数理解析研究所講究録
621巻 1987年 50-59


# Δ-genera and sectional genera of local rings

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The classification of (embedded) projective varieties, especially algebraic surfaces, by their *sectional genera* is quite a classical subject in algebraic geometry studied by Enriques, Castelnuovo, Roth and others. This old subject has been recently resurrected and extended to the classification of polarized varieties by their sectional genera (Fujita, Ionescu, Lanteri, Palleschi and others). T.Fujita, among others, introduced the notion of *Δ-genus* and *sectional genus* of a polarized variety, and studied the structure of polarized varieties with low genera.

Here we introduce the *Δ-genus* and the *sectional genus* for a general noetherian local ring, and our aim is to study the structure of local rings (or singularities) by these genera. So this note is a continuation of our previous work [3] on the genera of commutative rings.

By the way, the non-negativity of the sectional genus and the Δ-genus of a Cohen-Macaulay local ring traces back to Northcott (1960) and Abhyankar (1967). Moreover, the structure of Cohen-Macaulay local rings of Δ-genera zero and Gorenstein local rings of Δ-genera one has been studied by J.Sally in detail. Sally's work generalizes the study of rational surface singularities (due to Artin) and minimally elliptic surface singularities (due to Laufer and Wahl). Also, the sectional genera of curve singularities are studied by Kirby, Matlis and others.

Throughout this note, we denote by $(R, m, k)$ a Cohen-Macaulay local ring with $\dim(R) = d$. Let $I$ be an $m$-primary ideal of $R$. Then there exist integers $e_i$ $(0 \leq i \leq d)$ such that

$$\ell(R/I^{n+1}) = e_0\binom{n+d}{d} - e_1\binom{n+d-1}{d-1} + \ldots + (-1)^d e_d$$

for all $n \gg 0$. We called $e_i(I) = e_i$, $g(I) = e_d$, $p_a(I) = (-1)^d(e_0 - e_1 + \ldots + (-1)^d e_d - \ell(R/I))$ the i-th *Hilbert coefficient*, the *genus*, the *arithmetic genus* of $I$ respectively (cf. [3]).

*Definition* 1. We define the *Δ-genus* $g_\Delta(I)$ and the *sectional genus* $g_s(I)$ of $I$ by

$$g_\Delta(I) = e(I) + (d - 1)\ell(R/I) - \ell(I/I^2),$$
$$g_s(I) = e_1(I) - e(I) + \ell(R/I) \quad \text{if} \quad d \geq 1.$$

We also put $g_\Delta(R) = g_\Delta(m) = e(R) + \dim(R) - \mathrm{emb}(R) - 1$ and $g_s(R) = g_s(m) = e_1(R) - e(R) + 1$. Put $G(I) = \bigoplus_{n\geq 0} I^n/I^{n+1}$ and $G(R) = G(m)$. If $F(G(I), t) := \sum_{n\geq 0} \ell(I^n/I^{n+1})t^n = \Phi_I(t)/(1 - t)^d$ with $\Phi_I(t) \in \mathbf{Z}[t]$, then we have $e_i(I) = (d^i\Phi_I/dt^i|_{t=1})/i!$.

*Example*. If $R$ is a hypersurface of degree $e$ (equivalently, $g_\Delta(R) = e - 2$, $e(R) = e$), then $e_i(R) = \binom{e}{i+1}$ $(0 \leq i \leq d)$,

$g_s(R) = p_a(R) = \binom{e-1}{2}$ and $g(R) = \binom{e}{v}$, where $v = \mathrm{emb}(R)$.

*Lemma* 2 (Valla, 1979). Assume that $k$ is an infinite field and let $J$ be a minimal reduction of $I$. Then $g_\Delta(I) = \ell(I^2/IJ)$. Hence $g_\Delta(I) = 0$ if and only if $\delta(I) \leqq 1$, and in this case $G(I)$ is Cohen-Macaulay. Here we put $\delta(I) = \min\{n \mid JI^n = I^{n+1}$ for some minimal reduction $J$ of $I\}$ (the *reduction exponent* of $I$, cf. [2], [3]).

As a general strategy, we reduce problems to the case of curve singularities. So, first, we treat the case of dimension one.

*Theorem* 3 (see [3]). Assume that $d = 1$ and put $S = \bigcup_{n=1}^{\infty} (I^n : I^n)_{Q(R)}$. Then

(1) $S$ is a finitely generated $R$-module and $\ell(R/I^n) = e(I)n - \ell(S/R) + \ell(I^nS/I^n)$ for all $n \geqq 0$.

(2) $e(I) = \ell(S/IS)$, $g(I) = \ell(S/R)$, $p_a(I)$ $(= g_s(I)) = g(I) - e(I) + \ell(R/I) = \ell(IS/I)$, $g(I) \geqq p_a(I) \geqq g_\Delta(I) \geqq 0$, $p_a(I) - g_\Delta(I) = \ell(I^2S/I^2)$, and $g(I) - p_a(I) = \ell(I/J)$ if $k$ is an infinite field and $J$ is a minimal reduction of $I$.

(3) $g(I) = 0 \Leftrightarrow g(I) = p_a(I) \Leftrightarrow \delta(I) = 0 \Leftrightarrow I$ is a principal ideal.

(4) $p_a(I) = 0 \Leftrightarrow g_\Delta(I) = 0 \Leftrightarrow \delta(I) \leqq 1 \Leftrightarrow S = (I : I)$.

(5) $p_a(I) = g_\Delta(I) \Leftrightarrow \delta(I) \leqq 2 \Leftrightarrow S = (I^2 : I^2)$.

(6) $\delta(I) = n(I) + 1 = \mathrm{reg}(G(I)) = \min\{n \mid S = (I^n : I^n)\}$

$\leq p_a(I) + 1$. Moreover, $\ell(I^n/I^{n+1}) < e(I)$ for all $n \leq n(I)$. Here $n(I) = \min\{m \mid \ell(I^n/I^{n+1}) = e(I) \text{ for all } n > m\}$ (the *postulation number* of $I$) and $\operatorname{reg}(G(I)) = \min\{n \mid [H_P^i(G(I))]_j = 0 \text{ if } i + j > n\}$, $P = G(I)_+$ (the *regularity* of $G(I)$) (cf. [1], [3]).

*Proposition* 4. We have $g_s(I) \geq 0$ and the following conditions are equivalent:

(1) $g_s(I) = 0$.

(2) $g_\Delta(I) = 0$.

(3) $\operatorname{reg}(G(I)) \leq 1$.

(4) $\ell(R/I^{n+1}) = e(I)\binom{n+d-1}{d} + \ell(R/I)\binom{n+d-1}{d-1}$ for all $n \geq 0$.

If these conditions are satisfied, then $e_i(I) = 0$ $(2 \leq i \leq d)$, $p_a(I) = 0$, $g(I) = 0$ if $d \geq 2$, and $G(I)$ is Cohen-Macaulay.

*Proof*. We may assume that $k$ is an infinite field. The fact $g_s(I) \geq 0$ is proved in [3]. We give another proof: Take a superficial system of parameters $x_1,\dots,x_d \in I$ with respect to $I$. Then we have $g_s(I) = g_s(I/(x_1,\dots,x_{d-1})) \geq 0$ by Theorem 3. The assertions (2) $\Leftrightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (1) follow from Lemma 2 and [3]. So we have only to show the assertion (1) $\Rightarrow$ (3). Take $x_1,\dots,x_d$ as before, and put $J = (x_1,\dots,x_d)$ and $I_i = (x_1,\dots,\hat{x}_i,\dots,x_d)$, $1 \leq i \leq d$. Then we have $g_s(I/I_i) = g_s(I) = 0$. Hence, by Theorem 3, $x_i(I/I_i) = (I/I_i)^2$, i.e., $x_iI + I_i = I^2 + I_i$. Take any element $y$ of $I^2$. Then for any $j \neq 1$, we have $y = x_1y_1 + \dots + x_dy_d$

$= x_1z_1 + \dots + x_dz_d$ for some $y_i$, $z_i$ such that $y_1, z_j \in I$. Hence $x_1(y_1 - z_1) + \dots + x_d(y_d - z_d) = 0$, and this implies that $y_j - z_j \in J \subset I$. Therefore $y_j \in I$ for all $j$, $1 \leqq j \leqq d$, and we have $y \in JI$. Hence $I^2 = JI$, i.e., $\delta(I) \leqq 1$. Q.E.D.

*Theorem* 5. Put $\mathrm{emb}(R) = v$ and $e(R) = e$. Then

(1) $0 \leqq g_\Delta(R) \leqq g_s(R) \leqq \binom{e-1}{2}$. If $G(R)$ is Cohen-Macaulay, then $\mathrm{reg}(G(R)) \leqq g_\Delta(R) + 1$.

(2) $g_s(R) = 0 \iff g_\Delta(R) = 0 \iff \mathrm{reg}(G(R)) \leqq 1$. In this case, $r(R) = e - 1$. ($r(R)$ denotes the Cohen-Macaulay type of R.)

(3) $g_s(R) = g_\Delta(R) \iff \mathrm{reg}(G(R)) \leqq 2$

$\iff \Phi_R(t) = 1 + (v - d)t + (e + d - v - 1)t^2$

$\iff \ell(R/m^{n+1}) = e\binom{n+d}{d} - (2e - v + d - 2)\binom{n+d-1}{d-1} + (e + d - v - 1)\binom{n+d-2}{d-2}$ for all $n \geqq 2$.

(4) $g_s(R) = 1 \iff g_\Delta(R) = 1$ and $G(R)$ is Cohen-Macaulay

$\iff g_\Delta(R) = 1$ and $\mathrm{reg}(G(R)) = 2$

$\iff \Phi_R(t) = 1 + (v - d)t + t^2$

$\iff \ell(R/m^{n+1}) = e\binom{n+d-1}{d} + \binom{n+d-2}{d-2}$ for all $n \geqq 2$.

(5) $g_s(R) = \binom{e-1}{2}$ if and only if $R$ is a hypersurface.

*Proof*. We may assume that $k$ is an infinite field.
(1) Take a superficial system of parameters $x_1,\dots,x_d$ such that $x_i \in m - m^2$, and put $R' = R/(x_1,\dots,x_{d-1})$. Then $g_s(R) = g_s(R')$, $g_\Delta(R) = g_\Delta(R')$ and $e(R) = e(R')$. Hence the assertion follows from Theorem 3. The proof of (5) is similar.
(2) follows from Proposition 4. (3) If $\mathrm{reg}(G(R)) \leqq 2$, then

$G(R)$ is Cohen-Macaulay by Sally. Hence we have $\Phi_R(t) = 1 + (v - d)t + (e + d - v - 1)t^2$. So we have only to show that if $g_s(R) = g_\Delta(R)$, then $\delta(R) \leqq 2$. Take $x_1, \dots, x_d$ as in (1), and put $q = (x_1, \dots, x_d)$, $q_i = (x_1, \dots, \hat{x}_i, \dots, x_d)$, $R_i' = R/q_i$. Then $g_s(R_i') - g_\Delta(R_i') = g_s(R) - g_\Delta(R) = 0$. Hence $\delta(R_i') \leqq 2$ by Theorem 3. Therefore $x_i m^2 + q_i = m^3 + q_i$. Take any element $y$ of $m^3$. Then for any $j \neq 1$, $y = x_1 y_1 + \dots + x_d y_d = x_1 z_1 + \dots + x_d z_d$ for some $y_i$, $z_i$ such that $y_1$, $z_j \in m^2$. As in the proof of Proposition 4, we have $y_j - z_j \in q$, and $y_j$ is in $(m^2, q)$ for all $i$. Hence $y = u + w$ with $u \in q^2$, $w \in m^3$. Since $x_1, \dots, x_d$ is analytically independent, we have $u \in q^2 \cap m^3 = q^2 m$, and this implies that $y \in qm^2$. Therefore we have $m^3 = qm^2$, i.e., $\delta(R) \leqq 2$. (4) follows from (3). Q.E.D.

*Example.* (1) $R$ is a cubic hypersurface $\Longleftrightarrow$ $g_\Delta(R) = 1$, $e(R) = 3$ $\Longleftrightarrow$ $g_s(R) = 1$, $e(R) = 3$.

(2) If $R = k[[t^4, t^5, t^7]]$ or $k[[t^4, t^6, t^7]]$, then $g_s(R) = g_\Delta(R) = 1$. In general, if $H$ is a numerical semigroup and $R = k[[H]]$, then $\bar{p}_a(R) = 1$ if and only if $H = H_{e,r} := \{0, e, e + 1, \dots, e + r, e + r + 2, \dots\}$ with $e \geqq 3$, $0 \leqq r < e - 1$, and in this case $p_a(R) = g_\Delta(R) = 1$.

(3) If $g_s(R) = 2$, then $g_\Delta(R) = 2$ or $g_\Delta(R) = 1$. In the first case, $G(R)$ is Cohen-Macaulay, $\Phi_R(t) = 1 + (v - d) + 2t^2$, and $e_2(R) = 2$, $e_i(R) = 0$ $(3 \leqq i \leqq d)$. In the second case, $G(R)$ is not Cohen-Macaulay and $r(R) = e(R) - 2$. For example, if $R = k[[t^4, t^5, t^{11}]]$, then $g_s(R) = 2$, $g_\Delta(R) = 1$, and if

$R = k[[t^5, t^6, t^7]]$, then $g_s(R) = g_\Delta(R) = 2$.

*Theorem* 6. Assume that $R$ is Gorenstein. Then

(1) $g_s(R) = 0 \iff g_\Delta(R) = 0 \iff R$ is a regular local ring or a quadric hypersurface.

(2) $g_s(R) = 1 \iff g_\Delta(R) = 1 \iff g_s(R) = g_\Delta(R) \geqq 1 \iff \mathrm{reg}(G(R)) = 2$. In this case, $G(R)$ is Gorenstein.

(3) $g_s(R)$ does not attain 2.

(4) $g_s(R) = 3 \iff g_\Delta(R) = 2 \iff \Phi_R(t) = 1 + (v - d)t + t^2 + t^3$. In this case, $G(R)$ is Cohen-Macaulay, is not Gorenstein, $\mathrm{reg}(G(R)) = 3$ and $e_2(R) = 4$, $e_3(R) = 1$, $e_i(R) = 0$ $(4 \leqq i \leqq d)$.

(5) If $g_s(R) = 4$, then $g_\Delta(R) = 3$.

*Proof*. (1) follows from Theorem 5. (2) If $R$ is Gorenstein and $g_\Delta(R) = 1$, then $G(R)$ is Gorenstein by Sally. (3) If $g_s(R) = 2$, then we have $2 = g_s(R) \geqq g_\Delta(R) \geqq 2$. Hence we have $g_s(R) = g_\Delta(R) = 2$, which is a contradiction. (4) If $g_s(R) = 3$, then $2 \leqq g_\Delta(R) < g_s(R) = 3$. Hence $g_\Delta(R) = 2$. Conversely, if $g_\Delta(R) = 2$, then $G(R)$ is Cohen-Macaulay by Sally, and we have $\mathrm{reg}(G(R)) = g_\Delta(R) + 1 = 3$. Hence $\Phi_R(t) = 1 + (v - d)t + t^2 + t^3$ and we have $g_s(R) = 3$. (4) Since $3 \leqq g_\Delta(R) < g_s(R) = 4$, we have $g_\Delta(R) = 3$. Q.E.D.

*Example*. (1) $R$ is a quartic hypersurface $\iff g_\Delta(R) = 2$, $e(R) = 4 \iff R$ is Gorenstein and $g_s(R) = 3$, $e(R) = 4$.

(2) R is a complete intersection of type (2, 2) $\Leftrightarrow$ R is Gorenstein and $g_\Delta(R) = 1$, $e(R) = 4$ $\Leftrightarrow$ R is Gorenstein and $g_s(R) = 1$, $e(R) = 4$.

(3) $R = k[[t^5, t^6, t^9]]$ is Gorenstein and $g_s(R) = 3$, $g_\Delta(R) = 2$.

(4) $R = k[[t^6, t^7, t^8]]$ is Gorenstein and $g_s(R) = 4$, $g_\Delta(R) = 3$.

Next, we consider the normal genera. Henceforth we assume that R is analytically unramified. Then there exist integers $\bar{e}_i$ $(0 \leqq i \leqq d)$ such that

$$\ell(R/\overline{I^{n+1}}) = \bar{e}_0\binom{n+d}{d} - \bar{e}_1\binom{n+d-1}{d-1} + \ldots + (-1)^d\bar{e}_d$$

for all $n \gg 0$, where $\bar{J}$ denotes the integral closure of J. We called $\bar{e}_i(I) = \bar{e}_i$, $\bar{g}(I) = \bar{e}_d$, $\bar{p}_a(I) = (-1)^d(\bar{e}_0 - \bar{e}_1 + \ldots + (-1)^d\bar{e}_d - \ell(R/\bar{I}))$ the i-th *normal Hilbert coefficient*, the *normal genus*, the *normal arithmetic genus* of I, respectively (cf. [3]).

*Definition* 7. We define the *normal* $\Delta$*-genus* $\bar{g}_\Delta(I)$ and the *normal sectional genus* $\bar{g}_s(I)$ of I by

$$\bar{g}_\Delta(I) = e(I) + (d-1)\ell(R/\bar{I}) - \ell(\bar{I}/\overline{I^2}),$$
$$\bar{g}_s(I) = \bar{e}_1(I) - e(I) + \ell(R/\bar{I}) \quad \text{if } d \geqq 1.$$

*Lemma* 8. Assume that k is an infinite field and $\bar{I} = I$, and let J be a minimal reduction of I. Then $\bar{g}_\Delta(I) = \ell(\overline{I^2}/IJ)$, $\bar{g}_\Delta(I) - g_\Delta(I) = \ell(\overline{I^2}/I^2)$. Hence $\bar{g}_\Delta(I) = 0$ $\Leftrightarrow$

$\delta(I) \leqq 1$ and $\overline{I^2} = I^2$.

Put $\bar{\delta}(I) = \min\{n \mid$ there exists a minimal reduction $J$ of $\bar{I}$ such that $J\overline{I^m} = \overline{I^{m+1}}$ for all $m \geqq n\}$. Hence if $\bar{I} = I$, then $\bar{\delta}(I) \leqq 1 \Leftrightarrow \delta(I) \leqq 1$ and $\overline{I^n} = I^n$ for all $n \geqq 0$. We have $\bar{g}_s(I) \geqq g_s(I) \geqq 0$. It is easy to see that

$\bar{\delta}(I) \leqq 1$ for all $I$

$\Longrightarrow$ $\bar{g}_\Delta(I) = 0$ for all $I$

$\Longleftrightarrow$ $I\bar{I} = I^2$ for all $I$

$\Longrightarrow$ for all $I$ such that $\bar{I} = I$, we have $\delta(I) \leqq 1$ and $\overline{I^n} = I^n$ for all $n \gg 0$

$\Longleftrightarrow$ for all $I$ such that $\bar{I} = I$, we have $\bar{g}_s(I) = 0$ and $\bar{e}_i(I) = e_i(I) = 0$, $2 \leqq i \leqq d$. If $\bar{\delta}(I) \leqq 1$ (resp. $\bar{\delta}(m) \leqq 2$), then $\bar{G}(I) = \bigoplus_{n \geqq 0} \overline{I^n}/\overline{I^{n+1}}$ (resp. $\bar{G}(m)$) is Cohen-Macaulay. If $d = 1$, then $\bar{g}_s(I) = 0 \Leftrightarrow \bar{\delta}(I) \leqq 1 \Leftrightarrow \bar{R} = (\bar{I} : \bar{I})$.

*Theorem* 9. If $d = 2$, then $\bar{g}_s(I) = 0 \Leftrightarrow \bar{g}(I) = 0 \Leftrightarrow \bar{\delta}(I) \leqq 1$.

*Proof*. We already know that $\bar{g}(I) = 0 \Leftrightarrow \bar{\delta}(I) \leqq 1 \Longrightarrow \bar{g}_s(I) = 0$ (cf. [3]). If $\bar{g}_s(I) = 0$, then $\bar{g}_s(I) = g_s(I) = 0$, and this implies that $\bar{e}_1(I) = e_1(I)$ and $e_2(I) = 0$. Hence for all $n \gg 0$, we have $0 \leqq \ell(R/I^n) - \ell(R/\overline{I^n}) = e_2(I) - \bar{e}_2(I) = -\bar{e}_2(I) \leqq 0$. Therefore $\bar{g}(I) = \bar{e}_2(I) = 0$. Q.E.D.

*Example*. (1) Put $H = \langle e, e+1, e(e-1)-1 \rangle$, $e \geqq 4$

and $R = k[[H]]$. Then $\bar{g}_s(R) = e(e-3)/2$. For example, if $H = \langle 4, 5, 11 \rangle$, then $R$ is not Gorenstein, $\bar{g}_s(R) = g_s(R) = 2$, $g_\Delta(R) = 1$, $\delta(R) = 3$, $\bar{\delta}(R) = 2$, $G(R)$ is not Cohen-Macaulay, $\bar{G}(R)$ is Cohen-Macaulay, $r(R) = 2$, $\Phi_R(t) = 1 + 2t + t^3$ and $\mathrm{Proj}(R(m))$ is normal.

(2) If $R$ is Gorenstein, $d = 2$, $\bar{g}_s(R) = 1$ and $e(R) \geq 3$, then we have $g_s(R) = \bar{g}(R) = g(R) = g_\Delta(R) = 1$.

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(October 1986)