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<td>Goto, Shiro</td>
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Kyoto University
An example of one-dimensional Cohen-Macaulay local rings that possess only finitely many isomorphism classes of indecomposable maximal Buchsbaum modules

Shiro Goto (Nihon University)
後藤四郎（日大・文理）

In our paper [GN] with K. Nishida we proved the following

Theorem 1. Let \( P = k[[X_1, X_2, \ldots, X_n]] \) be a formal power series ring over an algebraically closed field \( k \) of char \( k \neq 2 \). Let \( R = P/I \), where \( I \) is an ideal of \( P \) and suppose \( \dim R = d \geq 2 \). Then the following two conditions are equivalent.

1. \( R \) is a regular local ring.
2. \( R \) is a Cohen-Macaulay ring that possesses only finitely many isomorphism classes of indecomposable maximal Buchsbaum modules.

When this is the case, the syzygy modules of the residue class field \( k \) of \( R \) are the representatives of indecomposable maximal Buchsbaum modules and so there are exactly \( d \) non-isomorphic indecomposable maximal Buchsbaum modules over \( R \).

Our contribution in the above theorem is the implication (2) \( \Rightarrow \) (1). The last assertion and the implication (1) \( \Rightarrow \) (2) are due to [G] (see also [EG, Theorem 3.2]), where some consequences of the result are discussed too.

I would like to note here that the assumption \( \dim R \geq 2 \) in Theorem 1 is not superfluous. When \( \dim R = 1 \), maximal Buchsbaum \( R \)-modules \( M \) are characterized by the condition that

\[
\dim R^N = 1 \quad \text{and} \quad \mathfrak{m} \cdot H^0_m(M) = (0)
\]
(here \( H^i_\mathfrak{m}(\cdot) \) denotes the \( i \) th local cohomology functor of \( R \) relative to the maximal ideal \( \mathfrak{m} \) of \( R \)). This condition (is of course not too much weak but) seems not quite strong. Nevertheless in some sense surprisingly, there exist such Cohen-Macaulay local rings \( R \) of \( \dim R = 1 \) that are non-regular but possess only finitely many isomorphism classes of indecomposable maximal Buchsbaum modules. In my lecture I will explore the typical example \( R = k[[t^2, t^3]] \).

Now let \( k \) be a field and \( S = k[[t]] \) a formal power series ring over \( k \). We put \( R = k[[t^2, t^3]] \). Then \( R \) and \( S \) are the only indecomposable maximal Cohen-Macaulay \( R \)-modules (cf. [H, Satz 1.6]) and the \( R \)-module \( S \) has a resolution of the following form

\[
\cdots \rightarrow R^2 \rightarrow R^2 \rightarrow R^2 \rightarrow S \rightarrow 0,
\]

\[
\begin{pmatrix}
\alpha & \beta \\
\gamma & \delta
\end{pmatrix}
\]

where \( \varepsilon \left( \begin{pmatrix}
\alpha \\
\beta
\end{pmatrix} \right) = \alpha + \beta t \). Therefore we have an embedding \( \sigma : S \rightarrow R^2 \) which sends 1 (resp. \( t \) ) to \( \begin{pmatrix}
t^3 \\
t^4
\end{pmatrix} \) (resp. \( \begin{pmatrix}
t^3 \\
t^4
\end{pmatrix} \) ) and makes the diagram

\[
\begin{array}{ccc}
S & \xrightarrow{\sigma} & R^2 & \xrightarrow{\varepsilon} & S \\
\downarrow t & & \downarrow \rho & & \downarrow -t \\
S & \xrightarrow{\sigma} & R^2 & \xrightarrow{\varepsilon} & S
\end{array}
\]

commutative, where \( \rho = \begin{pmatrix}
0 & -t^2 \\
-1 & 0
\end{pmatrix} \). Let \( A = (a_{ij}) \) and \( B = (b_{ij}) \) be \( m \times m \) matrices with entries in \( k \) and let
\[ \phi : (R^2)^m \to (R^2)^m \] be the homomorphism defined by
\[
\phi((x_i)) = \left( \sum_{j=1}^{m} a_{ij} x_j + \sum_{j=1}^{m} b_{ij} \rho(x_j) \right).
\]
Then we clearly have

**Lemma 2.** The diagram

\[
\begin{array}{ccc}
S^m & \xrightarrow{\sigma^m} & (R^2)^m \\
\downarrow A+tB & & \downarrow \phi \\
S^m & \xrightarrow{\epsilon^m} & S^m
\end{array}
\]

is commutative (here \( \sigma^m \) and \( \epsilon^m \) respectively denote the direct sum of \( m \) copies of \( \sigma \) and \( \epsilon \)).

Let \( m \) (\( = t^2 S \)) denote the maximal ideal of \( R \) and let \( N \) be an \( R \)-submodule of \( S \) such that \( m \) is contained in \( N \). We put \( M = R^2/\sigma(N) \). Then

**Proposition 3.** \( M \) is an indecomposable maximal Buchsbaum \( R \)-module with \( H^0_m(M) = S/N \).

**Proof.** Considering the exact sequence

\[ 0 \to S/N \to M \to S \to 0, \]
we get \( H^0_m(M) = S/N \) as \( m.(S/N) = (0) \) and as \( S \) is Cohen-Macaulay; so \( M \) is a maximal Buchsbaum \( R \)-module. Assume that \( M = M_1 \oplus M_2 \) for some non-zero submodules \( M_1 \) and \( M_2 \). Then \( M_1 \)'s are cyclic, since \( M \) is generated by two elements. If \( \dim R M_i = 1 \) for \( i = 1, 2 \), the isomorphisms \( S \cong M/H^0_m(M) \cong M_1/H^0_m(M_1) \oplus M_2/H^0_m(M_2) \) claim that \( S \) is decomposable. Hence

\[ \square \]
dim_{R}M_{i} = 0 \text{ for some } i, \text{ say } i = 2. \text{ Then } M_{2} \text{ is contained in } H_{m}(M) \text{ and so } S \text{ is a homomorphic image of } M_{i} \text{— this is impossible, because } M_{i} \text{ is cyclic while } S \text{ is not. Thus we see } M \text{ is indecomposable.}

We define

\[ M_{1} = R^{2}/\sigma(m) \quad M_{2} = R^{2}/\sigma(R) \quad \text{and } M_{3} = R^{2}/\sigma(tS) \quad \text{.} \]

By Proposition 3 M_{i}'s are indecomposable maximal Buchsbaum R-modules and \( M_{1} \neq M_{i} \quad (i = 2, \ 3) \), since

\[ \dim_{k}H_{m}^{0}(M_{i}) = \begin{cases} 2 & (i = 1) \\ 1 & (i = 2, \ 3) \end{cases} \]

\( M_{2} \) is of homological dimension 1 but \( M_{3} \) is not; so \( M_{2} \neq M_{3} \).

The goal of my lecture is the following

Theorem 4. \( M_{i}, M_{2}, M_{3}, S \) and \( R \) are the indecomposable maximal Buchsbaum R-modules.

To prove this theorem we need one more lemma 5, the proof of which is routine (use the induction on the size of matrices \( C \)) and shall be omitted.

Lemma 5. Let \( C \) be an \( m \times n \) matrix with entries in \( S/t^{2}S \).

Then there exist an invertible \( m \times m \) matrix \( P \) with entries in \( S/t^{2}S \) and an invertible \( n \times n \) matrix \( Q \) with entries in \( k \) such that \( PCQ \) has the following form
Proof of Theorem 4.

Let $M$ be an indecomposable maximal Buchsbaum $R$-module such that $M \not\cong R$. Let $V = \mathcal{H}^0_\mathfrak{m}(M)$. Then $\mathfrak{m}V = (0)$.

Claim. $V$ is contained in $\mathfrak{m}M$ and $M/V \cong S^m$ for some $m \geq 1$.

For let $W$ be the intersection of $V$ and $\mathfrak{m}M$ and write $V = W \oplus W'$. Then $W' \cap \mathfrak{m}M = (0)$ and we have an embedding $W' \to M \to M/\mathfrak{m}M$, which naturally splits. Hence $W' = (0)$ as $M$ is indecomposable and thus $V$ is contained in $\mathfrak{m}M$. Since $M/V$ is Cohen-Macaulay, the second assertion is clear.

By the above claim we get a commutative diagram

\[
\begin{array}{ccccccc}
0 & \rightarrow & \mathfrak{m} & \rightarrow & (R^2)^m & \rightarrow & M & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & S^m & \rightarrow & (R^2)^m & \rightarrow & S^m & \rightarrow & 0 \\
\end{array}
\]
with exact rows and columns. Here we consider $N$ to be an $R$-submodule of $S^m$ and the homomorphism $i : N \to S^m$ to be the inclusion map. Hence $mS^m$ is contained in $N$, as $V \cong S^m/N$. Let $\tau : S^m \to S^m/mS^m = (S/t^2S)^m$ denote the canonical epimorphism. We put $U = \tau(N)$ and $n = \dim_k U$. If $n = 0$, then $N = mS^m$ and so $M = (R^2/\sigma(m))^m$. Consequently, we get $m = 1$ and $M = M_1$.

Now suppose that $n \geq 1$ and let $v_1, v_2, \ldots, v_n$ be a $k$-basis of $U$. Let us apply Lemma 5 to the $m \times n$ matrix $C = (v_1, v_2, \ldots, v_n)$. Then Lemma 5 asserts that by some automorphism $P$ of $(S/t^2S)^m$, $U$ is mapped onto the $k$-subspace $U'$ which is spanned by the columns of an $m \times n$ matrix of the following form:

$$
\begin{pmatrix}
1 & & & t \\
& 0 & & 0 \\
& & \ddots & \vdots \\
& & & 0 \\
0 & & & 0 \\
& & & 0 \\
& & & 0 \\
\end{pmatrix}
\ \text{mod } t^2S.
$$

Let $L$ be the $R$-submodule of $S^m$ generated by the columns of the above matrix $(\#)$ and put $N' = mS^m + L$. Then clearly $U' = \tau(N')$.

We write $P = A + tB \ \text{mod } t^2S$ with $m \times m$ matrices $A$.
and \( B \) with entries in \( k \). Then since the following diagram

\[
\begin{array}{ccc}
S^m & \xrightarrow{\tau} & (S/t^2S)^m \\
\downarrow A+tB & & \downarrow P \\
S^m & \xrightarrow{\tau} & (S/t^2S)^m
\end{array}
\]

is commutative and since \( U' = \tau(N') \), we get that \( N' = (A+tB)N \).

Let us now recall the diagram in Lemma 2:

\[
0 \rightarrow S^m \xrightarrow{\sigma} (R^2)^m \xrightarrow{\varepsilon} S^m \rightarrow 0
\]

\[
\downarrow A+tB \quad \downarrow \phi \quad \downarrow A-tB
\]

\[
0 \rightarrow S^m \xrightarrow{\sigma} (R^2)^m \xrightarrow{\varepsilon} S^m \rightarrow 0
\]

Then as the rows of this diagram are exact and as both the matrices \( A+tB \) and \( A-tB \) are invertible, the middle \( \phi \) has to be an isomorphism whence, via \( \phi \), we find that

\[
M = (R^2)^m/\sigma^m(N)
\]

\[
\cong (R^2)^m/\sigma^m(N')
\]

Consequently we may assume that \( N = N' \). The condition that \( M \) is indecomposable now causes a very tight restriction on the form of the matrix (\#) above. We readily see that \( m = 1 \) and the matrix (\#) must be one of

\[
(1 \ t), \ (1) \ \text{and} \ (t).
\]

Thus \( M = R^2/\sigma(S) \ (= S) \), \( M = R^2/\sigma(R) \ (= M_2) \), or \( M = R^2/\sigma(tS) \ (= M_3) \) as claimed. This completes the proof of Theorem 4.

References


