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Sharp's Conjecture

the case of local rings with dim nonCM ≤ 1 or dim ≤ 5

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We continue to discuss a conjecture of Sharp on the existence of a dualizing complex from [2] and [3]. For terminologies, definitions and preliminaries, we refer the reader to [2], [3] and [4]. Throughout the note A denotes a $d$-dimensional local ring with the maximal ideal $m$. In this note we show the following two theorems.

Theorem 1. If $A$ has a dualizing complex and $\dim \text{nonCM}(A) \leq 1$, then $A$ is a homomorphic image of a Gorenstein ring.

Theorem 2. If $A$ has a dualizing complex and $\dim A \leq 5$, then $A$ is a homomorphic image of a Gorenstein ring.

In order to show Theorem 1, we make use of Faltings' Macaulayfication ([6]) and the theory of unconditioned strong $d$-sequences ([7]). If we had Theorem 1, Theorem 2 can be proven by a similar method to that given in [2, §2] and [3, §3].

Now we recall Faltings' Macaulayfication theorem.
Satz 3 ([6, Satz 3]). Sei $B$ lokaler Ring der Dimension $n+1$, $n$ sein maximales Ideal, $I = (x_1, \ldots, x_n) \subseteq \mathfrak{m}$ ein Ideal mit $\dim B/I = 1$ und $y \in \mathfrak{m}$ mit $\dim B/I + yB = 0$.

Es gelte: i) $B$ ist Quotient eines regulären Ringes.

ii) Für alle minimalen $p \in \text{Spek}(B)$ ist $\dim B/p = n+1$.

iii) Für alle $p \in \text{Spek}(B)$ mit $p \not\in I$ ist $B_p$ Cohen-Macaulay-Ring.

iv) $x_i \in (\text{Ann } H^j_I(B))^{2n}$ für alle $i$ und alle $j < n$.

v) $x_i \in (\text{Ann } H^j_B(B))^{2n}$ für alle $i$ und alle $j \leq n$.

Sei $X$ die Aufblasung des Ideals $I$ in $\text{Spek}(B)$ und $J = I^\mathfrak{m} + y\mathfrak{m} \subseteq \mathfrak{m}$ ein $\mathfrak{m}$-Ideal, welches die Faser von $X$ über $\mathfrak{m}$ definiert.

Auf $X$ gelte: vi) $JH^1_n(\mathfrak{m}) = 0$.

Sei $Y$ die Aufblasung von $X$ im Ideal $J$.

Dann ist $Y$ Cohen-Macaulay.

Now we assume that $A$ has a dualizing complex and $\dim \text{nonCM}(A) \leq 1$. We treat the case of $\text{Min}(A) = \text{Assh}(A)$. We note that

i) in Satz 3 can be replaced by that $B$ has a dualizing complex ([6, Bemerkungen]). In this case we can take elements $x_1, \ldots, x_{d-1}$, $y$ from $\mathfrak{m}$, for which iii), iv) v) and vi) in Satz 3 hold ([6, Bemerkung a) S.190] and [5]). Furthermore we may assume that $r$

(in Satz 3) is no less than $d-1$ and that $x_1, \ldots, x_{d-1}$ form an

unconditioned strong $d$-sequence in $A_p$ for every minimal prime

ideal $p$ of $I = (x_1, \ldots, x_{d-1})$ ([7, 6.19]). Let $L = I^p(I^p + yA)$,

$R = \mathfrak{n} \oplus L^n \subseteq A[LT] \subseteq A[T]$ with an indeterminate $T$ and $N = \mathfrak{m}R + R_+$.

Claim: $H^p_N(R)$ is finitely generated for $p \not\in d+1$.

It is sufficient to see that $R_p$ is Cohen-Macaulay for every
homogeneous prime ideal $P \neq N$. Put $P = P \cap A$. First suppose $P \neq m$. If $P \not\subset I$, $R_P = A[T]$ is Cohen-Macaulay as so is $A_P$. If $P \not\subset I$, $R_P = A_n^2(R_P^n)$ is Cohen-Macaulay as $x_1, \ldots, x_{d-1}$ is an unconditioned strong $d$-sequence in $A_P$ ([7, 4.1 and 7.10], cf. [3, 1.19]). Now let $P = m$. As $L^r = (x_1^{2r}, \ldots, x_{d-1}^{2r}, yx_1^r, \ldots, yx_{d-1}^r)$, $L^{r-1}$ and $P \not\subset R_+$, we have $x_1^{2r} \not\in P$ for some $i$ or $yx_j^r \not\in P$ for some $j$. Let $P \not\subset x_1^{2r}$. We put $t = x_1^{2r}T$, $S = R[1/t]$, $B = S_0$ and $Q = PS \cap B (\supseteq mB)$. Since $S = B[t, 1/t]$ and $t$ is algebraically independent over $B$, $S_{PS}$ is Cohen-Macaulay if and only if so is $B_Q$. Hence it is sufficient to show that $B_M$ is Cohen-Macaulay for every maximal ideal $M$ of $B$ containing $mB$. $B = S_0 = A[x/x_1^{2r} | x \in L] = A[x_2/x_1, \ldots, x_{d-1}/x_1, y/x_1^r]$. Satz 3 asserts that $B$ is Cohen-Macaulay. In the case of $P \not\subset yx_1^r$, the proof is similar to the above.

Hence we have Theorem 1 (cf. [2, Proof of 3.10] and [3, Proof of 4.11]).

We mention that the same theorem as Satz 3 (hence as Theorem 1) holds for a semi-local ring $(B, n_1, \ldots, n_t)$ if all $n_1, \ldots, n_t$ appear in the same degree term of a fundamental dualizing complex and every maximal chain of prime ideals has the same length.

Corollary to Theorem 1. If $A$ has a dualizing complex and $A$ is $(S_{d-2})$, then $A$ is a homomorphic image of a Gorenstein ring.

Now we prove Theorem 2. Let $d = 5$. (See [2, §2] or [3, §3] for the case of $d \leq 4$.) Suppose that the assertion is false. Then, by [2, 2.1] or [3, 3.1], there is a 5-dimensional local ring $A$ such that $A$ has a dualizing complex, is not a homomorphic image.
of a Gorenstein ring and is $(S_2)$. $A$ is not $(S_3)$ by Corollary above. Then $T(A) := \{ p \in \text{Spec}(A) \mid \text{depth } A_p = 2 < \dim A_p \}$ is not empty. Let $I$ be an ideal such that $V(I) = \text{nonCM}(A)$. As $A$ is $(S_2)$, height $I \geq 3$. There is an $A$-regular sequence $a, b$ in $I$. We have $T(A) \subset \text{Ass}(A/(a,b))$. We put $s(A) = \max \{ \dim A_p \mid p \in T(A) \}$, $T_0(A) = \{ p \in T(A) \mid \dim A_p = s(A) \}$ and $T_1(A) = T(A) \setminus T_0(A)$. Consider all such local rings, and take a local ring $A$ from them whose $s(A)$ is the smallest. As $A$ is $(S_2)$, $H^2_{\mathcal{E}A_p} (A_p)$ is of finite length for every $p$ in $\text{Spec}(A)$ with $\dim A_p \geq 3$. Hence there is a non zero divisor $x \in \bigcap \{ p \mid p \in T_0(A) \} \setminus \bigcup \{ p \mid p \in T_1(A) \}$ such that $x H^2_{\mathcal{E}A_p} (A_p) = 0$ for every $p$ in $T_0(A)$. Let $C = \text{Hom}_{A/xA}(K_{A/xA}, K_{A/xA})$. By the fact we mentioned before Corollary to Theorem 1, there exists a Gorenstein semi-local ring $G$ such that $\text{Max}(G) = \{ n \cap G \mid n \in \text{Max}(C) \}$, every maximal chain of prime ideals in $G$ has length 5, the length of a fundamental dualizing complex of $G$ is equal to 5 and $C$ is a homomorphic image of $G$. Let $B$ be the fibre product of $A + C$ and $G + C$. We have an exact sequence of $B$-modules $0 \rightarrow B \rightarrow A \oplus G \rightarrow C \rightarrow 0$. By the same argument as in Proof of [2, 2.3] or [3, 3.2], it is known that $B$ is a 5-dimensional local ring with the maximal ideal $m \cap B$ and $B$ has a dualizing complex. As $A$ is a homomorphic image of $B$, $B$ is a local ring and not $(S_3)$. Hence $T(B) \neq \emptyset$, and $s(B) \geq s(A)$ by the choice of $A$. Take $P$ from $T_0(B)$. We have $\text{depth } B_P = 2$. If $C_P = 0$, $B_P \cong A_P$ as $G$ is Gorenstein. Hence $PA \in T_0(A)$, a contradiction as $PA \notin x$. Therefore $C_P \neq 0$. Put $\dim C_P = t$. Then $\dim B_P = \dim A_P = \dim G_P = t + 1 = s(B) \geq s(A) > 2$. From the exact
sequence $0 \rightarrow B_P \rightarrow A_P \oplus G_P \rightarrow C_P \rightarrow 0$, we have depth $A_P = 2$ as depth $B_P = 2$, depth $G_P = t + 1 > 2$ and depth $C_P \geq 2$. Therefore $PA \in T_0(A)$ and $s(B) = s(A)$. Hence $xh^2_{PA_P}(A_P) = 0$ and $h^2_{PA_P}(A_P) \rightarrow h^2_{PA_P}(A_P/xA_P)$ is injective. It is known that $A_P/xA_P$ is $(S_2)$ at every non-maximal prime ideal. Hence $h^2_{PA_P}(A_P/xA_P) \rightarrow h^2_{PA_P}(C_P)$ is injective (cf. [1, Proposition 2]). From the exact sequence

$$0 = h^1_{PB_P}(C_P) \rightarrow h^2_{PB_P}(B_P) \rightarrow h^2_{PB_P}(A_P \oplus G_P) \rightarrow h^2_{PA_P}(A_P) \rightarrow h^2_{PB_P}(C_P),$$

we have $h^2_{PB_P}(B_P) = 0$, which contradicts depth $B_P = 2$. Now the proof is completed.

References


