Z})\otimes_{\mathbf{Z}} R,$$

$$M_{\mathrm{even}}(\Gamma_2;R) = \sum_{k:\mathrm{even}} M_k(\Gamma_n;R),$$

where $R$ is a commutative ring with 1. We note $M_k(\Gamma_n;\mathbf{C}) = M_k(\Gamma_n)$ by Eichler [3] and Baily [1]. Let $H_2$ be the Siegel upper half space of degree two. We denote the variable $Z$ on $H_2$ as $Z = \begin{pmatrix} z_1 & z_3 \\ z_3 & z_2 \end{pmatrix}$ and define the differential operator $\Theta$ by

$$\Theta = \begin{pmatrix} \partial_1 & \frac{1}{2}\partial_3 \\ \frac{1}{2}\partial_3 & \partial_2 \end{pmatrix} \text{ where } \partial_j = \frac{1}{2\pi i}\frac{\partial}{\partial z_j}$$

For a odd prime $\ell$, wee see that $|\Theta|$ induces a map from $M_k(\Gamma_n;\mathbf{F}_\ell)$ to $\mathbf{F}_\ell[q_3,q_3^{-1}][[q_1,q_2]]$ where $q_j = \exp(2\pi i z_j)$. Our aim is to show that $|\Theta|$ preserves $M_{\mathbf{even}}(\Gamma_2;\mathbf{F}_\ell)$ under the suitable conditions on $\ell$ (Theorem 3.2).

We proceed as follows. First, we construct the order two derivation $\partial : M_{\mathbf{even}}(\Gamma_2;\mathbf{C}) \to M_{\mathbf{even}}(\Gamma_2;\mathbf{C})$ with certain integrality. This result already enables us to prove certain congruence formulas. Using this derivation, we reduce our problem to one for an element of the Maass space in Section 3.

**Notation and Terminologies.**

By a holomorphic periodic function on $H_n$, we mean a holomorphic function $f$ on $H_n$ satisfying $f(Z+S) = f(Z)$ for all $Z \in H_n$ and all symmetric half-integral matrices $S$ of size $n$. In this case, we denote by $a(N,f)$ the Fourier coefficients of $f$ at $N$. Let $K$ be a algebraic number filed and $R$ a localization of the integer ring of $K$ at a prime ideal $\mathfrak{l}$ of $K$. If a holomorphic periodic function $f$ satisfies $a(N,f) \in R$ for all $N$, we denote by $f$ mod $\mathfrak{l}$ coefficient wise reduction mod $\mathfrak{l}$ of $f$. Suppose a holomorphic periodic function $g$ satisfies $a(N,g) \in R$ for all $N$. We write $f \equiv g$ mod $\mathfrak{l}$ if $a(N,f) \equiv a(N,g)$ mod $\mathfrak{l}$ for all $N$. For simplicity we denote by $(n_1,n_2,n_3)$ the matrix $\begin{pmatrix} n_1 & \frac{1}{2}n_3 \\ \frac{1}{2}n_3 & n_2 \end{pmatrix}$.

## 2. Construction of Derivation

Let $R$ be a subring of $\mathbf{C}$ containing $1, \frac{1}{2}, \frac{1}{3}$. By Igusa [5], we see that the ring of even weight modular forms of degree two with $R$-coefficients is isomorphic to a polynomial ring with four variables:

$$M_{\mathrm{even}}(\Gamma_2;R) = R[\varphi_4,\varphi_6,\chi_{10},\chi_{12}]. \tag{2.1}$$

Here, generators are determined by the following conditions.

$$\begin{aligned}
\varphi_4 &\in M_4(\Gamma_2;\mathbf{Z}) & a(0,\varphi_4) &= 1,\\
\varphi_6 &\in M_6(\Gamma_2;\mathbf{Z}) & a(0,\varphi_6) &= 1,\\
4\chi_{10} &\in S_{10}(\Gamma_2;\mathbf{Z}) & a((1,1,1),\chi_{10}) &= -1,\\
12\chi_{12} &\in S_{12}(\Gamma_2;\mathbf{Z}) & a((1,1,1),\chi_{12}) &= 1.
\end{aligned}$$

In general, we denote by $\varphi_k$ Siegel's Eisenstein series of degree two and weight $k$ for an even integer $k$ normalized as $a(0,\varphi_k) = 1$. It is known by Nagaoka [9] that $\varphi_{\ell-1}\equiv 1 \mod \ell$ if $\ell$ is a regular prime or if $\ell$ does not divide the numerator of Bernoulli number $B_{\ell-3}$.

We denote by $M_k^\infty(\Gamma_n)$ the $\mathbf{C}$-vector space of $C^\infty$-Siegel modular forms of degree $n$ and of weight $k$. Let $\delta = \delta_k$ be the Maass operator:

$$\delta_k\colon M_k^\infty(\Gamma_n) \to M_{k+2}^\infty(\Gamma_n).$$

In the degree two case, $\delta_k f$ for $f\in M_k^\infty(\Gamma_2)$ is given by

$$\delta_k f = k\left(k-\frac{1}{2}\right)|\eta|^{-1}f+\frac{1}{4}\left(k-\frac{1}{2}\right)|\eta|^{-1}\mathrm{Tr}(\eta\Theta f)+|\Theta|f, \tag{2.2}$$

where $\eta = -4\pi \mathrm{Im} Z$. See Maass [7, Sect. 19] for details. Although the Maass operator commutes with the Hecke operators, it does not keep holomorphy. So we construct another operator.

For simplicity we put $M_{\mathrm{even}}(\Gamma_2) = M_{\mathrm{even}}(\Gamma_2;\mathbf{C})$. Let $d: M_{\mathrm{even}}(\Gamma_2) \to C^\infty(H_2)$ be the C-derivation defined by

$$\begin{aligned} d\varphi_4 &= \frac{2}{7}(-4\varphi_6 - |\Theta|\varphi_4), & d\varphi_6 &= \frac{2}{11}(-6\varphi_4^2 - |\Theta|\varphi_6), \\ d\chi_{10} &= -\frac{2}{19}|\Theta|\chi_{10}, & d\chi_{12} &= -\frac{2}{23}|\Theta|\chi_{12}. \end{aligned} \tag{2.3}$$

(Note that $\varphi_4$, $\varphi_6$, $\chi_{10}$, and $\chi_{12}$ are algebraically independent.) For an even integer $k$, define the C-linear map $\partial_k: M_k(\Gamma_2) \to C^\infty(H_2)$ by

$$d_k = \frac{2}{2k-1}(\partial_k - |\Theta|), \tag{2.4}$$

where $d_k$ is the restriction of $d$ to $M_k(\Gamma_2)$, and put $\partial = \bigoplus_{k:\mathrm{even}} \partial_k$.

**Theorem 2.1.** *Let $k$ be an even integer. Let $R$ be a subring of* C *containing* 1, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{7}$, $\frac{1}{11}$, $\frac{1}{19}$, *and* $\frac{1}{23}$. *Then we have the following.*

(a) *If $f \in M_k(\Gamma_2)$, then $\partial f \in M_{k+2}(\Gamma_2)$.*

(b) *If $f \in S_k(\Gamma_2)$, then $\partial f \in S_{k+2}(\Gamma_2)$.*

(c) *If $f \in M_k(\Gamma_2;R)$, then $a(N, \partial f - |\Theta| f) \in \left(k-\frac{1}{2}\right)R$ for all symmetric half-integral matrices $N$.*

*Proof.* The assertion (a) holds for four generators $\varphi_4 \sim \chi_{12}$ by (2.3) and (2.4). To prove (a), it is enough to show that $\partial(fg) \in M_{k+j+2}(\Gamma_2)$ under the conditions $f \in M_k(\Gamma_2)$, $\partial f \in M_{k+2}(\Gamma_2)$, $g \in M_j(\Gamma_2)$, and $\partial g \in M_{j+2}(\Gamma_2)$. Since $d$ is a derivation, we have

$$\frac{2}{2(k+j)-1}\partial(fg) = \frac{2}{2j-1}f\partial g+\frac{2}{2k-1}g\partial f+H$$

where

$$H = \frac{2}{2(k+j)-1}\,|\Theta|(fg)-\frac{2}{2k-1}g\,|\Theta|f-\frac{2}{2j-1}f\,|\Theta|g$$

Clearly $H$ is a holomorphic function on $H_2$. On the other hand,

$$H = \frac{2}{2(k+j)-1}\delta_{k+j}(fg)-\frac{2}{2k-1}g\delta_k f-\frac{2}{2j-1}f\delta_j g$$

by (2.2). So, $H\in M^{\infty}_{k+j+2}(\Gamma_2)$. Hence $H\in M_{k+j+2}(\Gamma_2)$ and therefore $\partial(fg)\in M_{k+j+2}(\Gamma_2)$. The proof of (b) is similar. Noting (2.1), we have only to prove (c) for $f = \varphi_4^a\varphi_6^b\chi_{10}^c\chi_{12}^d$ where $a$, $b$, $c$, and $d$ are positive integers. In this case, we have

$$\begin{aligned}(\partial-|\Theta|)f &= \frac{(2k-1)a}{7}\varphi_4^{a-1}\varphi_6^b\chi_{10}^c\chi_{12}^d(\partial-|\Theta|)\varphi_4\\ &+ \frac{(2k-1)b}{11}\varphi_4^a\varphi_6^{b-1}\chi_{10}^c\chi_{12}^d(\partial-|\Theta|)\varphi_6\\ &+ \frac{(2k-1)c}{19}\varphi_4^a\varphi_6^b\chi_{10}^{c-1}\chi_{12}^d(\partial-|\Theta|)\chi_{10}\\ &+ \frac{(2k-1)d}{23}\varphi_4^a\varphi_6^b\chi_{10}^c\chi_{12}^{d-1}(\partial-|\Theta|)\chi_{12}\end{aligned} \tag{2.5}$$

Noting the condition on $R$, we see that (c) holds. Q.E.D.

A modular form $f$ is said to be an eigenform if it is a non-zero common eigen function of all Hecke operators. We denote by $\lambda(m,f)$ the eigenvalue of the $m$-th Hecke operator $T(m)$ and put $\mathbf{Q}(f) = \mathbf{Q}(\lambda(m,f)\,|\,m\geq 1)$.

**Corollary 2.2.** *Let $\ell$ be a prime number other than 2, 3, 7, 11, 19, 23. Let $K$ be an algebraic number field, $\mathfrak{l}$ its prime ideal lying over*

$\mathfrak{l}$ *and* $R$ *the localization of the integer ring of* $K$ *at* $\mathfrak{l}$. *Let* $k$ *be an even integer satisfying* $\mathfrak{l} \mid 2k-1$ *and let* $f \in M_k(\Gamma_2;R)$ *be an eigenform. Then at least one of the following holds.*

(a) $|N|\, a(N,f) \equiv 0 \bmod \mathfrak{l}$ *for all* $N>0$.

(b) *There exist an eigenform* $g \in S_{k+2}(\Gamma_2)$ *such that*

$$\mathrm{N}_{\mathbf{Q}(f)\mathbf{Q}(g)/\mathbf{Q}(f)}(m^2\lambda(m,f)-\lambda(m,g)) \equiv 0 \bmod \mathfrak{l}$$

*for all* $m \geq 1$ *where* $\mathrm{N}_{\mathbf{Q}(f)\mathbf{Q}(g)/\mathbf{Q}(f)}$ *is the norm map.*

*Proof.* Suppose (a) does not hold. Using $m^2\delta T(m) = T(m)\delta$ (cf. [10, (3.11) below]) and the uniqueness of the Fourier coefficients, we can define $T(m)|\Theta|f$ by $m^2|\Theta|T(m)f$. Clearly, $|\Theta|f$ is an eigenform when $f$ is a non-constant eigenform. Since $a(N,\partial f) \equiv |N|\, a(N,f) \bmod \mathfrak{l}$ by Theorem 2.1(c), $\partial f$ is a mod $\mathfrak{l}$ eigenform in the sense of Deligne and Serre [2, Sect. 6]. It is easy to see that $\partial f$ mod $\mathfrak{l}$ belongs to $S_{k+2}(\Gamma_2;R/\mathfrak{l})$. Now existence of $g$ follows from Deligne and Serre [2, Lemma 6.11]. Q.E.D.

*Remark* 2.3. Suppose the multiplicity one condition holds for a Hecke eigenform $f \in M_k(\Gamma_n)$. Then, by Kurokawa [6, Theorem 2], there is a non-zero constant $c$ satisfying $cf \in M_k(\Gamma_n;\mathrm{Z}(f))$ where $\mathrm{Z}(f)$ is the integer ring of $\mathbf{Q}(f)$. We note the multiplicity one condition certainly holds for degree two Eisenstein series (including Klingen type ones.)

*Remark* 2.4. In fact $\partial$ is an order two derivation. But the current author has not obtained the result taking advantage of this property.

## 3. Stability of mod $\ell$ modular forms

Let $M_k^{\mathrm{I}}(\Gamma_2)$ be the Maass space of weight $k$ and put $S_k^{\mathrm{I}}(\Gamma_2) = S_k(\Gamma_2)\cap M_k^{\mathrm{I}}(\Gamma_2)$, $M_k^{\mathrm{I}}(\Gamma_2;R) = M_k(\Gamma_2;R)\cap M_k^{\mathrm{I}}(\Gamma_2)$, and $S_k^{\mathrm{I}}(\Gamma_2;R) = S_k(\Gamma_2;R)\cap M_k^{\mathrm{I}}(\Gamma_2)$ for a subring $R$ of $\mathbf{C}$. The Fourier coefficients of $f\in S_k^{\mathrm{I}}(\Gamma_2)$ satisfies the relation

$$a((n_1,n_2,n_3),f) = \sum_{d\mid g,\ d>0} d^{k-1}a\left(\left(1,\frac{n_1n_2}{d^2},\frac{n_3}{d}\right),f\right)$$

for $(n_1,n_2,n_3)>0$ with $g = \gcd(n_1,n_2,n_3)$. Therefore, $a(N,f)$ for all $N>0$ are determined by $a((1,*,1),f)$ and $a((1,*,0),f)$. We also note that if the latter are integral with respect to a discrete valuation, so are the former. For a holomorphic periodic function $f$ on $H_2$ and an integer $j,h,\nu\geq 0$, we put

$$\varepsilon_\nu A_\nu(j,h,f) = \sum_{t\in\mathbf{Z}} a((j,h,t),f)\,t^\nu,$$

where $\varepsilon_0=1$ and $\varepsilon_\nu=2$ for $\nu>0$. By Maass [8], the Maass space $S_k^{\mathrm{I}}(\Gamma_2)$ is isomorphic to $S_k(\Gamma_1)\oplus S_{k+2}(\Gamma_1)$ as a $\mathbf{C}$-vector space:

$$\begin{array}{ccccc} S_k^{\mathrm{I}}(\Gamma_2) & \cong & S_k(\Gamma_1) & \oplus & S_{k+2}(\Gamma_1) \\ f & \to & (F_0 & , & F_2) \end{array} \tag{3.1}$$

where

$$F_0(z) = \sum_{h=1}^{\infty} A_0(1,h,f)\,q^h,$$

$$F_2(z) = \sum_{h=1}^{\infty}\left(A_0(1,h,f)-\frac{h}{k}A_2(1,h,f)\right)q^h,$$

with $q = \exp(2\pi iz)$.

**Lemma 3.1.** *Let $k$ an even integer and $\ell \geq 5$ be a prime number satisfying $\ell \nmid k(k+1)(k+2)$ and $\varphi_{\ell-1} \equiv 1 \bmod \ell$. Let $f \in M_k^{\mathrm{I}}(\Gamma_2;\mathbf{Z}_{(\ell)})$. Then there exists a cuspform $F \in S_k^{\mathrm{I}}(\Gamma_2;\mathbf{Z}_{(\ell)})$ whose Fourier coefficients satisfy*

$$a(N,F) \equiv |N|\, a(N,f) \bmod \ell.$$

*Proof.* (Outline.) It can be verified that

$$A_0(1,h,|\Theta|f) = \left(1-\frac{1}{2k}\right) hA_0(1,h,f)-\frac{1}{2}\left(A_2(1,h,f)-\frac{h}{k}A_0(1,h,f)\right).$$

By Serre [11], it holds that

$$\theta M_k(\Gamma_1;\mathbf{F}_\ell) \subset S_{k+\ell+1}(\Gamma_1;\mathbf{F}_\ell)$$

where $\theta f = q\dfrac{d}{dq}f$ is a derivation of $\mathbf{F}_\ell[[q]]$. So, there is $F_0 \in S_{k+\ell+1}(\Gamma_1;\mathbf{Z}_{(\ell)})$ satisfying

$$A_0(1,h,|\Theta|f) \equiv a(h,F_0) \bmod \ell.$$

A similar computation shows that there is $F_2 \in S_{k+\ell+3}(\Gamma_1;\mathbf{F}_\ell)$ satisfying

$$A_2(1,h,|\Theta|f)-\frac{h}{k+\ell+1}A_0(1,h,|\Theta|f) \equiv a(h,F_2) \bmod \ell.$$

(Cf. Eichler and Zagier [4, Theorem 3.1].) Then $F \in S_{k+\ell+1}^{\mathrm{I}}(\Gamma_2;\mathbf{Z}_{(\ell)})$ corresponding $(F_0,F_2)$ in (3.1) has all the required properties. Q.E.D.

**Theorem 3.2.** *Let $\ell>23$ be a prime number satisfying $\varphi_{\ell-1} \equiv 1 \bmod \ell$ and $k$ an even integer. Then, $|\Theta| M_k(\Gamma_2;\mathbf{F}_\ell) \subset S_{k+\ell+1}(\Gamma_2;\mathbf{F}_\ell)$. Especially, $M_{\mathrm{even}}(\Gamma_2;\mathbf{F}_\ell)$ is stable under $|\Theta|$.*

*Proof.* Let $a$, $b$, $c$, and $d$ be non-negative even integers satisfying $4a+6b+10c+12d = k$. Put $f = \varphi_4^a\varphi_6^b\chi_{10}^c\chi_{12}^d$. We have only to prove that

$$|\Theta|f \bmod \ell \subset S_{k+\ell+1}(\Gamma_2;\mathbf{F}_\ell). \tag{3.2}$$

As in (2.5), we have

$$\begin{aligned}
|\Theta|f \equiv\ & \varphi_{\ell-1}\partial f \\
& - \frac{(2k-1)a}{7}(\varphi_4^{a-1}\varphi_6^b\chi_{10}^c\chi_{12}^d\varphi_{\ell-1}\partial\varphi_4-\varphi_4^{a-1}\varphi_6^b\chi_{10}^c\chi_{12}^d|\Theta|\varphi_4) \\
& - \frac{(2k-1)b}{11}(\varphi_4^a\varphi_6^{b-1}\chi_{10}^c\chi_{12}^d\varphi_{\ell-1}\partial\varphi_6-\varphi_4^a\varphi_6^{b-1}\chi_{10}^c\chi_{12}^d|\Theta|\varphi_6) \\
& - \frac{(2k-1)c}{19}(\varphi_4^a\varphi_6^b\chi_{10}^{c-1}\chi_{12}^d\varphi_{\ell-1}\partial\chi_{10}-\varphi_4^a\varphi_6^b\chi_{10}^{c-1}\chi_{12}^d|\Theta|\chi_{10}) \\
& - \frac{(2k-1)d}{23}(\varphi_4^a\varphi_6^b\chi_{10}^c\chi_{12}^{d-1}\varphi_{\ell-1}\partial\chi_{12}-\varphi_4^a\varphi_6^b\chi_{10}^c\chi_{12}^{d-1}|\Theta|\chi_{12}) \quad \bmod \ell
\end{aligned}$$

by Lemma 3.1. Since four generators $\varphi_4\sim\chi_{12}$ belong to the Maass space, $|\Theta|f \bmod \ell \subset M_{k+\ell+1}(\Gamma_2;\mathbf{F}_\ell)$. On the other hand, $a(N,|\Theta|f) = 0$ for $|N|=0$. So (3.2) holds. Q.E.D.

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