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Kyoto University
Sharp's Conjecture

the case of local rings with \( \dim \text{nonCM} \leq 1 \) or \( \dim \leq 5 \)

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We continue to discuss a conjecture of Sharp on the existence of a dualizing complex from [2] and [3]. For terminologies, definitions and preliminaries, we refer the reader to [2], [3] and [4]. Throughout the note \( A \) denotes a \( d \)-dimensional local ring with the maximal ideal \( \mathfrak{m} \). In this note we show the following two theorems.

Theorem 1. If \( A \) has a dualizing complex and \( \dim \text{nonCM}(A) \leq 1 \), then \( A \) is a homomorphic image of a Gorenstein ring.

Theorem 2. If \( A \) has a dualizing complex and \( \dim A \leq 5 \), then \( A \) is a homomorphic image of a Gorenstein ring.

In order to show Theorem 1, we make use of Faltings' Macaulayfication ([6]) and the theory of unconditioned strong \( d \)-sequences ([7]). If we had Theorem 1, Theorem 2 can be proven by a similar method to that given in [2, §2] and [3, §3].

Now we recall Faltings' Macaulayfication theorem.
Satz 3([6, Satz 3]). Sei B lokaler Ring der Dimension \( n+1 \), \( n \) sein maximales Ideal, \( I = (x_1, \ldots, x_n) \subset n \) ein Ideal mit \( \dim B/I = 1 \) und \( y \in n \) mit \( \dim B/I+yB = 0 \).

Es gelte: i) \( B \) ist Quotient eines regulären Ringes.

ii) Für alle minimalen \( p \in \text{Spek}(B) \) ist \( \dim B/p = n+1 \).

iii) Für alle \( p \in \text{Spek}(B) \) mit \( p \not\in I \) ist \( B_p \) Cohen-Macaulay-Ring.

iv) \( x_i \in (\text{Ann } H^j_I(B))^2 n \) für alle \( i \) und alle \( j < n \).

v) \( x_i \in (\text{Ann } H^j_n(B))^2 n \) für alle \( i \) und alle \( j \leq n \).

Sei \( X \) die Aufblasung des Ideals \( I \) in \( \text{Spek}(B) \) und \( J = I^r \O_X^r + y \O_X^r \) ein \( \O_X^r \)-Ideal, welches die Faser von \( X \) über \( n \) definiert.

Auf \( X \) gelte: vi) \( \text{JH}^1_n(\O_X) = 0 \).

Sei \( Y \) die Aufblasung von \( X \) im Ideal \( J \).

Dann ist \( Y \) Cohen-Macaulay.

Now we assume that \( A \) has a dualizing complex and \( \dim \text{nonCM} (A) \leq 1 \). We treat the case of \( \text{Min}(A) = \text{Assh}(A) \). We note that

i) in Satz 3 can be replaced by that \( B \) has a dualizing complex ([6, Bemerkungen]). In this case we can take elements \( x_1, \ldots, x_{d-1}, y \) from \( m \), for which iii), iv) v) and vi) in Satz 3 hold ([6, Bemerkung a) S.190] and [5]). Furthermore we may assume that \( r \) (in Satz 3) is no less than \( d-1 \) and that \( x_1, \ldots, x_{d-1} \) form an unconditioned strong \( d \)-sequence in \( A \) for every minimal prime ideal \( p \) of \( I = (x_1, \ldots, x_{d-1}) \) ([7, 6.19]). Let \( L = I^r(I^r + yA) \), \( R = \bigoplus_{l=0} L^n = A[L,T] \subset A[T] \) with an indeterminate \( T \) and \( N = mR + R^+ \).

Claim: \( H^p_N(R) \) is finitely generated for \( p \neq d+1 \).

It is sufficient to see that \( R_p \) is Cohen-Macaulay for every
homogeneous prime ideal \( P \neq N \). Put \( P = P \cap A \). First suppose \( P \neq m \). If \( P \nsubseteq I \), \( R \cong \mathbb{A}^{[T]}_P \) is Cohen-Macaulay as \( A \). If \( P \supseteq I \), \( R \cong \mathbb{A}^{[T]}_P \) is Cohen-Macaulay as \( x_1, \ldots, x_{d-1} \) is an unconditioned strong \( d \)-sequence in \( A \) ([7, 4.1 and 7.10], cf. [3, 1.19]). Now let \( P = m \). As \( L' = (x_1^{2r}, \ldots, x_{d-1}^{2r}, yx_1^r, \ldots, yx_{d-1}^r) \), \( L'^{-1} \) and \( P \nsubseteq R_+ \), we have \( x_1^{2r} \notin P \) for some \( i \) or \( yx_j^r \notin P \) for some \( j \). Let \( P \neq x_1^{2r} \). We put \( t = x_1^{2r}T \), \( S = R[1/t] \), \( B = S_0 \) and \( Q = PS \cap B (\supseteq mB) \). Since \( S = B[t, 1/t] \) and \( t \) is algebraically independent over \( B \), \( S_{PS} \) is Cohen-Macaulay if and only if so is \( B_Q \). Hence it is sufficient to show that \( B_M \) is Cohen-Macaulay for every maximal ideal \( M \) of \( B \) containing \( mB \). \( B = S_0 = A[x/x_1^{2r} | x \in L] = A[x_2/x_1, \ldots, x_{d-1}/x_1, y/x_1^r] \). Satz 3 asserts that \( B \) is Cohen-Macaulay. In the case of \( P \not\subseteq yx_1^r \), the proof is similar to the above.

Hence we have Theorem 1 (cf. [2, Proof of 3.10] and [3, Proof of 4.11]).

We mention that the same theorem as Satz 3 (hence as Theorem 1) holds for a semi-local ring \( (B, n_1, \ldots, n_t) \) if all \( n_1, \ldots, n_t \) appear in the same degree term of a fundamental dualizing complex and every maximal chain of prime ideals has the same length.

Corollary to Theorem 1. If \( A \) has a dualizing complex and \( A \) is \( (S_d^{-2}) \), then \( A \) is a homomorphic image of a Gorenstein ring.

Now we prove Theorem 2. Let \( d = 5 \). (See [2, §2] or [3, §3] for the case of \( d \leq 4 \).) Suppose that the assertion is false. Then, by [2, 2.1] or [3, 3.1], there is a 5-dimensional local ring \( A \) such that \( A \) has a dualizing complex, is not a homomorphic image.
of a Gorenstein ring and is \((S_2)\). \(A\) is not \((S_3)\) by Corollary above. Then \(T(A) := \{ p \in \text{Spec}(A) \mid \text{depth } A_p = 2 < \dim A_p \} \) is not empty. Let \(I\) be an ideal such that \(V(I) = \text{nonCM}(A)\). As \(A\) is \((S_2)\), height \(I \geq 3\). There is an \(A\)-regular sequence \(a, b\) in \(I\). We have \(T(A) \subseteq \text{Ass}(A/(a,b))\). We put \(s(A) = \max \{ \dim A_p \mid p \in T(A) \}\), \(T_0(A) = \{ p \in T(A) \mid \dim A_p = s(A) \}\) and \(T_1(A) = T(A) \setminus T_0(A)\). Consider all such local rings, and take a local ring \(A\) from them whose \(s(A)\) is the smallest. As \(A\) is \((S_2)\), \(H^2_{A_p} (A_p)\) is of finite length for every \(p \in \text{Spec}(A)\) with \(\dim A_p \geq 3\). Hence there is a non zero divisor \(x \in \bigcap \{ p \mid p \in T_0(A) \} \setminus \bigcup \{ p \mid p \in T_1(A) \}\) such that \(x H^2_{A_p} (A_p) = 0\) for every \(p \in T_0(A)\). Let \(C = \text{Hom}_{A/xa}(K_{A/xa}, K_{A/xa})\). By the fact we mentioned before Corollary to Theorem 1, there exists a Gorenstein semi-local ring \(G\) such that \(\text{Max}(G) = \{ p \cap G \mid p \in \text{Max}(C) \}\), every maximal chain of prime ideals in \(G\) has length 5, the length of a fundamental dualizing comlex of \(G\) is equal to 5 and \(C\) is a homomorphic image of \(G\). Let \(B\) be the fibre product of \(A + C\) and \(G + C\). We have an exact sequence of \(B\)-modules \(0 \rightarrow B \rightarrow A \oplus G \rightarrow C \rightarrow 0\). By the same argument as in Proof of [2, 2.3] or [3, 3.2], it is known that \(B\) is a 5-dimensional local ring with the maximal ideal \(m \cap B\) and \(B\) has a dualizing complex. As \(A\) is a homomorphic image of \(B\), \(B\) is not a homomorphic image of a Gorenstein ring and not \((S_3)\). \(B\) is \((S_2)\). Hence \(T(B) \neq \emptyset\), and \(s(B) \geq s(A)\) by the choice of \(A\). Take \(p\) from \(T_0(B)\). We have \(\text{depth } B_p = 2\). If \(C_p = 0\), \(B_p \cong A_p\) as \(G\) is Gorenstein. Hence \(pA \in T_0(A)\), a contradiction as \(pA \notin x\). Therefore \(C_p \neq 0\). Put \(\dim C_p = t\). Then \(\dim B_p = \dim A_p = \dim G_p = t + 1 = s(B) \geq s(A) > 2\). From the exact
sequence $0 \rightarrow B_P \rightarrow A_P \oplus G_P \rightarrow C_P \rightarrow 0$, we have depth $A_P = 2$ as depth $B_P = 2$, depth $G_P = t + 1 > 2$ and depth $C_P \geq 2$. Therefore $P \in T_0(A)$ and $s(B) = s(A)$. Hence $xH^2_{PA_P}(A_P) = 0$ and $H^2_{PA_P}(A_P) \rightarrow H^2_{PA_P}(A_P/xA_P)$ is injective. It is known that $A_P/xA_P$ is $(S_2)$ at every non-maximal prime ideal. Hence $H^2_{PA_P}(A_P/xA_P) \rightarrow H^2_{PA_P}(C_P)$ is injective (cf. [1, Proposition 2]). From the exact sequence

$$0 = H^1_{PB_P}(C_P) \rightarrow H^2_{PB_P}(B_P) \rightarrow H^2_{PB_P}(A_P \oplus G_P) \cong H^2_{PA_P}(A_P) \rightarrow H^2_{PB_P}(C_P),$$

we have $H^2_{PB_P}(B_P) = 0$, which contradicts depth $B_P = 2$. Now the proof is completed.

References


