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Lojasiewicz type inequalities and Newton diagrams

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§ 1. Introduction

Let $\mathbb{C}$ be the complex number field and f be an analytic function near the origin $0 \in \mathbb{C}^n$. Let $x_1, \ldots, x_n$ be a coordinate system of $\mathbb{C}^n$ near 0. Assume that f has an isolated singularity at 0. In other words, in some neighborhood of 0,

$$\frac{\partial f}{\partial x_1}(x) = \ldots = \frac{\partial f}{\partial x_n}(x) = 0 \quad \text{if and only if } x = 0.$$

Then there are positive numbers α, C such that the following Lojasiewicz type inequality (L_α) holds near 0.

$$(L_\alpha) \qquad |grad\ f(x)| \geq C|x|^{\alpha},$$

where $grad\ f(x) = (\frac{\partial f}{\partial x_1}(x), \ldots, \frac{\partial f}{\partial x_n}(x))$,

and $|\ \ |$ is the usual euclid norm.

This inequality has appeared as a characterization of C^0-sufficiency of jets.

Theorem (1,1) [Chang-Lu,1]

Let f be an analytic function near 0. If (L_α) holds near 0 for some $\alpha < r$, then $j^r f$ is a C^0-sufficient jet in holomorphic functions.

Originally this theorem was proved by Kuo in real case(see [2]). S.Koike pointed out me that the converse of this theorem is true (see[0]).

Set $\alpha_0(f)$ the minimal number of α such that (L_α) holds near 0. In [Lichtin, 3,4], using the Newton diagram of f, he gave an estimation of $\alpha_0(f)$ in case for n=2. But he didn't give similar analysis in case $n\geq 3$. In this note, we give an estimation of $\alpha_0(f)$ using the Newton diagram of f for general n. (Theorem (3.3)). In §5. we treat real case.

To estimate $\alpha_0(f)$, we use a *simplicial* finite subdivision of the *dual Newton diagram* $\Gamma^*(f)$ of f.(see §2., for definition) We don't use so-called *unimodular* subdivision of $\Gamma^*(f)$, which plays an important role in the theory of torus embedding. We don't need any knowledge of torus embedding in order to prove our theorem. The key step of our proof is to analyze a face of the Newton polygon of f, which is not compact, nor coordinate, i.e. which is corresponding to I_0, in our later notation.

§ 2. Newton polygon

(2.1) Let f be an analytic function near $0\in\mathbb{C}^n$, and let $\sum_{\nu} a_\nu x^\nu$ be the Taylor expansion of f at 0. Set

$\mathbb{R}_+ = \{ x \in \mathbb{R} \mid x \geq 0 \}$,

$\Gamma_+(f)$ = the convex hull of $\{ \nu + \mathbb{R}_+^n \mid a_\nu \neq 0 \}$,

$\Gamma(f)$ = union of compact faces of $\Gamma_+(f)$, and

$\Gamma^{(k)}(f) = \{$ k-dimensional face of $\Gamma(f)\}$.

We call $\Gamma_+(f)$(resp. $\Gamma(f)$) the *Newton polygon* of f (resp. the *Newton boundary* of f).

(2.2) Let $a = (a_1, \ldots, a_n) \in \mathbb{R}_+^n$ and $\alpha = (\alpha_1, \ldots, \alpha_n) \in (\mathbb{R}_+^n)^*$, where $(\mathbb{R}^n)^*$ is the dual space of $\mathbb{R}^n$. Set

$\langle a , \alpha \rangle := a_1\alpha_1 + \ldots + a_n\alpha_n$,

$\ell(\alpha) := \min \{ \langle a,\alpha\rangle \mid a \in \Gamma_+(f)\}$,

$\gamma(\alpha) := \{ a \in \Gamma_+(f) \mid \langle a,\alpha\rangle = \ell(\alpha) \}$, and

$\Gamma^*(f) := (\mathbb{R}_+{}^n)^*/\sim$,

where the equivalent relation $\alpha \sim \alpha'$ defined by $\gamma(\alpha) = \gamma(\alpha')$.

We call $\gamma(\alpha)$ the face of $\Gamma_+(f)$ *supported by* α, and $\Gamma^*(f)$ the *dual Newton diagram* of f. Naturally we can identify an equivalent class with a polyhedral cone $\sigma = \mathbb{R}_+a^1(\sigma) + \ldots + \mathbb{R}_+a^k(\sigma)$, where $a^1(\sigma)$, ..., $a^k(\sigma)$ are some integral vectors. We may assume that $b = a^j(\sigma)$ if $pb=a^j(\sigma)$ for some non-negative integer p. i.e. the greatest common divisior of components of $a^j(\sigma)$ is 1. We say that σ is a *k-simplex* if $a^1(\sigma)$, ...,$a^k(\sigma)$ are linearly independent.

(2.3) Using above identification, we can consider $\Gamma^*(f)$ as a rational polyhedral finite subdivision of the first quadrant. Let Σ be a simlicial finite subdivision of $\Gamma^*(f)$. In other word, Σ is a finite set of simpleces that gives a subdivision of $\Gamma^*(f)$. Let $\Sigma^{(k)}$ be the set of all k-simpleces of Σ. Let $\mathbb{C}^n(\sigma)$ be a copy of $\mathbb{C}^n$ for each $\sigma \in \Sigma^{(n)}$, and $y_\sigma := (y_{\sigma,1}, \ldots, y_{\sigma,n})$ be a coordinate system of $\mathbb{C}^n(\sigma)$. For a matrix $A = (a_i^j) \in \mathrm{Mat}(n,n;\mathbb{Z})$, set

$$A_y = (y_1^{a_1^1} \cdots\cdots y_n^{a_1^n}, \ \ldots, \ y_1^{a_n^1} \cdots\cdots y_n^{a_n^n}).$$

Define the mapping π_σ: $\mathbb{C}^n(\sigma) \longrightarrow \mathbb{C}^n$ by $\pi_\sigma(y_\sigma) := {}^{a(\sigma)}y_\sigma$, where $a(\sigma) = (a^1(\sigma), \ldots, a^n(\sigma))$. Set

$W_\sigma := \{ y_\sigma \in \mathbb{C}^n(\sigma) \mid |y_{\sigma,j}| \leq 1 \}$,

W := disjoint union of W_σ for $\sigma \in \Sigma^{(n)}$, and

$V := \{ x \in \mathbb{C}^n \mid |x_i| \leq 1 \}$.

Define a mapping $\pi: W \longrightarrow V$ by $\pi(y_\sigma) := \pi_\sigma(y_\sigma)$ for $y_\sigma \in W_\sigma$.

For a subset I of $\{1, \ldots, n\}$, set

$E_I = E_{\sigma,I} = \{ y_\sigma \in W_\sigma \mid y_{\sigma,i} = 0, \text{ for any } i \in I \}$, and

$E_I^* = E_{\sigma,I}^* := \{ y_\sigma \in E_{\sigma,I} \mid y_{\sigma,j} \neq 0, \text{ for any } j \in \{1,\ldots,n\}-I \}$.

(2.4) **Lemma**

1) $\pi^{-1}(0) \cap W_\sigma$ is compact.

2) π is surjective.

Proof) 1) Since $\pi_\sigma^{-1}(0)$ is a union of some coordinate spaces, 1) is obvious.

2) For any $x \in V$, set $x_i = r_i \cdot e^{2\pi\sqrt{-1}\theta_i}$, $r_i \geq 0$, $0 \leq \theta_i < 1$.

Set $y_{\sigma,j} = r_{\sigma,j} \cdot e^{2\pi\sqrt{-1}\theta_{\sigma,i}}$, $r_{\sigma,i} \geq 0$, $0 \leq \theta_{\sigma,i} < 1$.

Since $x_i = y_{\sigma,1}^{a_i^1(\sigma)} \cdots\cdots y_{\sigma,n}^{a_i^n(\sigma)}$, we obtain that

$$r_i = r_{\sigma,1}^{a_i^1(\sigma)} \cdots\cdots r_{\sigma,n}^{a_i^n(\sigma)}, \tag{2.4.1}$$

and $\theta_i \equiv a_i^1(\sigma)\theta_{\sigma,1} + \ldots + a_i^n(\sigma)\theta_{\sigma,n} \pmod{\mathbb{Z}}$. (2.4.2)

Since $a(\sigma)$ has the maximal rank, the equations (2.4.2) have a solution. We have to solve (2.4.1) for some σ under the condition $r_{\sigma,j} \leq 1$. If $r_i \neq 0$, for $i=1,\ldots,n$, then we obtain that

"(2.4.1) <=> $\log r_i = \sum_{j=1}^{n} a_i^j(\sigma) \log r_{\sigma,j}$ on $r_{\sigma,1} \ldots r_{\sigma,n} \neq 0$."

Therefore "$\left(-\log r_1, \ldots, -\log r_n\right) \in \sigma$

<=> $\left(-\log r_{\sigma,1}, \ldots, -\log r_{\sigma,n}\right) \in$ the first quadrant

<=> $r_{\sigma,j} \leq 1$ for $j = 1, \ldots, n$."

Since Σ is a subdivision of the first quadrant, there are σ and r_σ, satisfying (2.4.1). Since $(\mathbb{C}-0)^n \cap V$ is dense in V, and in view of 1), (2.4.1) have a solution with $r_{\sigma,j} \leq 1$. (q.e.d)

(2.5) Define f_γ by $\sum_{\nu\in\gamma} a_\nu x^\nu$ for $\gamma \subset (\mathbb{R}_+^n)^*$. Note that f_γ is a polynomial if γ is compact. We say that f is *non-degenerate* if the equations

$$\frac{\partial f_\gamma}{\partial x_1} = \dots = \frac{\partial f_\gamma}{\partial x_n} = 0$$

have no common solution on $x_1 \cdot \ldots \cdot x_n \neq 0$ for any compact face γ of $\Gamma_+(f)$.

§3. Result.

(3.1) Let H_γ denote the hypersurface with $\gamma \subset H_\gamma$ for $\gamma \in \Gamma^{(n-1)}(f)$. Let $m_i(\gamma)$ denote the i-coordinate of the point (i-axis) $\cap$ H_γ. Set

$\mathbf{m}(\gamma) := \max \{ m_1(\gamma), \ldots , m_n(\gamma) \}$, and

$m_0(f) := \max \{ \mathbf{m}(\gamma) \mid \gamma \in \Gamma^{(n-1)}(f) \}$.

(3.2) We consider the following condition for the Newton polygon.

(3.2.1) **Condition** $\displaystyle\bigcup_{\gamma \in \Gamma^{(n-1)}(f)} \gamma = \Gamma(f)$.

(3.3) **Theorem.**

Let f be an analytic function near 0. Assume that f has an isolated singularity at 0 and f is non-degenerate in the sense of (2.5), and $\Gamma_+(f)$ satisfies the condition (3.2.1). Then

$\alpha_0(f) \leq m_0(f) - 1$.

(3.4) **Corollary.**

Let f be as above, and let r *be the smallest integer with* $r > m_0(f)-1$. *Then $j^r f$ is a C^0-sufficient jet.*

(3.5) **Remark.** Theorem (3.3) asserts nothing new when the function f is convenient ("convenient" means that the Newton polygon $\Gamma_+(f)$ meet each coordinate axis).(See [5].) But when f is not convenient, this is a new result.

(3.6) **Example.** Set $f(x_1,x_2,x_3) = x_1^5 + x_2^5 + x_2x_3^5$.
Then $\alpha_0(f) \leq m_0(f)-1 = 25/4 - 1 = 21/4$. In this case it is easy to

show that the equal mark holds. Define g by $f + x_3{}^{100}$. Then we get $\alpha_0(g) = \alpha_0(f) = 21/4$, and $m_0(g) = 100$. So, in general, the equal mark does not always hold.

§4. Proof.

(4.1) For $a = (a_1, \ldots, a_n)$, set

$m(a) := \min\{ a_1, \ldots, a_n \}$, and

$M(a) := \max\{ a_1, \ldots, a_n \}$.

(4.2) Let Σ be a simplicial subdivision of $\Gamma^*(f)$. Set

$\Sigma_+^{(1)} = \{ a \in \Sigma^{(1)} | \ m(a) > 0 \}$.

We consider the following conditions for Σ.

Condition(4.2.1). $\Sigma^{(1)} = \{$1-simplex of $\Gamma^*(f)\}$.

Condition(4.2.2). For a subset A of $\Sigma^{(1)}$,

1) $A \cap \Sigma_+^{(1)} \neq \phi$, if $\bigcap_{a \in A} \gamma(a)$ is compact, and

2) $\ell(a) \geq M(a)$ or $\ell(a) = 0$ for any $a \in \Sigma^{(1)}$.

(4.2.3) It is easy to show that the condition (4.2.1) implies the condition (4.2.2) under the assumption (3.2.1). (See (4.7).)

(4.3)**Proposition.**

Suppose that f has an isolated singularity at 0 and is non-degenerate in the sense of (2.5), and let Σ be a simlicial subdivision of $\Gamma^*(f)$ satisfying the condition (4.2.2). Then

$\alpha_0(f) \leq \max \{\ell(a)/m(a) | a \in \Sigma_+^{(1)}\} - 1$.

(4.3.1) Since $\ell(a^j(\sigma))/m(a^j(\sigma)) = \mathbf{m}(\gamma(a^j(\sigma)))$, it is easy to show that (4.3) implies (3.3). Then, in this section, we shall prove proposition(4.3).

(4.4) Set $x = {}^{A}y$, where $A = (a_i^j)$.

$N_+ := \{ j \in \{1, \ldots, n\} \mid m(a^j) > 0 \}$,

$N_- := \{ j \in \{1, \ldots, n\} \mid \ell(a^j) = 0 \}$.

$N_0 := \{1, \ldots, n\} - N_+ - N_-$.

For a subset I of $\{ 1, \ldots, n \}$, set

$I_+ := I \cap N_+$, $I_0 := I \cap N_0$, $I_- = I \cap N_-$,

$N_I := \{i \mid a_i^j \neq 0$, there is a number $j \in I_0\}$, and

$M_I := \{i \mid a_i^j = \ell(a)$, for $j \in I_0\}$.

Set $e_i = (0, \ldots, 0, \overset{i}{1}, 0, \ldots, 0)$, and

$\gamma_I = \bigcap_{j \in I} \gamma(a^j)$.

(4.5) Define $g_k(y_\sigma)$ and $g'_k(y_\sigma)$ by

$$\left(x_k \cdot \frac{\partial f}{\partial x_k} \right)(\pi_\sigma(y_\sigma)) = \prod_{j=1}^{n} y_{\sigma,j}^{\ell(a^j(\sigma))} \cdot g_k(y_\sigma), \text{ and}$$

$$\frac{\partial f}{\partial x_k}(\pi_\sigma(y_\sigma)) = \prod_{j \in N_+} y_{\sigma,j}^{\ell(a^j(\sigma)) - a_k^j(\sigma)} \cdot g'_k(y_\sigma).$$

Then

$$g_k(y_\sigma) = \sum_{\nu} \nu_k \, a_\nu \, y_{\sigma,1}^{\langle \nu, a^1 \rangle - \ell(a^1)} \cdot \ldots \cdot y_{\sigma,n}^{\langle \nu, a^n \rangle - \ell(a^n)},$$

$$g'_k(y_\sigma) = \sum_{\nu} \nu_k \, a_\nu \prod_{j \in N_+} y_{\sigma,j}^{\langle \nu, a^j \rangle - \ell(a^j)} \cdot \prod_{j \in N_0 \cup N_-} y_{\sigma,j}^{\langle \nu, a^j \rangle - a_k^j}.$$

Note that g_k and g'_k are analytic functions.

(4.6) Since

$$\prod_{j \in N_0 \cup N_-} y_{\sigma,j}^{a_k^j(\sigma)} \cdot g'_k(y_\sigma) = \prod_{j \in N_0 \cup N_-} y_{\sigma,j}^{\ell(a^j(\sigma))} \cdot g_k(y_\sigma),$$

$$\{ g'_1 = \ldots = g'_n = 0 \} = \{ g_1 = \ldots = g_n = 0 \},$$

on $\prod_{j \in N_0 \cup N_-} y_{\sigma,j} \neq 0$.

(4.7)**Lemma.** If $\gamma(a) \cap \{\nu_1 \cdot \cdots \cdot \nu_n \neq 0\} \neq \phi$, then $\ell(a) \geq M(a)$.

Proof. It is enough to prove that

(4.7.1) if $\gamma(a) \cap \{ \nu_i \neq 0 \} \neq \phi$, then $\ell(a) \geq a_i$.

By the assumption, there is a $\nu = (\nu_1, \ldots, \nu_n) \in \gamma(a) \cap \mathbb{Z}^n$ with $\nu_i \neq 0$. Then $\ell(a) = a_1 \cdot \nu_1 + \ldots + a_n \cdot \nu_n \geq a_i \cdot \nu_i \geq a_i$.

(q.e.d.)

(4.8)**Lemma.** Suppose that f has an isolated singularity at 0. Let I be a subset of $\{1,\ldots,n\}$. Assume that γ_I is not compact. Then there are a number $i \in \{1, \ldots, n\}$ and a point $\nu \in \Gamma_+(f) \cap \mathbb{Z}^n$ such that

$\langle \nu, a^j \rangle = a^j_i$ for any $j \in I$.

Proof. Assume that any $i = 1, \ldots, n$, there is a number $j \in I$ such that $\langle \nu, a^j \rangle > a^j_i$.

Since $\frac{\partial f}{\partial x_k}(A_y) = \sum_{\nu} \nu_k\, a_\nu\, y_1^{\langle \nu, a^1 \rangle - a^1_k} \cdot \cdots \cdot y_n^{\langle \nu, a^n \rangle - a^n_k}$,

$\frac{\partial f}{\partial x_k} \circ \pi$ is identically zero on E_I. So , $\pi(E_I)$ is a singular locus of f. Because f has an isolated sigularity at 0, $\pi(E_I) = \{0\}$. Therefore $\bigcap_{i \in I} \gamma(a^i)$ is compact. **(q.e.d.)**

(4.8.1) Under the same assumption of (4.8), in view of (4.7), we get that $\ell(a^j) = M(a^j)$ for any $j \in I$.

(4.8.2) Under the same assumption of (4.8), for any $\nu \in \gamma_I$, one of the following properties hold.

(4.8.2.1) $\nu_k = 0$ for any $k \in M_I$.

(4.8.2.2) There is a unique $k \in M_I$ such that

$\nu_k = 1$ and $\nu_{k'}$: . for any $k' \in N_I - \{k\}$.

(4.8.3) Moreover assume that the condition (4.2.1). Since $\gamma_{I_0} \cap \{\nu_1 \cdot \ldots \cdot \nu_n \neq 0\} \neq \phi$, and γ_{I_0} isn't compact, the consequences of (4.8)-(4.8.2) hold for I_0.

Lemma(4.9) Assume that γ_I isn't compact, and that

$$\gamma_I \cap \{\nu_1 \cdot \cdots \cdot \nu_n \neq 0\} \neq \phi.$$

Then there is an analytic function f_i, for each $i \in N_I$, such that

$$f(x_1, \ldots, x_n) = \sum_{i \in N_I} x_i \cdot f_i(x_1, \ldots, x_n).$$

Proof. If there is a $\nu \in \Gamma_+(f)$ with $\nu_i = 0$ for any $i \in N_I$, then $\langle \nu, a^j \rangle = 0$ for any $j \in I_0$. This is a contradiction. (q.e.d.)

(4.10.1) Set $h_i = f_i | \{x_j = 0 | j \in N_I\}$. Since f has an isolated singularity at 0, then we obtain that

$$\{ h_i = 0 \mid i \in N_I \} = \{0\} \quad \text{on } \{ x_j = 0 \mid j \in N_I \}.$$

In particular, at least one of h_i isn't identically zero. Moreover if the coefficient field is $\mathbb{C}$, then we get that $\#L_I \geq n - \#N_I$, where $L_I = \{i \in N_I \mid h_i$ isn't identically zero$\}$.

(4.10.2) $N_I \supset M_I \supset L_I$.

Proof. It is clear that $N_I \supset M_I$. For any $j \in I_0$, $a^j_k = 0$ if $k \in N_I$. Suppose h_i is not identically 0, then the weighted degree of $x_i h_i$ with respect to a^j equals to a^j_i, and thus equals to $\ell(a^j)$. Therefore, $i \in M_I$. (q.e.d.)

(4.11)**Lemma**. For any $y \in E^*_I$, the following conditions are equivalent.

1) $g_k(y) = 0$.

2) $\partial f_{\gamma_I} / \partial x_k(\tilde{x}) = 0$,

where $\tilde{x} = {}^A\tilde{y}$, $\tilde{y} = (\tilde{y}_1, \ldots, \tilde{y}_n)$, $\tilde{y}_j = 1$, if $j \in I$; $\tilde{y}_j = y_j$, otherwise.

Proof. Since $g_k|E_I = \sum_{\nu\in\gamma_I} \nu_k\cdot a_\nu \prod_{j\in I} y_j^{\langle\nu,a^j\rangle-\ell(a^j)}$

$$= \sum_{\nu\in\gamma_I} \nu_k\cdot a_\nu \prod_{j=1}^{n} \tilde{y}_j^{\langle\nu,a^j\rangle-\ell(a^j)},$$

we obtain that " $g_k(y) = 0$ for $y \in E_I^*$

$$\Leftrightarrow \sum_{\nu\in\gamma_I} \nu_k\cdot a_\nu \prod_{j=1}^{n} \tilde{y}_j^{\langle\nu,a^j\rangle} = 0$$

$$\Leftrightarrow \sum_{\nu\in\gamma_I} \nu_k\cdot a_\nu\, \tilde{x}^\nu = 0$$

$$\Leftrightarrow \partial f_{\gamma_I}/\partial x_k(\tilde{x}) = 0."$$ (q.e.d.)

(4.12)Lemma. For any $y \in E_I^*$, the following conditions are equivalent.

1) $g'_k(y) = 0$.

2) $\partial f_{\Gamma_{I,k}}/\partial x_k(\tilde{x}) = 0$,

where $\Gamma_{I,k} := \{\nu \in \Gamma_+(f) \mid \langle\nu,a^j\rangle = \ell(a^j)$, for any $j\in I_+$, $\langle\nu,a^j\rangle = a_k^j$, for any $j \in I_0\cup I_-\}$.

Proof. Since $g'_k|E_I =$

$$\sum_{\nu\in\Gamma_{I,k}} \nu_k\cdot a_\nu \prod_{j\in N_+-I_+} y_j^{\langle\nu,a^j\rangle-\ell(a^j)} \prod_{j\in N_0\cup N_--I_0\cup I_-} y_j^{\langle\nu,a^j\rangle-a_k^j}$$

$$= \sum_{\nu\in\Gamma_{I,k}} \nu_k\cdot a_\nu \prod_{j\in N_+} \tilde{y}_j^{\langle\nu,a^j\rangle-\ell(a^j)} \prod_{j\in N_0\cup N_-} \tilde{y}_j^{\langle\nu,a^j\rangle-a_k^j}$$

$$= \sum_{\nu\in\Gamma_{I,k}} \nu_k\cdot a_\nu \prod_{j=1}^{n} \tilde{y}_j^{\langle\nu,a^j\rangle-\ell(a^j)}\cdot \prod_{j\in N_0\cup N_-} \tilde{y}_j^{\ell(a^j)-a_k^j},$$

we obtain that

" $g'_k(y) = 0$ for $y \in E_I^*$

$$\Leftrightarrow \sum_{\nu\in\Gamma_{I,k}} \nu_k\cdot a_\nu \prod_{j=1}^{n} \tilde{y}_j^{\langle\nu,a^j\rangle} = 0$$

$$\Leftrightarrow \partial f_{\Gamma_{I,k}}/\partial x_k(\tilde{x}) = 0."$$ (q.e.d.)

(4.13) Lemma. Assume that f has an isolated singularity at 0.

1) If $k \not\in M_I$, then $\Gamma_{I,k} = \phi$.

2) If $k \in M_I$, then $\partial f_{\Gamma_{I,k}}/\partial x_k = \partial f_{\tilde{\gamma}_I}/\partial x_k$,

where $\tilde{\gamma}_I = \gamma_{I_+ \cup I_0} \cap \delta_{S_I - N_I} \cap \gamma(a')$,

$\delta_J = \{ \nu_j = 0 \mid j \in J \}$, $J \subset \{1, \ldots, n\}$,

$S_I = \{ j \in \{1, \ldots, n\} \mid$ there is a number $i \in I_-$ with $a^i = e_j \}$, and

$a'_i = d$, if $i \in M_I$; $d-1$, if $i \in N_I - M_I$; 0, otherwise,

d = sufficiently large integer.

Proof. 1) By the definition of ℓ and $\Gamma_{I,k}$ and (4.8), 1) is obvious.

2) Set $H_k = \partial f_{\tilde{\gamma}_I}/\partial x_k$. We obtain that

$$\begin{aligned}
\partial f_{\Gamma_{I,k}}/\partial x_k &= \frac{\partial}{\partial x_k}(f_{\gamma_{I_+ \cup I_0} \cap \delta_{S_I - \{k\}} \cap \{\nu_k = 1, \text{ if } k \in S_I\}}) \\
&= \frac{\partial}{\partial x_k}(f_{\gamma_{I_+ \cup I_0} \cap \delta_{S_I - \{k\}} \cap \{\nu_k = 1\}}) \\
&= \frac{\partial}{\partial x_k}(x_k((f_k)_{\delta_{N_I}}))_{\gamma_{I_+ \cup I_0} \cap \delta_{S_I - N_I}}) \\
&= \frac{\partial}{\partial x_k}((x_k h_k)_{\gamma_{I_+ \cup I_0} \cap \delta_{S_I - N_I}}) \\
&= H_k
\end{aligned}$$

The definition of $\Gamma_{I,k}$ implies the first equality. Since $k \in M_I$, (4.8.2) implies the second one. The third one follows from (4.8.2) and (4.9). (q.e.d.)

(4.14) Lemma. Assume that f is non-degenerate in the sense of (2.5), and that f has an isolated singularity at 0, and that $I_0 \neq \phi$. Then

$$\{ \frac{\partial f_\gamma}{\partial x_k} = 0, \text{ for any } k \in M_I \} \subset \{ x_1 \cdot \cdots \cdot x_n = 0 \},$$

where $\gamma = \tilde{\gamma}_I$.

Proof. For the sake of simplicity, we assume that

$$L_I = \{1, \ldots, s\},\ N_I = \{1, \ldots, s, s+1, \ldots, c\}.$$

By (4.10.1), we get that $s \geq 1$. By (4.10.2), $L_I \subset M_I$.

Assume that there is a $(x^0_{c+1}, \ldots, x^0_n)$ such that

$x^0_{c+1} \cdot \cdots \cdot x^0_n \neq 0$, and that

$H_k(x^0_{c+1}, \ldots, x^0_n) = 0$, for any $k \in M_I$.

Note that $f_\gamma(x) = \sum_{k=1}^{s} x_k H_k(x_{c+1}, \ldots, x_n)$. By the assumption of non-degeneracy of subfaces of γ,

$$\{\sum_{j\in J} x_j \cdot \frac{\partial H_j}{\partial x_i}(x^0_{c+1}, \ldots, x^0_n) = 0 \text{ for } i = c+1, \ldots, n\} \subset \{\prod_{j\in J} x_j = 0\},$$

for any subset J of L_I.

In other words, $rank \left(\frac{\partial H_j}{\partial x_i}(x^0_{c+1}, \ldots, x^0_n)\right) = s$.

On the other hand, since $H_1, \ldots, H_s$ are weighted homogeneous polynomials for some weight $(a_{c+1}, \ldots, a_n)$,

$$\sum_{i=c+1}^{n} a_i \cdot x^0_i \cdot \frac{\partial H_j}{\partial x_i}(x^0_{c+1}, \ldots, x^0_n) = 0, \text{ for } j = 1, \ldots, s.$$

This asserts that $rank \left(\frac{\partial H_j}{\partial x_i}(x^0_{c+1}, \ldots, x^0_n)\right) < s$. This is a contradiction. (q.e.d.)

(4.14.1) As a consequence of this proof, we obtain that

$\#L_I \leq n - \#N_I$.

(4.15) In this paragraph, we assume that f has an isolated singularity and is non-degenerate. Let Σ be a simplicial subdivision of $\Gamma^*(f)$ satisfying the conditions (4.2.1) and (4.2.2).

(4.15.1)Claim. The function $\sum_{k=1}^{n} |g'_k(y)|^2$ is positive on $\pi^{-1}(0)$.

Proof. Assume that there is a point $y \in E^*_I$ such that

$\sum_{k=1}^{n} |g_k'(y)|^2 = 0$. If $\prod_{j\in N_0\cup N_-} y_j \neq 0$, then $g_k(y) = 0$, for $k=1,\dots,n$, because of (4.6). By Lemma (4.11), this contradicts non-degeneracy of f. Assume that $\prod_{j\in N_0\cup N_-} y_j = 0$. Lemma(4.12) and (4.14) assert that non-degeneracy implies positivity of $\sum_{k\in M_I} |g_k'(y)|^2$. By (4.13), $\sum_{k\in M_I} |g_k'(y)|^2 = \sum_{k=1}^{n} |g_k'(y)|^2$. So this is a cotradiction. (q.e.d.)

(4.15.2)

$$|\mathrm{grad}\ f|^2({}^{a(\sigma)}y_\sigma) = \sum_{k=1}^{n} \left|\frac{\partial f}{\partial x_k}({}^{a(\sigma)}y_\sigma)\right|^2$$
$$= \sum_{k=1}^{n} \prod_{j\in N_+} |y_{\sigma,j}|^{2(\ell(a^j(\sigma))-a_k^j(\sigma))}\cdot|g_k'(y_\sigma)|^2$$
$$\geq \prod_{j\in N_+} |y_{\sigma,j}|^{2(\ell(a^j(\sigma))-m(a^j(\sigma)))} \sum_{k=1}^{n} |g_k'(y_\sigma)|^2.$$

(4.15.3)

$$|x|^2({}^{a(\sigma)}y_\sigma) = \sum_{k=1}^{n} |y_{\sigma,1}^{a_k^1(\sigma)} \cdots\cdots y_{\sigma,n}^{a_k^n(\sigma)}|^2$$
$$= \prod_{j\in N_+} |y_{\sigma,j}|^{2m(a^j(\sigma))} \sum_{k=1}^{n} |y_{\sigma,1}^{a_k^1(\sigma)-m(a^1(\sigma))} \cdots\cdots y_{\sigma,n}^{a_k^n(\sigma)-m(a^n(\sigma))}|^2$$

Note that the condition (4.2.2.2) implies

$$\{|x|^2({}^{a(\sigma)}y)=0\} = \{y_{\sigma,j}=0 \text{ for } j \text{ with } m(a^j(\sigma))>0\}.$$

(4.15.4) By (4.15.1)-(4.15.3) and (2.4), finally we obtain that

$$\alpha_0(f) \leq \max\ \{\ell(a^j(\sigma))/m(a^j(\sigma)),\ \text{for } \sigma,\ j \text{ with } m(a^j(\sigma)) > 0\} - 1,$$

where $a(\sigma)=(a^1(\sigma),\dots,a^n(\sigma))$ for $\sigma \in \Sigma^{(n)}$.

This completes the proof of proposition (4.3).

§ 5. Real case.

In this section we treat a real function $f:(\mathbb{R}^n,0) \longrightarrow (\mathbb{R},0)$. We can define the number $\alpha_0(f)$ in the same way as the complex case. Similar charcterization of C^0-sufficiency of real jet was proved by Kuo [2].

(5.1) Definition.

Let γ be a compact face of $\Gamma_+(f)$ and I_γ be a subset of $\{1,\dots,n\}$ depending on γ. We call

$N := \{(\gamma,I_\gamma) \mid \gamma\text{: a compact face of } \Gamma_+(f)\}$

a *Newton data* of f if the following properties (5.1.1) and (5.1.2) are satisfied.

(5.1.1) $\{\partial f_\gamma/\partial x_i = 0 \mid i\in I_\gamma\} \subset \{x_1\cdot\dots\cdot x_n=0\}$.

(5.1.2) $I_\tau \subset I_\gamma$ if $\tau<\gamma$.

(5.1.3) Note that the real analogue of (2.5) implies the existence of a Newton data.

(5.2) Example. $f(x_1,x_2) = x_1{}^3 + x_2{}^{2k}x_1$ $(k\geq 1)$.

Set $\gamma_1=\gamma(e_1)$, $\gamma_2=\gamma(e_1+2e_2)$, $\gamma_3=\gamma(e_2)$, and $\gamma_{ij}=\gamma_i\cap\gamma_j$. Then $N = \{(\gamma_2,\{1\}),(\gamma_{12},\{1\}),(\gamma_{23},\{1\})\}$ is a Newton data of f.

(5.3) Theorem.

Suppose that f has an isolated singularity at 0 and a Newton data N. And suppose that $\Gamma_+(f)$ satisfies the condition (3.2.1). Then

$\alpha_0(f) \leq m(N)$,

where $m(N) = \max\{(\ell(a)-a_i)/m(a) \mid a\text{:1-simplex of } \Gamma_+(f) \text{ with } m(a)>0,\ i\in I_{\gamma(a)}\}$.

This theorem follows immediately from the following

(5.4) Proposition.

Suppose that f has an isolated singularity at 0 and a Newton data N. Let Σ be a simplicial finite subdivision of $\Gamma^*(f)$ satisfying the condition (4.2.2). Then

$$\alpha_0(f) \le m(N,\Sigma) := \max\{(\ell(a)-a_i)/m(a) \mid a\in\Sigma_+^{(1)},\ i\in I_{\gamma(a)}\}.$$

Proof. The proof is almost similar to complex case. **But we have to** modify the construction of a simplicial subdivision of $\Gamma^*(f)$.

(5.5) Notation.

$W_{\mathbb{R}} = W \cap \mathbb{R}^n$, $W_{\sigma,\mathbb{R}} = W_\sigma \cap \mathbb{R}^n$, $V_{\mathbb{R}} = V \cap \mathbb{R}^n$, $\pi_{\mathbb{R}}=\pi|W_{\mathbb{R}}$, **and so on.**

It is easy to show the following two lemmata.

(5.6) Lemma.

Let Σ be a simplicial finite subdivision of $\Gamma^*(f)$ satisfying

(5.6.1) For any $i\in\{1,\dots,n\}$, there is a number $j\in\{1,\dots,n\}$ such that $a_i^j(\sigma)$ is odd, for any $\sigma \in \Sigma^{(n)}$. Then $\pi_{\mathbb{R}}: W_{\mathbb{R}} \longrightarrow V_{\mathbb{R}}$ is a surjection.

(5.7) Lemma. For $a_i\ge0$, $b_i>0$, $c_i\ge0$, such that one of c_i is positive, the following inequalities hold.

$$\max\{a_i/b_i \mid i=1,\dots,n\} \ge (\sum c_i a_i)/(\sum c_i b_i) \ge \min\{a_i/b_i \mid i=1,\dots,n\}.$$

(5.8) For any $\sigma \in \Sigma^{(n)}$, define $k(\sigma)$ and $p(\sigma)$ by

$n - k(\sigma) = \#\{a^1(\sigma),\dots,a^n(\sigma)\}\cap\{e_1,\dots,e_n\}$, and

$p(\sigma) = \#\{j \mid a_j^i(\sigma) \in 2\mathbb{Z}, \text{ for any } i\}$.

After suitable renumbering, we may assume that

$a^i(\sigma) = e_i$, $i=k(\sigma)+1,\dots,n$,

$\ell(a^i(\sigma)) > 0$, $i=1,\dots,k(\sigma)$,

$a_j^i(\sigma) \in 2\mathbb{Z}$ for $i=1,\dots,n$; $j=1,\dots,p(\sigma)\le k(\sigma)$, and

there is a number i such that $a_j^i(\sigma)$ is odd for any $j=p(\sigma)+1,\dots,k(\sigma)$.

(5.9) Choose $a_\sigma \in \{a^1(\sigma),\dots,a^n(\sigma)\}$ such that $m(a_\sigma) > 0$, for σ with $k(\sigma)>0$. Choose $b_\sigma \in ((2\mathbb{Z}+1)^{p(\sigma)}\times\mathbb{Z}^{n-p(\sigma)}) \cap \sigma$, for σ with $k(\sigma)>0$. Then there is a simplicial finite subdivision Σ_μ ($\mu=1,2,\dots$) of Σ

satisfying

$\Sigma_\mu^{(1)} = \Sigma^{(1)} \cup \{b_\sigma \mid p(\sigma)>0$, and $k(\sigma)=0\} \cup \{(\mu-1)a_\sigma+b_\sigma \mid p(\sigma)>0$, and $k(\sigma)>0\}$.

Since Σ_μ satisfies (5.6.1), the mapping corresponding to Σ_μ is surjective. Then we obtain

$$\alpha_0(f) \leq \inf\ \{m(N,\Sigma_\mu) \mid \mu=1,2,\ldots.\}.$$

By (5.7) and the construction of Σ_μ, it is easy to show that

$$\inf\ \{m(N,\Sigma_\mu) \mid \mu=1,2,\ldots\} = m(N).$$

Note that

$$\begin{aligned} |\mathrm{grad}\ f|^2(a(\sigma)y_\sigma) &= \sum_{k=1}^{n} \left|\frac{\partial f}{\partial x_k}(a(\sigma)y_\sigma)\right|^2 \\ &\geq \sum_{k\in I_\sigma} \left|\frac{\partial f}{\partial x_k}(a(\sigma)y_\sigma)\right|^2 \\ &= \sum_{k\in I_\sigma} \prod_{j\in N_+} |y_{\sigma,j}|^{2(\ell(a^j(\sigma))-a_k^j(\sigma))} \cdot |g_k'(y_\sigma)|^2 \\ &\geq \prod_{j\in N_+} |y_{\sigma,j}|^{2(\ell(a^j(\sigma))-m_\sigma(a^j(\sigma)))} \sum_{k\in I_\sigma} |g_k'(y_\sigma)|^2, \end{aligned}$$

where $m_\sigma(a^j(\sigma)) = \min\ \{a_i^j(\sigma) \mid i\in I_\sigma\}$.

Comparing it with (4.15.3), we obtain proposition(5.4).

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