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<tr>
<td>著者</td>
<td>Nishida, Koji</td>
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<tr>
<td>引用</td>
<td>数理解析研究所講究録 (1987), 621: 125-135</td>
</tr>
<tr>
<td>発行日</td>
<td>1987-04</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/99891">http://hdl.handle.net/2433/99891</a></td>
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<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
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Kyoto University
Rings with only finitely many isomorphism classes
of indecomposable maximal Buchsbaum modules

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1. Introduction.

Throughout this report $R$ is a ring of the form

$$k \llbracket X_1, \ldots, X_n \rrbracket / I,$$

where $k$ is an algebraically closed field of characteristic $k \neq 2$ and $I$ is

an ideal of $k \llbracket X_1, \ldots, X_n \rrbracket$. We denote by $m$ (resp. $d$) the

maximal ideal of $R$ (resp. the dimension of $R$). The Jacobson

radical of a (non-commutative) ring $A$ is denoted by $J(A)$.

The purpose of this report is to give a sketch of proof of the

following result which is a joint work [16] with S. Goto.

Theorem 1. If $d \geq 2$, then the following two conditions

are equivalent.

(1) $R$ is a regular local ring.

(2) $R$ possesses only finitely many isomorphism classes of indicom-

posable maximal Buchsbaum modules. (See [6] for the notion of maximal

Buchsbaum module.)
When this is the case, the syzygy modules of the residue class field \( k \) of \( R \) are the representatives of indecomposable maximal Buchsbaum modules and so there are exactly \( d \) non-isomorphic indecomposable maximal Buchsbaum modules over \( R \).

Our contribution in the above theorem is the implication \((2) \Rightarrow (1)\).

The last assertion and the implication \((1) \Rightarrow (2)\) are due to [6]. We actually construct infinitely many non-isomorphic indecomposable maximal Buchsbaum \( R \)-modules when \( R \) is not a regular local ring.

We would like to note here that the assumption \( d \geq 2 \) in Theorem 1 is not superfluous. There actually exist non-regular Cohen-Macaulay local rings \( R \) of \( \dim R = 1 \) that possesses only finitely many isomorphism classes of indecomposable maximal Buchsbaum modules.

The typical example is the ring

\[
R = k [\prod_{i=1}^{2} X_i, Y]/(x^3 + y^2)
\]

\((k, \text{any field})\), which has exactly 5 indecomposable maximal Buchsbaum modules ([16, Theorem (5.3)]). So the result of one-dimensional case seems more complicated.
2. Key lemma.

The following lemma plays an important role in the proof of Theorem 1.

Lemma 2. Let \( L \) be an indecomposable maximal Cohen-Macaulay (abbr. MCM) \( R \)-module and let \( J = J(\text{End}_R L) \). If \( d \geq 2 \) and if one of the following conditions

(a) \( \dim_k L/JL \geq 2 \)

(b) \( \dim_k JL/(J^2L + mL) \geq 2 \)

is satisfied, then \( R \) has a family \( \{ M_\lambda \}_{\lambda \in k} \) of indecomposable maximal Buchsbaum modules such that \( M_\lambda \neq M_\mu \) for \( \lambda \neq \mu \).

Sketch of proof. Choose elements \( f \) and \( g \) of \( L \) (resp. \( JL \)), when the condition (a) (resp. (b)) is satisfied, so that the classes \( \bar{f} \) and \( \bar{g} \) of \( f \) and \( g \) in \( L/JL \) (resp. \( JL/(J^2L + mL) \)) are linearly independent over \( k \). For each \( \lambda \in k \), we put \( h_\lambda = f + \lambda g \) and define

\[ M_\lambda = JL + Rh_\lambda \quad (\text{resp. } M = J^2L + mL + Rh_\lambda). \]

Then \( \{ M_\lambda \}_{\lambda \in k} \) meets the needs of this lemma.

Proposition 3. If \( R \) satisfies the condition (2) of Theorem 1, then \( R \) is a simple hypersurface.
Proof. Let $K_R$ be the canonical module of $R$. $K_R$ is an indecomposable MCM $R$-module. If $R$ were not a Gorenstein ring, then by Lemma 2 we can construct from $L = K_R$ infinitely many non-isomorphic indecomposable maximal Buchsbaum $R$-modules, because $\text{End}_R K_R = R$ and because $\dim K_R / m K_R \geq 2$ by [7, Satz 6.10]. Hence $R$ must be Gorenstein. Since $R$ is finite CM-representation type, by [8, Satz 1.2 1.2] and [3, Theorem A] $R$ is a simple hypersurface.

Proposition 4. If $R$ is a normal ring of $\dim R = 2$ and if $R$ satisfies the condition (2) of Theorem 1, then $R$ is a UFD.

Proof. Assume that $R$ is not a UFD and take a non-principal prime ideal $\mathfrak{p}$ of $R$ so that $\dim R_\mathfrak{p} = 1$. Then $\mathfrak{p}$ is an indecomposable MCM $R$-module and $\text{End}_R \mathfrak{p} = R$, $\dim K_{\mathfrak{p}} / m_{\mathfrak{p}} \mathfrak{p} \geq 2$. By Lemma 2 we can construct from $L = \mathfrak{p}$ infinitely many non-isomorphic indecomposable maximal Buchsbaum modules — this is a contradiction.

The rest of this report is devoted to show the following proposition briefly.

Proposition 5. If $R$ is a simple hypersurface of $\dim R \geq 2$, then $R$ possesses infinitely many non-isomorphic indecomposable
maximal Buchsbaum modules.

From Proposition 3 and Proposition 5 we get the implication

(2) \implies (1) of Theorem 1.

3. The case where \( d \geq 3 \).

It is well known that a \( d \)-dimensional simple hypersurface is isomorphic to a singularity of form

\[
\frac{k \langle X, Y, Z_1, \ldots, Z_{d-1} \rangle}{(f(X, Y) + Z_1^2 + \ldots + Z_{d-1}^2)},
\]

where \( f(X, Y) \) is one of the following ([9]):

\[
\begin{align*}
(A_n) & \quad x^{n+1} + y^{n+1} \quad (n \geq 1) \\
(D_n) & \quad x^{n+1} + xy^2 \quad (n \geq 4) \\
(E_6) & \quad x^3 + y^4 \quad (ch k \neq 3) \\
 & \quad x^3 + y^4, \quad x^3 + x^2y^2 + y^4 \quad (ch k = 3) \\
(E_7) & \quad x^3 + xy^3 \quad (ch k \neq 3) \\
 & \quad x^3 + xy^3, \quad x^3 + x^2y^2 + xy^3 \quad (ch k = 3) \\
(E_8) & \quad x^3 + y^5 \quad (ch k \neq 3, 5) \\
 & \quad x^3 + y^5, \quad x^3 + x^2y^3 + y^5, \quad x^3 + x^2y^2 + y^5 \quad (ch k = 3) \\
 & \quad x^3 + y^5, \quad x^3 + xy^4 + y^5 \quad (ch k = 5).
\end{align*}
\]

In the case where \( d \geq 3 \),

\[
\begin{align*}
R_0 = k \langle X, Y, Z_1, \ldots, Z_{d-3} \rangle / (f(X, Y) + Z_1^2 + \ldots + Z_{d-3}^2)
\end{align*}
\]
is also a simple hypersurface. Hence there exists an indecomposable MCM $R_0$-module $M$ which is not free. By Knörrer's Periodicity Theorem ([10, Theorem 3.1]) we can take an indecomposable MCM $R$-module $L$ so that $L/(z_{d-1}, z_{d-2})L = M \oplus N$, where $z_{d-1}$ (resp. $z_{d-2}$) is the class of $Z_{d-1}$ (resp. $Z_{d-2}$) in $R$ and $N$ is the first syzygy module of $M$. Let

$$\xi : L \longrightarrow M \oplus N$$

be the canonical epimorphism. Then we can prove that

$$\xi(JL) \subseteq J_1M \oplus J_2N,$$

where $J$, $J_1$ and $J_2$ denote $J(\text{End}_R L)$, $J(\text{End}_R M)$ and $J(\text{End}_R N)$ respectively. So $\xi$ induces the epimorphism

$$\bar{\xi} : L/JL \longrightarrow M/J_1M \oplus N/J_2N,$$

and we get $\dim_k L/JL \geq 2$. Hence Proposition 5 is deduced from Lemma 2 in the case where $d \geq 3$.

4. The case where $d = 2$.

Let $R$ be a 2-dimensional simple hypersurface which satisfies the condition (2) of Theorem 1. By Proposition 4 $R$ must be of type
Hence $R$ is a ring of the form
\[ k \square x, y, z \square / (x^3 + y^2G + y^5 + z^2), \]

where $G$ is either 0 or one of the following
\[ x^2y, x^2 \quad (\text{ch } k = 3), \]
\[ xy^2 \quad (\text{ch } k = 5). \]

Let $x$, $y$, $z$ and $g$ respectively denote the class of $X$, $Y$, $Z$ and $G$ in $R$. Then the maximal ideal $\mathfrak{m}$ of $R$ is $(x, y, z)R$.

Let $L'$ denote the second syzygy module of $R/\mathfrak{m}$:
\[ 0 \rightarrow L' \rightarrow R^3 (x, y, z) \rightarrow R^2 \rightarrow R/\mathfrak{m} \rightarrow 0. \]

Then $L'$ is a MCM $R$-module of rank 2 and is generated by
\[ \begin{pmatrix} x^2 \\ y^4 + yg \\ z \end{pmatrix}, \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix}, \begin{pmatrix} -z \\ 0 \\ x \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ -z \\ y \end{pmatrix}. \]

Let $\phi: R^3 \rightarrow R^2$ be the homomorphism defined by $\phi\left( \begin{pmatrix} a \\ b \\ c \end{pmatrix} \right) = \begin{pmatrix} a \\ c \end{pmatrix}$, and put $L = \phi(L')$. Then $L$ is also a MCM $R$-module of rank 2 and is generated by
\[ f_1 = \begin{pmatrix} x^2 \\ z \end{pmatrix}, \quad f_2 = \begin{pmatrix} -y \\ 0 \end{pmatrix}, \quad f_3 = \begin{pmatrix} -z \\ x \end{pmatrix} \quad \text{and} \quad f_4 = \begin{pmatrix} 0 \\ y \end{pmatrix}. \]

We can see that $L$ is indecomposable and $\dim_k L / JL = 1$, where
\[ J = J(\text{End}_R L) \].
To show that \( \dim_k J_L/(J^2 L + mL) \geq 2 \) we consider the ring 
\[ T = R/yR \ ( = k \{ X, Z \} / (X^3 + Z^2) ) \]. Let \( \bar{T} \) denote the normalization of \( T \) and put \( t = -z/x \). Then
\[ \bar{T} = k \{ t \} , \ x = -t^2 , \ \text{and} \ z = t^3 . \]
Let \( \bar{L} = L/yL \) and recall that any indecomposable maximal Cohen-Macaulay \( T \)-module is isomorphic to \( T \) or \( \bar{T} \) ([8, Satz 1.6]). Then we have that
\[ \bar{L} \cong \bar{T} \oplus \bar{T} , \]
as \( \text{rank}_T \bar{L} = 2 \) and as \( \bar{L} \) is minimally generated by the four elements
\[ \{ \bar{F}_i \} _1 \leq i \leq 4 \] (here \( \bar{\cdot} \) denotes the reduction mod \( yL \)). It is easily checked that \( \bar{F}_2 \) and \( \bar{F}_3 \) form a \( \bar{T} \)-free basis of \( \bar{L} \).

Since \( \text{End}_T \bar{L} = \text{End}_{\bar{T}} \bar{L} \), we shall identify \( \text{End}_T \bar{L} \) with 
\[ C = M_2(\bar{T}) \] (the matrix algebra) via the \( \bar{T} \)-free basis \( \bar{F}_2 \) and \( \bar{F}_3 \).

Let \( A = \text{End}_R L \) and put \( \bar{A} = A/yA \). Then \( \bar{A} \) may be canonically considered to be a subalgebra of \( \text{End}_T \bar{L} \) and we have a homomorphism
\[ \psi : A \longrightarrow \text{End}_T \bar{L} = C \]
of \( R \)-algebras. Thus via \( \psi \) we may write each element of \( A \) as a 
\[ 2 \times 2 \] matrix with coefficients in \( \bar{T} \). Then we have the following
Fact. Let $\xi \in J^2$ and write $\Psi(\xi) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Then $c \in tT$, $a, d \in t^2T$, and $b \in t^3T$.

Let $E : L \longrightarrow \tilde{L}$ denote the canonical epimorphism. By the above Fact we can see that $E(J^2L + mL) \subseteq W$, where $W = t^2Tt_2 + tTt_3$.

So $E$ induces the epimorphism

$$\bar{E} : L/(J^2L + mL) \longrightarrow \tilde{L}/W \cong \tilde{T}/t^2T \cdot \tilde{T}/tT.$$ 

Hence $\dim_k L/(J^2L + mL) \geq 3$, by which we have

$$\dim_k JL/(J^2L + mL) \geq 2,$$

since $\dim_k L/JL = 1$. This completes the proof of Theorem 1.
References


