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Wave Forms for $\Gamma_0(p)$

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§ 1. Introduction.

Let H be the upper half plane and $G=SL_2(R)$. We fix a prime number p through this paper. Let $\Gamma=\Gamma_0(p)$ and χ a Dirichlet character modulo p satisfying $\chi(-1)=1$. We regard χ as a character of $\Gamma_0(p)$ by $\chi(g)=\chi(d)$ $(g=\begin{bmatrix} a, b \\ c, d \end{bmatrix})$. Let Δ denote the Laplace-Beltrami operator in $L^2(\Gamma\backslash H,\chi)$ and $0<\lambda_1<\lambda_2<\lambda_3<\cdots\cdots$, the spectra of Δ in $L^2_0(\Gamma\backslash H,\chi)$.

The purpose of this note is to investigate the relevance of the space W(1/4) of wave forms having a minimal principal eigenvalue 1/4 to Selberg's type zeta function. For its sake, we use Selberg's trace formula theory [8]. Selberg [9] conjugated there are no exceptional eigenvalues of Δ in $L^2_0(\Gamma\backslash H,\chi)$ for congruence subgroups and principal characters. If it is true, 1/4 is a possible minimal eigenvalue

for such cases. The zeta function $z(s,\chi)$ over all primitive hyperbolic conjugate classes of Γ which is defined in (4.5) was first appeared in Selberg [8]. For cocompact subgroups, it was studied in Huber [4]. In Ishikawa-Tanigawa [6], they saw that such a zeta function which is slightly modified with signature is continued as meromorphic on the whole plane and its residue at zero gives the dimension of the space of cusp forms of weight one for $\Gamma_0(p)$.

We give a view of Eisenstein series and its constant terms in § 2, and a definition of wave forms in § 3. In § 4, we consider the trace formula for a certain kernel K^*_s with a parameter s (Re(s)>1), which is defined in (4.3). Continuing each term of the trace formula as a meromorphic function of s, we get an analytic continuation of $z(s,\chi)$ in Theorem 1. In § 5, we give the sketch of the proof of Theorem 1. Moreover, the dimension formula of W(1/4) becomes remained as a residue at zero. It turns out in Theorem 2 that its dimension depends only on the residue of $z(s,\chi)$ at zero and the number of Γ-inequivalent cusps.

§ 2. Eisenstein series.

There are two Γ-inequivalent cusps in Γ which are represented by ∞ and 0. Denote by Γ_κ the stabilizer of a cusp κ in Γ, then

$$\Gamma_\infty=\{\pm\begin{bmatrix}1,n\\0,1\end{bmatrix};\ n\in Z\},\quad \Gamma_0=\{\pm\begin{bmatrix}1,0\\nP,1\end{bmatrix};\ n\in Z\}.$$

Fix elements $\sigma_\infty=1$, $\sigma_0=\begin{bmatrix}0,&-1/\sqrt{P}\\\sqrt{P},&0\end{bmatrix}\in G$ such that $\sigma_\kappa(\infty)=\kappa$ and $\sigma_\kappa^{-1}\Gamma_\kappa\sigma_\kappa=\Gamma_\infty$ ($\kappa\in\{\infty,0\}$). The Eisenstein series $E_\kappa(z,t,\chi)$ at-

tached to a cusp κ is defined by

(2.1) $$E_\kappa(z,t,\chi)=\sum_{g\in\Gamma_\kappa\backslash\Gamma}(\mathrm{Im}\,\sigma_\kappa^{-1}gz)^t\,\bar{\chi}(g),$$

where t is a complex variable with Re(t)>1. We can write them as

(2.2) $$L(2t,\bar{\chi})\,E_\infty(z,t,\chi)=\frac{1}{2}\sum_{(c,d)\in Z\times Z,\ d\not\equiv 0(\mathrm{mod}\ p)}\bar{\chi}(d)y^t/|cpz+d|^{2t}$$

(2.3) $$L(2t,\chi)\,E_0(z,t,\chi)=\frac{1}{2}\sum_{(a,b)\in Z\times Z,}{}'\ \chi(a)y^t/p^t|az+b|^{2t}.$$

Note that if χ is a principal character, $L(2t,\chi)=L(2t,\bar{\chi})$ $=\zeta(2t)(1-p^{-2t})$.

Lemma 1. <u>The constant term of the Fourier expansion of</u> $E_\kappa(\sigma_\mu z,t,\chi)$ <u>is given in the form</u>

(2.4) $$\delta_{\kappa\mu}y^{t+m_{\kappa\mu}(t,\chi)}y^{1-t},$$

<u>where</u> $\delta_{\kappa\mu}$ <u>is Kronecker's delta. The matrix of the constant terms can be expressed as follows:</u>

(2.5) $$M(t,\chi)=\frac{\Lambda(2t-1)}{\Lambda(2t)}(p^{2t}-1)^{-1}\begin{bmatrix} p-1, & p^t-p^{1-t} \\ p^t-p^{1-t}, & p-1 \end{bmatrix}\quad \underline{\text{if } \chi \text{ is principal,}}$$

$$M(t,\chi)=p^{-t}\begin{bmatrix} 0, & \dfrac{\Lambda(2t-1,\bar{\chi})}{\Lambda(2t,\bar{\chi})} \\ \dfrac{\Lambda(2t-1,\chi)}{\Lambda(2t,\chi)}, & 0 \end{bmatrix}\quad \underline{\text{if } \chi \text{ is not principal.}}$$

<u>where</u> $\Lambda(t)=\pi^{-(t/2)}\Gamma(t/2)\zeta(t)$, $\Lambda(t,\chi)=(p/\pi)^{t/2}\Gamma(t/2)L(t,\chi)$.

Above formulae follow from (2.2) and (2.3) immediately. The

functional equation of $\Lambda(t)$ or $\Lambda(t,\chi)$ gives

(2.6) $$M(t,\chi)M(1-t,\chi)=1.$$

By the general theory of Eisenstein series [7], it is known that there is a functional equation

(2.7) $$^t[E_\infty(z,t,\chi),\ E_0(z,t,\chi)]$$
$$=M(t,\chi)\ ^t[E_\infty(z,1-t,\chi),\ E_0(z,1-t,\chi)],$$

and $E_\kappa(z,t,\chi)$ can be continued as meromorphic to the whole t-plane.

§ 3. Wave forms.

Let $L^2(\Gamma\backslash H,\chi)$ be the space of automorphic functions f with respect to Γ, χ such that f is square integrable over $\Gamma\backslash H$, and $L^2_0(\Gamma\backslash H,\chi)$ the space of f in $L^2(\Gamma\backslash H,\chi)$ satisfying the cuspidal condition:

(3.1) $$\int_0^1 f(\sigma_\kappa z)\,dx=0, \qquad \text{for almost all } y\ (z=x+iy).$$

Let $W(\lambda)$ denote the subspace in $L^2_0(\Gamma\backslash H,\chi)$ satisfying

(3.2) $$\Delta f=\lambda f, \qquad \left(\Delta=-y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\right).$$

Then $L^2_0(\Gamma\backslash H,\chi)$ decomposes into the direct sum of $W(\lambda_i)$ whose dimension is of finite ([2]), where $\{\lambda_i\}$ $(0<\lambda_1<\lambda_2<\lambda_3<\cdots\cdots)$ is the set of all eigenvalues of Δ in $L^2_0(\Gamma\backslash H,\chi)$. It follows from Lemma 1 that the discrete part of the orthcomplement of $L^2_0(\Gamma\backslash H,\chi)$ appears only when χ is a principal character. In such a case, it coincides with C which is spanned by the constant functions in $L^2(\Gamma\backslash H,\chi)$.

§ 4. Selberg's trace formula.

We introduce a point pair G-invariant kernel function of (a)-(b) type in the sense of [8]. For a complex number s with Re(s)>1, put

(4.1) $$k_s(z,z')=d(s)\,|\,yy'\,|^{(s+1)/2}/\,|\,(z-\bar{z}')/2i\,|^{s+1}$$

$$d(s)=\frac{1}{2}\,B(\frac{1}{2},\frac{s}{2})^{-1}.$$

Since the integral operator in $C^\infty(H)$ determined by $k_s(z,z')$ is G-invariant, its eigenvalue depends only on the spectrum λ of Δ ; so we use $h_s(\lambda)$ for it. By a direct calculation, we have

Lemma 2. <u>Selberg's transformation is given as follows,</u>

(4.2) $$h_s(\lambda)=2^s\,B(\frac{s}{2}+ir,\frac{s}{2}-ir)$$

<u>where</u> r <u>satisfies the equation</u> $\lambda=(1/4)+r^2$.

By the aid of the Eisenstein series, we define

(4.3) $$K^*_s(z,z')=K_s(z,z')-H_s(z,z')$$

$$K_s(z,z')=\sum_{g\in\Gamma/\{\pm 1\}} k_s(z,gz')\chi(g)$$

$$H_s(z,z')=\sum_{\kappa\in\{\infty,0\}}\frac{1}{4\pi}\int_{Re(t)=1/2} h_s(t-t^2)E_\kappa(z,t,\chi)\overline{E_\kappa(z',t,\chi)}d\,|\,t\,|$$

By the same argument as in [5, § 4], $K^*_s(z,z')$ is bounded for all $(z,z')\in H\times H$, and the integral operator K^*_s with kernel $K^*_s(z,z')$ is completely continuous and is zero operator on the continuous part $L^2_c(\Gamma\backslash H,\chi)$ of the orthcomplement of $L^2_0(\Gamma\backslash H,\chi)$. So we can regard K^*_s as an operator on $L^2_0(\Gamma\backslash H,\chi)\oplus\delta\,C$ where $\delta=1$ or 0, according to χ =principal or not. Let d_i denote dim $W(\lambda_i)$. We have

$$(4.4) \qquad \delta \cdot h_s(0) + \sum_{i=1}^{\infty} d_i \cdot h_s(\lambda_i) = \int_{\Gamma \backslash H} K^*_s(z,z)dz$$

It follows from the general theory [8] that the series above in (4.4) converges absolutely and uniformly in any compact subset for $Re(s)>1$. By Stirling's formula for Γ-functions, it will also converge absolutely and uniformly in any compact subset including no poles of $h_s(\lambda_i)$'s and of $\delta \cdot h_s(0)$ for whole s-plane. Then we have

Lemma 3. $\delta \cdot h_s(0) + \sum_{i=1}^{\infty} d_i \cdot h_s(\lambda_i)$ <u>can be continued as a meromorphic function of the whole s-plane with simple poles at</u> $\{\pm 2ir_i - 2k;\ k \geqq 0,\ k \in \underline{Z},\ i \geqq 1\} \cup \delta\ \{1-2k;\ k \geqq 0,\ k \in \underline{Z}\}$ <u>where</u> $\lambda_i = (1/4) + r_i^2$.

The integral in (4.4) will be decomposed into the sum of Γ-conjugate classes. Let $A(\pm 1, s)$, $A(E_i, s)$ or $A(\kappa, s)$ denote the contribution from the identity, elliptic conjugate classes of order i and 2i or cusp κ to its integral respectively.

Definition. For $Re(s)>1$, we define

$$(4.5) \qquad z(s,\chi)$$

$$= \sum_{g \in P} \sum_{m=1}^{\infty} \log(N(g))(N(g^m)^{1/2} - N(g^m)^{-1/2})^{-1}(N(g^m)^{1/2} + N(g^m)^{-1/2})^{-s}\chi(g^m)$$

where P denotes a set of primitive hyperbolic conjugate classes of Γ, $N(g)$ a norm λ^2 of g (λ being an eigenvalue of g with $|\lambda| > 1$).

Theorem 1. (1) For $\mathrm{Re}(s)>0$, we can rewrite Selberg's trace formula as

$$(4.6)\qquad \delta\cdot 2^s B(\tfrac{s+1}{2},\tfrac{s-1}{2})+\sum_{i=1}^{\infty} d_i\cdot 2^s B(\tfrac{s}{2}+ir,\tfrac{s}{2}-ir)$$

$$=A(\pm 1,s)+A(E_2,s)+A(E_3,s)+2^s z(s,\chi)+A(\infty,s)+A(0,s).$$

The expressions of $A(\pm 1,s)$, $A(E_2,s)$, $A(E_3,s)$ and $A(\kappa,s)$ are given in (5.2), (5.4), (5.5), (5.10) and (5.11).

(2) $A(\pm 1,s)$, $A(E_2,s)$, $A(E_3,s)$ and $A(\kappa,s)$ can be continued as a meromorphic function in the whole s-plane. Then Selberg's type zeta function $z(s,\chi)$ can be continued as meromorphic in the whole s-plane.

§ 5. The sketch of proof of Theorem 1.

First we will calculate the diagonal integral of K^*_s. In § § 5.1 -5.4, we will assume $\mathrm{Re}(s)>1$. For $g\in\Gamma$, put

$$(5.1)\qquad A(g,s)=\int_{\Gamma(g)\backslash H} k_s(z,gz)\,dz\,\chi(g),$$

where $\Gamma(g)$ denotes a normalizer of g in Γ.

5.1 Center.

$$(5.2)\qquad A(\pm 1,s)=d(s)\,v(\Gamma\backslash H)=sB(\tfrac{1}{2},\tfrac{1+s}{2})(p+1)/12,$$

where $v(\Gamma\backslash H)=\pi(p+1)/3$ is a volume of a fundamental domain of Γ.

5.2 Elliptic.

Lemma 4. For an elliptic element g, we have

$$(5.3)\quad A(g,s)=B(\tfrac{1}{2},\tfrac{1+s}{2})\frac{1}{[\Gamma(g):\{\pm 1\}]}F_1(1,\tfrac{1+s}{2},\tfrac{1+s}{2},1+s;\zeta^2;\bar{\zeta}^2)\chi(g),$$

<u>where</u> F_1 <u>denotes the hypergeometric function of two invariants,</u> ζ, $\bar{\zeta}$ <u>eigenvalues of</u> g.

Let E_2, E_3 be conjugate classes of order 4 or 3, 6 in Γ. It is known that $|E_2|=2$ if $p\equiv 1 \bmod 4$, $|E_2|=1$ if $p=2$ and $|E_2|=0$ otherwise, and $|E_3|=4$ if $p\equiv 1 \bmod 3$, $|E_3|=2$ if $p=3$ and $|E_3|=0$ otherwise. Therefore the elliptic contributions $A(E_2,s)$, $A(E_3,s)$ from E_2 and E_3 are given as

(5.4) $$A(E_2,s)=\frac{1}{4}B(\tfrac{1}{2},\tfrac{1+s}{2})\alpha_p^{(2)}\cdot\beta_\chi^{(2)}$$

(5.5) $$A(E_3,s)=\frac{1}{3}B(\tfrac{1}{2},\tfrac{1+s}{2})F_1(1,\tfrac{1+s}{2},\tfrac{1+s}{2},1+s;\omega;\bar{\omega})\alpha_p^{(3)}\cdot\beta_\chi^{(3)}$$

where $\omega=(-1+\sqrt{-3})/2$,

$$\alpha_p^{(2)}=\begin{cases}2 & \text{if } p\equiv 1 \bmod 4\\ 1 & \text{if } p=2\\ 0 & \text{if } p\equiv 3 \bmod 4\end{cases},\qquad \alpha_p^{(3)}=\begin{cases}2 & \text{if } p\equiv 1 \bmod 3\\ 1 & \text{if } p=3\\ 0 & \text{if } p\equiv 2 \bmod 3\end{cases},$$

$$\beta_\chi^{(2)}=\begin{cases}1 & \text{if } \chi(g)=1\ (g\in E_2)\\ -1 & \text{if } \chi(g)=-1\end{cases},\qquad \beta_\chi^{(3)}=\begin{cases}2 & \text{if } \chi(g)=1\ (g\in E_3)\\ -1 & \text{if } \chi(g)=\omega,\bar{\omega}\end{cases}.$$

5.3 <u>Hyperbolic.</u> Γ does not have any hyperbolic elements leaving a cusp fixed. We get the following lemma in the same way as [5, § 5.2].

Lemma 5.

(5.6) $$A(g,s)=2^s\log(\lambda_0^2)\,\chi(g)\times(N(g)^{1/2}-N(g)^{-1/2})^{-1}(N(g)^{1/2}+N(g)^{-1/2})^{-s}$$

<u>where</u> $\lambda_0>1$ <u>is an eigenvalue of a generator of</u> $\Gamma(g)$.

Therefore the hyperbolic contribution becomes

(5.7) $$2^s z(s,\chi).$$

5.4 Parabolic and Eisenstein series. The divergence part of the sum of $A(g,s)$ over Γ_∞ and Γ_0 is just canceled by that of H_s. Take $Y>>0$ and $D_Y=\{z;\ 0\leqq x\leqq 1,\ 0\leqq y\leqq Y\}$.

Lemma 6.

$$(5.8)\qquad \sum_{\substack{g\in\Gamma_\kappa/\{\pm 1\}\\ g\neq\pm 1}} \int_{\sigma_\kappa D_Y} k_s(z,gz)dz=\log(Y)+\frac{\gamma}{2}-2\psi\left(\frac{s}{2}\right)+o(1)\quad (Y\to\infty)$$

where γ denotes Euler's constant and ψ the digamma function.

For simplicity, we use $h_s^*(r)$ for $h_s(\lambda)$ $(\lambda=(1/4)+r^2)$. By the same calculation as in [7], we get

Lemma 7.

$$(5.9)\qquad -\int_{\sigma_\kappa D_Y} \frac{1}{4\pi}\int_{-\infty}^{\infty} h_s^*(r)\, E_\kappa(z,t,\chi)\,\overline{E_\kappa(z,t,\chi)}\,dr\,dz$$

$$=-\log(Y)-\frac{1}{4}h_s\left(\frac{1}{4}\right)m_{\kappa\kappa}\left(\frac{1}{2}\right)$$
$$+\frac{1}{4\pi}\int_{-\infty}^{\infty} h_s^*(r)\Big\{\sum_{\mu\in\{\infty,0\}} m_{\kappa\mu}(t,\chi)\overline{m'_{\kappa\mu}(t,\chi)}\Big\}dr+o(1)$$
$$(t=(1/2)+ir,\ Y\to\infty)$$

Combining Lemma 6, 7 with Lemma 1, we obtain

(i) If χ is principal, then

$$(5.10)\qquad A(\kappa,s)=\frac{\gamma}{2}+\log(\pi)-\frac{1}{2}\psi\left(\frac{s}{2}\right)+\frac{1}{4}h_s\left(\frac{1}{4}\right)$$
$$-\frac{1}{2\pi}\int_{-\infty}^{\infty} h_s^*(r)(1-p^{-1+2ir})^{-1}dr-\frac{1}{2\pi}\int_{-\infty}^{\infty} h_s^*(r)\psi\left(\frac{1}{2}+ir\right)dr$$
$$-\frac{1}{\pi}\int_{-\infty}^{\infty} h_s^*(r)A(r)dr,$$

where $A(r)=\dfrac{\zeta'(1+2ir)}{\zeta(1+ir)}+\dfrac{1}{2ir}$.

(ii) If χ is not principal, then

$$(5.11)\qquad A(\kappa,s)=\frac{\gamma}{2}+\log(\pi)-\frac{3}{2}\log(p)-\frac{1}{2}\psi\left(\frac{s}{2}\right)$$

$$-\frac{1}{2\pi}\int_{-\infty}^{\infty} h_s^*(r)\psi(\tfrac{1}{2}+ir)dr$$

$$-\frac{1}{2\pi}\int_{-\infty}^{\infty} h_s^*(r)\{\frac{L'(1+2ir,\chi)}{L(1+2ir,\chi)}+\frac{L'(1+2ir,\bar{\chi})}{L(1+2ir,\bar{\chi})}\}dr$$

Remark 1. For Re(s)>0, let define Dirichlet series by

$$(5.12)\qquad l(s)=-\sum_{n=1}^{\infty}\Lambda(n)n^{-1}(n+n^{-1})^{-s}$$

$$l(s,\chi)=-\sum_{n=1}^{\infty}\Lambda(n)n^{-1}(n+n^{-1})^{-s}\chi(n)$$

where $\Lambda(n)=0$ when n is not a power of a prime and $\Lambda(n)=\log(q)$ when n is a power of a prime q. For Re(s)>0, the integrals above can be expressed as follows:

$$-\frac{1}{2\pi}\int_{-\infty}^{\infty} h_s^*(r)(1-p^{-1+2ir})^{-1}dr$$

$$=-\log(p)-2^s\sum_{n=1}^{\infty}\Lambda(p^n)p^{-n}(p^n+p^{-n})^{-s},$$

$$-\frac{1}{\pi}\int_{-\infty}^{\infty} h_s^*(r)A(r)dr=-\frac{1}{2}h_s(\tfrac{1}{4})-2^{s+1}l(s),$$

$$-\frac{1}{2\pi}\int_{-\infty}^{\infty} h_s^*(r)\{\frac{L'(1+2ir,\chi)}{L(1+2ir,\chi)}+\frac{L'(1+2ir,\bar{\chi})}{L(1+2ir,\bar{\chi})}\}dr$$

$$=-2^s(l(s,\chi)+l(s,\bar{\chi})).$$

5.5 Each term of $A(*,s)$ which is expressed by the beta function, the digamma function or the hypergeometric function will be continued as meromorphic in the whole s-plane. The integrands in (5.10), (5.11) are evaluated as

$$\psi(\tfrac{1}{2}+ir)=O(\log|r|),\quad A(r)=O(\log(2(1+|r|))),$$

$$\frac{L'(1+2ir,\chi)}{L(1+2ir,\chi)}=O(\log(2p(1+|r|)))$$

when $|r|$ tends to infinity, so all terms of $A(\kappa, s)$ which are expressed by the integrals are holomorphic in the half plane $Re(s)>0$. Moreover they will be also continued as meromorphic in the whole s-plane. Then $z(s,\chi)$ becomes a meromorphic function there.

The end of the sketch of proof of Theorem 1.

Remark 2. Since $A(*,s)$'s are holomorphic in the half plane $Re(s)>0$, $z(s,\chi)$ has only possible simple poles in $(0,1]$. Such a pole occurs at $s=1$ only when χ is principal. Especially, if $z(s,\chi)$ has no poles in $(0,1)$, there are no exceptional eigenfunctions in $L^2_0(\Gamma\backslash H,\chi)$.

§ 6. The dimension formula of W(1/4).

Theorem 2. <u>The dimension of W(1/4) is given by the following formula:</u>

$$\dim W(1/4)=\frac{1}{4}\operatorname*{Res}_{s=0} z(s,\chi)+\frac{1}{2}(1+\delta). \tag{6.1}$$

<u>where</u> $\delta=1$ <u>when</u> χ <u>is principal and</u> $\delta=0$ <u>when</u> χ <u>is not principal.</u>

Proof. The left hand side of (4.5) has a simple pole at 0 with residue $4\cdot\dim W(1/4)$. In its right hand side, $A(\pm 1,s)$, $A(E_i,s)$ are regular at zero, but $A(\kappa,s)$ has a simple pole there with residue $1+\delta$. Therefore we get (6.1).

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