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An example of one-dimensional Cohen-Macaulay local rings that possess only finitely many isomorphism classes of indecomposable maximal Buchsbaum modules

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In our paper [GN] with K. Nishida we proved the following

Theorem 1. Let $P = k[[X_1, X_2, \ldots, X_n]]$ be a formal power series ring over an algebraically closed field $k$ of char $k \neq 2$. Let $R = P/I$, where $I$ is an ideal of $P$ and suppose $\dim R = d \geq 2$. Then the following two conditions are equivalent.

(1) $R$ is a regular local ring.

(2) $R$ is a Cohen-Macaulay ring that possesses only finitely many isomorphism classes of indecomposable maximal Buchsbaum modules.

When this is the case, the syzygy modules of the residue class field $k$ of $R$ are the representatives of indecomposable maximal Buchsbaum modules and so there are exactly $d$ non-isomorphic indecomposable maximal Buchsbaum modules over $R$.

Our contribution in the above theorem is the implication (2) $\Rightarrow$ (1). The last assertion and the implication (1) $\Rightarrow$ (2) are due to [G] (see also [EG, Theorem 3.2]), where some consequences of the result are discussed too.

I would like to note here that the assumption $\dim R \geq 2$ in Theorem 1 is not superfluous. When $\dim R = 1$, maximal Buchsbaum $R$-modules $M$ are characterized by the condition that

$$\dim_R^0 N = 1 \text{ and } m \cdot H^0_m (M) = (0)$$
(here $H^i_m(\cdot)$ denotes the $i^{th}$ local cohomology functor of $R$ relative to the maximal ideal $m$ of $R$). This condition (is of course not too much weak but) seems not quite strong. Nevertheless in some sense surprisingly, there exist such Cohen-Macaulay local rings $R$ of $\dim R = 1$ that are non-regular but possess only finitely many isomorphism classes of indecomposable maximal Buchsbaum modules. In my lecture I will explore the typical example $R = k[[t^2, t^3]]$.

Now let $k$ be a field and $S = k[[t]]$ a formal power series ring over $k$. We put $R = k[[t^2, t^3]]$. Then $R$ and $S$ are the only indecomposable maximal Cohen-Macaulay $R$-modules (cf. [H, Satz 1.6]) and the $R$-module $S$ has a resolution of the following form

$$\cdots \rightarrow R^2 \xrightarrow{\epsilon} R^2 \xrightarrow{\sigma} R^2 \xrightarrow{\rho} S \rightarrow 0,$$

where $\epsilon(\begin{bmatrix} a \\ b \end{bmatrix}) = a + bt$. Therefore we have an embedding $\sigma : S \rightarrow R^2$ which sends $1$ (resp. $t$) to $\begin{bmatrix} t^3 \\ -t^2 \end{bmatrix}$ (resp. $\begin{bmatrix} t^4 \\ -t^3 \end{bmatrix}$)

and makes the diagram

$$\begin{array}{ccc}
S & \xrightarrow{\sigma} & R^2 & \xrightarrow{\epsilon} & S \\
\downarrow t & & \downarrow \rho & & \downarrow -t \\
S & \xrightarrow{\sigma} & R^2 & \xrightarrow{\epsilon} & S
\end{array}$$

commutative, where $\rho = \begin{bmatrix} 0 & -t^2 \\ -1 & 0 \end{bmatrix}$. Let $A = (a_{ij})$ and $B = (b_{ij})$ be $m \times m$ matrices with entries in $k$ and let
\( \phi : (R^2)^m \to (R^2)^m \) be the homomorphism defined by

\[
\phi((x_j)) = (\sum_{j=1}^{m} a_{ij}x_j + \sum_{j=1}^{m} b_{ij}p(x_j)).
\]

Then we clearly have

Lemma 2. The diagram

\[
\begin{array}{c}
S^m \xrightarrow{\sigma^m} (R^2)^m \xrightarrow{\epsilon^m} S^m \\
\downarrow A+tB \downarrow \phi \downarrow A-tB \\
S^m \xrightarrow{\sigma^m} (R^2)^m \xrightarrow{\epsilon^m} S^m
\end{array}
\]

is commutative (here \( \sigma^m \) and \( \epsilon^m \) respectively denote the direct sum of \( m \) copies of \( \sigma \) and \( \epsilon \)).

Let \( m \) (= \( t^2S \)) denote the maximal ideal of \( R \) and let \( N \) be an \( R \)-submodule of \( S \) such that \( m \) is contained in \( N \). We put \( M = R^2/\sigma(N) \). Then

Proposition 3. \( M \) is an indecomposable maximal Buchsbaum \( R \)-module with \( H^0_m(M) = S/N \).

Proof. Considering the exact sequence

\[ 0 \to S/N \to M \to S \to 0, \]

we get \( H^0_m(M) = S/N \) as \( m.(S/N) = (0) \) and as \( S \) is Cohen-Macaulay; so \( M \) is a maximal Buchsbaum \( R \)-module. Assume that \( M = M_1 \oplus M_2 \) for some non-zero submodules \( M_1 \) and \( M_2 \). Then \( M_i \)'s are cyclic, since \( M \) is generated by two elements. If \( \dim_R M_i = 1 \) for \( i = 1, 2 \), the isomorphisms \( S \cong M/H^0_m(M) \cong M_1/H^0_m(M_1) \oplus M_2/H^0_m(M_2) \) claim that \( S \) is decomposable. Hence
\[ \text{dim}_R M_i = 0 \text{ for some } i, \text{ say } i = 2. \] Then \( M_2 \) is contained in \( H_m^0(M) \) and so \( S \) is a homomorphic image of \( M_1 \) — this is impossible, because \( M_1 \) is cyclic while \( S \) is not. Thus we see \( M \) is indecomposable.

We define

\[ M_1 = R^2/\mathfrak{a}(m), \quad M_2 = R^2/\mathfrak{a}(R), \quad \text{and} \quad M_3 = R^2/\mathfrak{a}(tS). \]

By Proposition 3 \( M_i \)'s are indecomposable maximal Buchsbaum \( R \)-modules and \( M_1 \not\cong M_i \) \( (i = 2, 3) \), since

\[ \text{dim}_k H_m^0(M_1) = 2 \quad (i = 1), \]

\[ = 1 \quad (i = 2, 3). \]

\( M_2 \) is of homological dimension 1 but \( M_3 \) is not; so \( M_2 \not\cong M_3 \). The goal of my lecture is the following

Theorem 4. \( M_1, M_2, M_3, S \) and \( R \) are the indecomposable maximal Buchsbaum \( R \)-modules.

To prove this theorem we need one more lemma 5, the proof of which is routine (use the induction on the size of matrices \( C \)) and shall be omitted.

Lemma 5. Let \( C \) be an \( m \times n \) matrix with entries in \( S/t^2S \). Then there exist an invertible \( m \times m \) matrix \( P \) with entries in \( S/t^2S \) and an invertible \( n \times n \) matrix \( Q \) with entries in \( k \) such that \( PCQ \) has the following form
Proof of Theorem 4.

Let $M$ be an indecomposable maximal Buchsbaum $R$-module such that $M \not\cong R$. Let $V = \frac{H^0(M)}{m}$. Then $mV = (0)$.

Claim. $V$ is contained in $mM$ and $M/V \cong S^m$ for some $m \geq 1$.

For let $W$ be the intersection of $V$ and $mM$ and write $V = W \cap W'$. Then $W' \cap mM = (0)$ and we have an embedding $W' + M \to M/mM$, which naturally splits. Hence $W' = (0)$ as $M$ is indecomposable and thus $V$ is contained in $mM$. Since $M/V$ is Cohen-Macaulay, the second assertion is clear.

By the above claim we get a commutative diagram

\[
\begin{array}{cccccc}
0 & \to & R^m & \to & (R^2)^m & \to & M \to 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \to & S^m & \to & (R^2)^m & \to & S^m \to 0 \\
\end{array}
\]
with exact rows and columns. Here we consider \( N \) to be an \( R \)-submodule of \( S^m \) and the homomorphism \( i : N \to S^m \) to be the inclusion map. Hence \( mS^m \) is contained in \( N \), as \( V \cong S^m / N \).

Let \( \tau : S^m \to S^m / mS^m = (S/t^2S)^m \) denote the canonical epimorphism. We put \( U = \tau(N) \) and \( n = \dim_k U \). If \( n = 0 \), then \( N = mS^m \) and so \( M = (R^2/\sigma(m))^m \). Consequently, we get \( m = 1 \) and \( M = M_1 \).

Now suppose that \( n \geq 1 \) and let \( v_1, v_2, \ldots, v_n \) be a \( k \)-basis of \( U \). Let us apply Lemma 5 to the \( m \times n \) matrix \( C = (v_1, v_2, \ldots, v_n) \). Then Lemma 5 asserts that by some automorphism \( \varphi \) of \( (S/t^2S)^m \), \( U \) is mapped onto the \( k \)-subspace \( U' \) which is spanned by the columns of an \( m \times n \) matrix of the following form:

\[
\begin{bmatrix}
1 & \ldots & 0 & \ldots & t \\
\vdots & & \vdots & & \vdots \\
0 & \ldots & 1 & \ldots & 0 \\
1 & \ldots & 0 & \ldots & 0 \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \ldots & 0 & \ldots & 0
\end{bmatrix}
\mod t^2S.
\]

Let \( L \) be the \( R \)-submodule of \( S^m \) generated by the columns of the above matrix \((#)\) and put \( N' = mS^m + L \). Then clearly \( U' = \tau(N') \).

We write \( P = A + tB \mod t^2S \) with \( m \times m \) matrices \( A \)
and $B$ with entries in $k$. Then since the following diagram

$$
\begin{array}{ccc}
S^m & \xrightarrow{\tau} & (S/t^2S)^m \\
\downarrow A+tB & & \downarrow P \\
S^m & \xrightarrow{\tau} & (S/t^2S)^m
\end{array}
$$

is commutative and since $U'=\tau(N')$, we get that $N'=(A+tB)N$.

Let us now recall the diagram in Lemma 2:

$$
\begin{array}{c}
0 \rightarrow S^m \xrightarrow{\sigma} (R^2)^m \xrightarrow{\epsilon} S^m \rightarrow 0 \\
\downarrow A+tB \quad \downarrow \phi \quad \downarrow A-tB \\
0 \rightarrow S^m \xrightarrow{\sigma} (R^2)^m \xrightarrow{\epsilon} S^m \rightarrow 0
\end{array}
$$

Then as the rows of this diagram are exact and as both the matrices $A+tB$ and $A-tB$ are invertible, the middle $\phi$ has to be an isomorphism whence, via $\phi$, we find that

$$
M = (R^2)^m/\sigma^m(N) \\
\cong (R^2)^m/\sigma^m(N').
$$

Consequently we may assume that $N=N'$. The condition that $M$ is indecomposable now causes a very tight restriction on the form of the matrix (#) above. We readily see that $m=1$ and the matrix (#) must be one of

$$(1\ t),\ (1)\ \text{and}\ (t).$$

Thus $M=R^2/\sigma(S)\ (=S)$, $M=R^2/\sigma(R)\ (=M_2)$, or $M=R^2/\sigma(tS)\ (=M_3)$ as claimed. This completes the proof of Theorem 4.

References


