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Rings with only finitely many isomorphism classes of indecomposable maximal Buchsbaum modules

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1. Introduction.

Throught this report R is a ring of the form

$$k[[X_1, \dots, X_n]] / I ,$$

where k is an algebraically closed field of ch $k \neq 2$ and I is an ideal of $k[[X_1, \dots, X_n]]$. We denote by $\underline{m}$ (resp. d) the maximal ideal of R (resp. the dimension of R). The Jacobson radical of a (non-commutative) ring A is denoted by J(A).

The purpose of this report is to give a sketch of proof of the following result which is a joint work [16] with S. Goto.

Theorem 1. If $d \geqq 2$, then the following two conditions are equivalent.

(1) R is a regular local ring.

(2) R possesses only finitely many isomorphism classes of indicomposable maximal Buchsbaum modules. (See [6] for the notion of maximal Buchsbaum module.)

When this is the case, the syzygy modules of the residue class field k of R are the representatives of indecomposable maximal Buchsbaum modules and so there are exactly d non-isomorphic indecomposable maximal Buchsbaum modules over R.

Our contribution in the above theorem is the implication (2) $\Rightarrow$ (1). The last assertion and the implication (1) $\Rightarrow$ (2) are due to [6]. We actually construct infinitely many non-isomorphic indecomposable maximal Buchsbaum R-modules when R is not a regular local ring.

We would like to note here that the assumption $d \geqq 2$ in Theorem 1 is not superfluous. There actually exist non-regular Cohen-Macaulay local rings R of $\dim R = 1$ that possesses only finitely many isomorphism classes of indecomposable maximal Buchsbaum modules. The typical example is the ring

$$R = k[[X, Y]]/(X^3 + Y^2)$$

(k, any field), which has exactly 5 indecomposable maximal Buchsbaum modules ([16, Theorem (5.3)]). So the result of one-dimensional case seems more complicated.

2. Key lemma.

The follwing lemma plays an important role in the proof of Theorem 1.

Lemma 2. Let L be an indecomposable maximal Cohen-Macaulay (abbr. MCM) R-module and let $J = J(\mathrm{End}_R L)$. If $d \geqq 2$ and if one of the following conditions

(a) $\dim_k L/JL \geqq 2$

(b) $\dim_k JL/(J^2L + \underline{m}L) \geqq 2$

is satisfied, then R has a family $\{M\}_{\lambda \in k}$ of indecomposable maximal Buchsbaum modules such that $M_\lambda \not\cong M_\mu$ for $\lambda \neq \mu$.

Sketch of proof. Choose elements f and g of L (resp. JL), when the condition (a) (resp. (b)) is satisfied, so that the classes $\bar{f}$ and $\bar{g}$ of f and g in L/JL (resp. $JL/(J^2L + \underline{m}L)$) are linearly independent over k . For each $\lambda \in k$, we put $h_\lambda = f + \lambda g$ and define

$$M_\lambda = JL + Rh_\lambda \quad (\text{resp. } M = J^2L + \underline{m}L + Rh_\lambda).$$

Then $\{M\}_{\lambda \in k}$ meets the needs of this lemma.

Proposition 3. If R satisfies the condition (2) of Theorem 1, then R is a simple hypersurface.

Proof. Let K_R be the canonical module of R. K_R is an indecomposable MCM R-module. If R were not a Gorenstein ring, then by Lemma 2 we can construct from $L = K_R$ infinitely many non-isomorphic indecomposable maximal Buchsbaum R-modules, because $\mathrm{End}_R K_R = R$ and because $\dim_k K_R/\underline{m}K_R \geqq 2$ by [7, Satz 6.10]. Hence R must be Gorenstein. Since R is finite CM-representation type, by [8, Satz 1.2 1.2] and [3, Theorem A] R is a simple hypersurface

Proposition 4. If R is a normal ring of $\dim R = 2$ and if R satisfies the condition (2) of Theorem 1, then R is a UFD.

Proof. Assume that R is not a UFD and take a non-principal prime ideal $\mathfrak{p}$ of R so that $\dim R_{\mathfrak{p}} = 1$. Then $\mathfrak{p}$ is an indecomposable MCM R-module and $\mathrm{End}_R \mathfrak{p} = R$, $\dim_k \mathfrak{p}/\underline{m}\mathfrak{p} \geqq 2$. By Lemma 2 we can costruct from $L = \mathfrak{p}$ infinitely many non-isomorphic indecomposable maximal Buchsbaum modules —— this is a contradiction.

The rest of this report is devoted to show the following proposition briefly.

Proposition 5. If R is a simple hypersurface of $\dim R \geqq 2$, then R possesses infinitely many non-isomorphic indecomposable maximal Buchsbaum modules.

From Proposition 3 and Proposition 5 we get the implication $(2) \Longrightarrow (1)$ of Theorem 1.

3. The case where $d \geqq 3$.

It is well known that a d-dimensional simple hypersurface is isomorphic to a singularity of form

$$k[[X, Y, Z_1, \ldots, Z_{d-1}]]/(f(X,Y) + Z_1^2 + \ldots + Z_{d-1}^2),$$

where $f(X,Y)$ is one of the following ([9]) :

(A_n) $X^2 + Y^{n+1}$ $(n \geqq 1)$

(D_n) $X^{n-1} + XY^2$ $(n \geqq 4)$

(E_6) $X^3 + Y^4$ $(\mathrm{ch}\ k \neq 3)$
$X^3 + Y^4$, $X^3 + X^2Y^2 + Y^4$ $(\mathrm{ch}\ k = 3)$

(E_7) $X^3 + XY^3$ $(\mathrm{ch}\ k \neq 3)$
$X^3 + XY^3$, $X^3 + X^2Y^2 + XY^3$ $(\mathrm{ch}\ k = 3)$

(E_8) $X^3 + Y^5$ $(\mathrm{ch}\ k \neq 3, 5)$
$X^3 + Y^5$, $X^3 + X^2Y^3 + Y^5$, $X^3 + X^2Y^2 + Y^5$ $(\mathrm{ch}\ k = 3)$
$X^3 + Y^5$, $X^3 + XY^4 + Y^5$ $(\mathrm{ch}\ k = 5)$.

In the case where $d \geqq 3$,

$$R_0 = k[[X, Y, Z_1, \ldots, Z_{d-3}]]/(f(X,Y) + Z_1^2 + \ldots + Z_{d-3}^2)$$

is also a simple hypersurface. Hence there exists an indecomposable MCM R_0-module M which is not free. By Knörrer's Periodicity Theorem ([10, Theorem 3.1]) we can take an indecomposable MCM R-module L so that $L/(z_{d-1}, z_{d-2})L = M \oplus N$, where z_{d-1} (resp. z_{d-2}) is the class of Z_{d-1} (resp. Z_{d-2}) in R and N is the first syzygy module of M. Let

$$\varepsilon : L \longrightarrow M \oplus N$$

be the canonical epimorphism. Then we can prove that

$$\varepsilon'(JL) \subset J_1M \oplus J_2N ,$$

where J, J_1 and J_2 denote $J(\mathrm{End}_R L)$, $J(\mathrm{End}_R M)$ and $J(\mathrm{End}_R N)$ respectively. So ε induces the epimorphism

$$\bar{\varepsilon} : L/JL \longrightarrow M/J_1M \oplus N/J_2N ,$$

and we get $\dim_k L/JL \geqq 2$. Hence Proposition 5 is deduced from Lemma 2 in the case where $d \geqq 3$.

4. The case where $d = 2$.

Let R be a 2-dimensional simple hypersurface which satisfies the condition (2) of Theorem 1. By Proposition 4 R must be of type

E_8 . Hence R is a ring of the form

$$k[[X,Y,Z]]/(X^3 + Y^2G + Y^5 + Z^2) ,$$

Where G is either 0 or one of the following

$$X^2Y , \quad X^2 \qquad (\text{ch } k = 3) ,$$
$$XY^2 \qquad (\text{ch } k = 5) .$$

Let x , y , z and g respectively denote the class of X , Y , Z and G in R . Then the maximal ideal $\underline{m}$ of R is $(x, y, z)R$.

Let L' denote the second syzygy module of $R/\underline{m}$:

$$0 \longrightarrow L' \longrightarrow R^3 \xrightarrow[(x\ y\ z)]{} R^2 \longrightarrow R/\underline{m} \longrightarrow 0$$

Then L' is a MCM R-module of rank 2 and is generated by

$$\begin{pmatrix} x^2 \\ y^4 + yg \\ z \end{pmatrix} , \quad \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix} , \quad \begin{pmatrix} -z \\ 0 \\ x \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ -z \\ y \end{pmatrix} .$$

Let $\phi : R^3 \longrightarrow R^2$ be the homomorphism defined by $\phi\left(\begin{pmatrix} a \\ b \\ c \end{pmatrix}\right) = \begin{pmatrix} a \\ c \end{pmatrix}$, and put $L = \phi(L')$. Then L is also a MCM R-module of rank 2 and is generated by

$$f_1 = \begin{pmatrix} x^2 \\ z \end{pmatrix} , \quad f_2 = \begin{pmatrix} -y \\ 0 \end{pmatrix} , \quad f_3 = \begin{pmatrix} -z \\ x \end{pmatrix} \quad \text{and} \quad f_4 = \begin{pmatrix} 0 \\ y \end{pmatrix} .$$

We can see that L is indecomposable and $\dim_k L/JL = 1$, where $J = J(\mathrm{End}_R L)$.

To show that $\dim_k JL/(J^2L + \underline{m}L) \geqq 2$ we consider the ring $T = R/yR$ ($= k[[X,Z]]/(X^3 + Z^2)$). Let $\bar{T}$ denote the normalization of T and put $t = -z/x$. Then

$$\bar{T} = k[[t]] , \quad x = -t^2 , \quad \text{and} \quad z = t^3 .$$

Let $\bar{L} = L/yL$ and recall that any indecomposable maximal Cohen-Macaulay T-module is isomorphic to T or $\bar{T}$ ([8, Satz 1.6]). Then we have that

$$\bar{L} \cong \bar{T} \oplus \bar{T} ,$$

as $\mathrm{rank}_T \bar{L} = 2$ and as $\bar{L}$ is minimally generated by the four elements $\{\bar{f}_i\}_{1 \leqq i \leqq 4}$ (here $\bar{\cdot}$ denotes the reduction mod yL). It is easily checked that $\bar{f}_2$ and $\bar{f}_3$ form a $\bar{T}$-free basis of $\bar{L}$.

Since $\mathrm{End}_T \bar{L} = \mathrm{End}_{\bar{T}} \bar{L}$, we shall identify $\mathrm{End}_T \bar{L}$ with $C = M_2(\bar{T})$ (the matrix algebra) via the $\bar{T}$-free basis $\bar{f}_2$ and $\bar{f}_3$. Let $A = \mathrm{End}_R L$ and put $\bar{A} = A/yA$. Then $\bar{A}$ may be canonically considered to be a subalgebra of $\mathrm{End}_T \bar{L}$ and we have a homomorphism

$$\psi : A \longrightarrow \mathrm{End}_T \bar{L} = C$$

of R-algebras. Thus via ψ we may write each element of A as a 2×2 matrix with coefficients in $\bar{T}$. Then we have the following

Fact. Let $\xi \in J^2$ and write $\psi(\xi) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.
Then $c \in t\bar{T}$, a , $d \in t^2\bar{T}$, and $b \in t^3\bar{T}$.

Let $\varepsilon : L \longrightarrow \bar{L}$ denote the canonical epimorphism. By the above Fact we can see that $\varepsilon(J^2L + \underline{m}L) \subset W$, where $W = t^2\bar{T}\bar{f}_2 + t\bar{T}\bar{f}_3$.
So ε induces the epimorphism

$$\bar{\varepsilon} : L/(J^2L + \underline{m}L) \longrightarrow \bar{L}/W \cong \bar{T}/t^2\bar{T} \oplus \bar{T}/t\bar{T} .$$

Hence $\dim_k L/(J^2L + \underline{m}L) \geqq 3$, by which we have

$$\dim_k JL/(J^2L + \underline{m}L) \geqq 2$$

since $\dim_k L/JL = 1$. This completes the proof of Theorem 1.

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