Title

Rings with only finitely many isomorphism classes of indecomposable maximal Buchsbaum modules

Author(s)

Nishida, Koji

Citation

数理解析研究所講究録 1987, 621: 125-135

Issue Date

1987-04

URL

http://hdl.handle.net/2433/99891

Type

Departmental Bulletin Paper

Textversion

publisher

Kyoto University
Rings with only finitely many isomorphism classes
of indecomposable maximal Buchsbaum modules

Koji Nishida (Chiba Univ.)

1. Introduction.

Throughout this report $R$ is a ring of the form

$$k[x_1, \ldots, x_n] / I,$$

where $k$ is an algebraically closed field of characteristic different from 2 and $I$ is an ideal of $k[x_1, \ldots, x_n]$. We denote by $m$ (resp. $d$) the maximal ideal of $R$ (resp. the dimension of $R$). The Jacobson radical of a (non-commutative) ring $A$ is denoted by $J(A)$.

The purpose of this report is to give a sketch of proof of the following result which is a joint work [16] with S. Goto.

Theorem 1. If $d \geq 2$, then the following two conditions are equivalent.

(1) $R$ is a regular local ring.

(2) $R$ possesses only finitely many isomorphism classes of indecomposable maximal Buchsbaum modules. (See [6] for the notion of maximal Buchsbaum module.)
When this is the case, the syzygy modules of the residue class field \( k \) of \( R \) are the representatives of indecomposable maximal Buchsbaum modules and so there are exactly \( d \) non-isomorphic indecomposable maximal Buchsbaum modules over \( R \).

Our contribution in the above theorem is the implication \((2) \Rightarrow (1)\).
The last assertion and the implication \((1) \Rightarrow (2)\) are due to [6]. We actually construct infinitely many non-isomorphic indecomposable maximal Buchsbaum \( R \)-modules when \( R \) is not a regular local ring.

We would like to note here that the assumption \( d \geq 2 \) in Theorem 1 is not superfluous. There actually exist non-regular Cohen-Macaulay local rings \( R \) of \( \dim R = 1 \) that possesses only finitely many isomorphism classes of indecomposable maximal Buchsbaum modules. The typical example is the ring

\[
R = k \left[ \begin{array}{c} X, Y \end{array} \right] /(x^3 + y^2)
\]

\((k, \text{any field}),\) which has exactly 5 indecomposable maximal Buchsbaum modules ([16, Theorem (5.3)]). So the result of one-dimensional case seems more complicated.
2. Key lemma.

The following lemma plays an important role in the proof of Theorem 1.

Lemma 2. Let $L$ be an indecomposable maximal Cohen-Macaulay (abbr. MCM) $R$-module and let $J = J(\text{End}_R L)$. If $d \geq 2$ and if one of the following conditions holds:

(a) $\dim_k L/JL \geq 2$

(b) $\dim_k JL/(J^2L + mL) \geq 2$

is satisfied, then $R$ has a family $\{ M_\lambda \}_{\lambda \in k}$ of indecomposable maximal Buchsbaum modules such that $M_\lambda \not\cong M_\mu$ for $\lambda \neq \mu$.

Sketch of proof. Choose elements $f$ and $g$ of $L$ (resp. $JL$), when the condition (a) (resp. (b)) is satisfied, so that the classes $[f]$ and $[g]$ of $f$ and $g$ in $L/JL$ (resp. $JL/(J^2L + mL)$) are linearly independent over $k$. For each $\lambda \in k$, we put $h_\lambda = f + \lambda g$ and define

$$M_\lambda = JL + Rh_\lambda \quad (\text{resp. } M = J^2L + mL + Rh_\lambda).$$

Then $\{ M_\lambda \}_{\lambda \in k}$ meets the needs of this lemma.

Proposition 3. If $R$ satisfies the condition (2) of Theorem 1, then $R$ is a simple hypersurface.
Proof. Let $K_R$ be the canonical module of $R$. $K_R$ is an indecomposable MCM $R$-module. If $R$ were not a Gorenstein ring, then by Lemma 2 we can construct from $L = K_R$ infinitely many non-isomorphic indecomposable maximal Buchsbaum $R$-modules, because $\text{End}_R K_R = R$ and because $\dim_k K_R / mK_R \geq 2$ by [7, Satz 6.10]. Hence $R$ must be Gorenstein. Since $R$ is finite CM-representation type, by [8, Satz 1.2 1.2] and [3, Theorem A] $R$ is a simple hypersurface.

Proposition 4. If $R$ is a normal ring of $\dim R = 2$ and if $R$ satisfies the condition (2) of Theorem 1, then $R$ is a UFD.

Proof. Assume that $R$ is not a UFD and take a non-principal prime ideal $\mathfrak{q}$ of $R$ so that $\dim R_\mathfrak{q} = 1$. Then $\mathfrak{q}$ is an indecomposable MCM $R$-module and $\text{End}_R \mathfrak{q} = R$, $\dim_k \mathfrak{q} / m\mathfrak{q} \geq 2$. By Lemma 2 we can construct from $L = \mathfrak{q}$ infinitely many non-isomorphic indecomposable maximal Buchsbaum modules — this is a contradiction.

The rest of this report is devoted to show the following proposition briefly.

Proposition 5. If $R$ is a simple hypersurface of $\dim R \geq 2$, then $R$ possesses infinitely many non-isomorphic indecomposable
maximal Buchsbaum modules.

From Proposition 3 and Proposition 5 we get the implication

\[(2) \implies (1)\] of Theorem 1.

3. The case where \(d \geq 3\).

It is well known that a \(d\)-dimensional simple hypersurface is isomorphic to a singularity of form

\[
k \left[ x, y, z_1, \ldots, z_{d-1} \right] / (f(x, y) + z_1^2 + \ldots + z_{d-1}^2),
\]

where \(f(x, y)\) is one of the following ([9]):

(A\(_n\)) \(x^n + y^{n+1}\) \(\quad (n \geq 1)\)

(D\(_n\)) \(x^{n-1} + xy^2\) \(\quad (n \geq 4)\)

(E\(_6\)) \(x^3 + y^4\) \(\quad (\text{ch } k \neq 3)\)
\[x^3 + y^4, \quad x^3 + x^2y^2 + y^4 \quad (\text{ch } k = 3)\]

(E\(_7\)) \(x^3 + xy^3\) \(\quad (\text{ch } k \neq 3)\)
\[x^3 + xy^3, \quad x^3 + x^2y^2 + xy^3 \quad (\text{ch } k = 3)\]

(E\(_8\)) \(x^3 + y^5\) \(\quad (\text{ch } k \neq 3, 5)\)
\[x^3 + y^5, \quad x^3 + x^2y^3 + y^5, \quad x^3 + x^2y^2 + y^5 \quad (\text{ch } k = 3)\]
\[x^3 + y^5, \quad x^3 + xy^4 + y^5 \quad (\text{ch } k = 5).\]

In the case where \(d \geq 3\),

\[
R_0 = k \left[ x, y, z_1, \ldots, z_{d-3} \right] / (f(x, y) + z_1^2 + \ldots + z_{d-3}^2)
\]
is also a simple hypersurface. Hence there exists an indecomposable MCM $R_0$-module $M$ which is not free. By Knörrer's Periodicity Theorem ([10, Theorem 3.1]) we can take an indecomposable MCM $R$-module $L$ so that $L/(z_{d-1}, z_{d-2})L = M \otimes N$, where $z_{d-1}$ (resp. $z_{d-2}$) is the class of $Z_{d-1}$ (resp. $Z_{d-2}$) in $R$ and $N$ is the first syzygy module of $M$. Let

$$\varepsilon: L \longrightarrow M \otimes N$$

be the canonical epimorphism. Then we can prove that

$$\varepsilon(JL) \subseteq J_1M \otimes J_2N,$$

where $J$, $J_1$ and $J_2$ denote $J(\text{End}_R L)$, $J(\text{End}_R M)$ and $J(\text{End}_R N)$ respectively. So $\varepsilon$ induces the epimorphism

$$\bar{\varepsilon}: L/JL \longrightarrow M/J_1M \otimes N/J_2N,$$

and we get $\dim_k L/JL \geq 2$. Hence Proposition 5 is deduced from Lemma 2 in the case where $d \geq 3$.

4. The case where $d = 2$.

Let $R$ be a 2-dimensional simple hypersurface which satisfies the condition (2) of Theorem 1. By Proposition 4 $R$ must be of type
$E_8$. Hence $R$ is a ring of the form

$$k[X, Y, Z] / (X^3 + Y^2G + Y^5 + Z^2),$$

where $G$ is either 0 or one of the following

$$X^2Y, \quad x^2 \quad (\text{ch } k = 3),$$

$$XY^2, \quad (\text{ch } k = 5).$$

Let $x, y, z$ and $g$ respectively denote the class of $X, Y, Z$ and $G$ in $R$. Then the maximal ideal $\mathfrak{m}$ of $R$ is $(x, y, z)R$.

Let $L'$ denote the second syzygy module of $R/\mathfrak{m}$:

$$0 \longrightarrow L' \longrightarrow R^3 \longrightarrow R/\mathfrak{m} \longrightarrow 0$$

Then $L'$ is a MCM $R$-module of rank 2 and is generated by

$$\begin{pmatrix} x^2 \\ y^4 + yg \\ z \end{pmatrix}, \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix}, \begin{pmatrix} -z \\ 0 \\ x \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ -z \\ y \end{pmatrix}.$$

Let $\phi: R^3 \longrightarrow R^2$ be the homomorphism defined by $\phi(\begin{pmatrix} a \\ b \\ c \end{pmatrix}) = \begin{pmatrix} a \\ c \end{pmatrix}$, and put $L = \phi(L')$. Then $L$ is also a MCM $R$-module of rank 2 and is generated by

$$f_1 = \begin{pmatrix} x^2 \\ z \end{pmatrix}, \quad f_2 = \begin{pmatrix} -y \\ 0 \end{pmatrix}, \quad f_3 = \begin{pmatrix} -z \\ x \end{pmatrix} \quad \text{and} \quad f_4 = \begin{pmatrix} 0 \\ y \end{pmatrix}.$$

We can see that $L$ is indecomposable and $\dim_k L/JL = 1$, where

$J = J(\text{End}_R L)$. 

-7-
To show that $\dim_k JL/(J^2L + mL) \geq 2$ we consider the ring $T = \mathbb{R}/yR \cong \mathcal{O}_{\mathbb{C}}/(x^3 + z^2)$. Let $\bar{T}$ denote the normalization of $T$ and put $t = -z/x$. Then
\[
\bar{T} = k[t] \cong k[x^2, z^2], \quad x = -t^2, \quad z = t^3.
\]
Let $\bar{L} = L/yL$ and recall that any indecomposable maximal Cohen-Macaulay $T$-module is isomorphic to $T$ or $\bar{T}$ ([8, Satz 1.6]). Then we have that
\[
\bar{L} \cong \bar{T} \otimes T,
\]
as $\text{rank}_T \bar{L} = 2$ and as $\bar{L}$ is minimally generated by the four elements $\{ \bar{F}_i \}_{1 \leq i \leq 4}$ (here $\bar{T}$ denotes the reduction mod $yL$). It is easily checked that $\bar{F}_2$ and $\bar{F}_3$ form a $\bar{T}$-free basis of $\bar{L}$.

Since $\text{End}_T \bar{L} = \text{End}_{\bar{T}} \bar{L}$, we shall identify $\text{End}_T \bar{L}$ with $C = M_2(\bar{T})$ (the matrix algebra) via the $\bar{T}$-free basis $\bar{F}_2$ and $\bar{F}_3$.

Let $A = \text{End}_R L$ and put $\bar{A} = A/yA$. Then $\bar{A}$ may be canonically considered to be a subalgebra of $\text{End}_T \bar{L}$ and we have a homomorphism
\[
\psi : A \rightarrow \text{End}_T \bar{L} = C
\]
of $R$-algebras. Thus via $\psi$ we may write each element of $A$ as a $2 \times 2$ matrix with coefficients in $\bar{T}$. Then we have the following
Fact. Let \( \xi \in J^2 \) and write \( \Psi(\xi) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \).

Then \( c \in t \bar{T}, \ a, \ d \in t^2 \bar{T}, \) and \( b \in t^3 \bar{T} \).

Let \( \mathcal{E} : L \rightarrow \bar{L} \) denote the canonical epimorphism. By the above Fact we can see that \( \mathcal{E}(J^2L + mL) \subset W \), where \( W = t^2 \bar{T}F_2 + t \bar{T}F_3 \).

So \( \mathcal{E} \) induces the epimorphism

\[
\bar{\mathcal{E}} : L/(J^2L + mL) \rightarrow \bar{L}/W \cong \bar{T}/t^2 \bar{T} \cdot \bar{T}/t \bar{T}.
\]

Hence \( \dim_k L/(J^2L + mL) \geq 3 \), by which we have

\[
\dim_k JL/(J^2L + mL) \geq 2
\]

since \( \dim_k L/JL = 1 \). This completes the proof of Theorem 1.
References


