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Baum-Connes Conjectures and their Applications

Hiroshi TAKAI

Department of Mathematics,
Tokyo Metropolitan University,
Fukazawa, Setagaya, Tokyo, JAPAN.

§1 Introduction   Interrelationship among different areas in mathematics gives a plenty of beneficence to themselves as a number of results support its justification. Concerning both geometry and analysis especially, there is no doubt that Atiyah-Singer index theory has a crucial role to develope their fields simultaneously.

Recently, Connes[3] has initiated a new index theory for both dynamical systems and foliated manifolds, which is really useful to cases with pathological ambient spaces whereas the index theory of Atiyah-Singer is no longer applicable to them. The main idea of his theory is based on K-theory of both C^*-algebras and twisted vector bundles. Its validity can be found in many manuscripts due to Connes, Kasparov, Pimsner-Voiculescu and Rosenberg etc. especially concerning differential dynamics and foliated manifolds, Baum-Connes [1] has conjectured the existance of a K-theoretic index formula between geometric and analytic K-theory, which may be considered as a ultimate one of a generalization of Atiyah-Singer index theorem. It has a quite important meaning involved as a central ingredient to develop topology, differential geometry and C^*-algebras etc. Precisely, their conjecture says that the geometric K-group is isomorphic to the analytic one under the K-index mapping for foliated manifolds or differential dynamical systems. If it is affirmative, as corollaries are deduced the conjectures due to Novikov, Gromov-Lawson-Rosenberg and Kadison etc in topology, differential geometry and C^*-algebras respectively. As a matter of fact, no theorem from general sights has been verified until now although various exampls supporing the conjecture have been constructed by several peoples.

In this report, we shall state the construction, the results obtained and some applications of the Baum-Connes conjecture, and we shall especially illustrate its affirmation for generalized Anosov foliations on infra-homogeneous spaces. The basic refferences are due to Baum-Connes[1],[2] , Connes[3], Kasparov[6] and Rosenberg[9]~[11].

§2 Construction Let (M,F) be a foliated smooth manifold and G the holonomy groupoid of F. Let $\Omega^{1/2}$ be the half density bundle over G tangential to $F\otimes F$ and $C_c(\Omega^{1/2})$ the $*$-algebra consisting of all continuous sections of $\Omega^{1/2}$ over G with compact support by the following algebraic operations:

$$(fg)(\gamma) = \int_{\gamma=\gamma_1\gamma_2} f(\gamma_1)g(\gamma_2)$$
$$f^*(\gamma) = \overline{f(\gamma^{-1})}$$

for all f,g in $C_c(\Omega^{1/2})$. Given any x in M, let H_x be the Hilbert space consisting of all L^2-sections of $\Omega^{1/2}$ over G. Let us consider the $*$-representation π_x of $C_c(\Omega^{1/2})$ on H_x defined by

$$(\pi_x(f)\xi)(\gamma) = \int_{\gamma=\gamma_1\gamma_2} f(\gamma_1)\xi(\gamma_2)$$

for all f in $C_c(\Omega^{1/2})$ and ξ in H_x. Applying π_x, a C^*-norm $\|\cdot\|$ on $C_c(\Omega^{1/2})$ can be defined by

$$\|f\| = \sup_{x\in M} \|\pi_x(f)\|$$

for all f in $C_c(\Omega^{1/2})$. We denote by $C_r^*(M,F)$ the completion of $C_c(\Omega^{1/2})$ with respect to $\|\cdot\|$, which is called a foliation C^*-algebra associated to (M,F).

We now consider the K-theory $K_a(M,F) = K(C_r^*(M,F))$ of $C_r^*(M,F)$, which is called the analytic K-theory of (M,F). On the other hand, the following construction is considered as a purely geometric one of K-theory associated to (M,F): Let X be a proper G-manifold and $\tilde{\nu}^*$ the dual bundle of the normal bundle $\tilde{\nu}$ of the foliation of X determined by the G-orbits. Let ρ be the canonical G-equivariant mapping from X to M and $\rho^*(\nu^*)$ the pull back of the dual bundle ν^* of the normal bundle ν of F. We consider a pair (X,ξ) of X and a G-vector bundle ξ over $\tilde{\nu}^*\oplus\rho^*(\nu^*)$, which is called a K-cocycle of (M,F). Let $\Gamma(M,F)$ be the set of all K-cocycles of (M,F). We then introduce an equivalence relation $\sim$ on $\Gamma(M,F)$ by the following fashion:

$(X_1,\xi_1) \sim (X_2,\xi_2)$ if and only if there exist a proper G-manifold X and G-mappings ϕ_j from X_j to X such that

$$\text{(i)} \quad \rho_j = \rho\cdot\phi_j \quad \text{and} \quad \text{(ii)} \quad \phi_1!(\xi_1) = \phi_2!(\xi_2) \quad ,$$

where ρ,ρ_j are the canonical G-mappings from X,X_j to M respectively, and $\phi_j!$ mean the Gysin mappings from G-vector bundles over $\tilde{\nu}_j^*\oplus\rho_j^*(\nu^*)$ to those over $\tilde{\nu}^*\oplus\rho^*(\nu^*)$. Denote by $K_g(M,F) = \Gamma(M,F)/\sim$ the set of all equivalent classes in $\Gamma(M,F)$ with respect to $\sim$. Then it is an abelian group equipped with the disjoint union of G-vector bundles. We call it the geometric K-theory of (M,F).

In what follows, we shall explain the K-index mapping μ from $K_g(M,F)$ to $K_a(M,F)$. Given any (X,ξ) in $\Gamma(M,F)$, let us consider the G-mapping j from X to $X\times M$ defined by $j(x) = (x,\rho(x)), x\in X$. Then it implies that $\rho = \pi\cdot j$ where π is the projection from $X\times M$ to M. Let $\tilde{j}$ be the canonical G-mapping from $\tilde{\nu}_X^*\oplus\rho^*(\nu^*)$ to $\tilde{\nu}_{X\times M}^*\oplus\pi^*(\nu^*)$. Then it is a projection whose fiber has a G-equivariant spin^c structure. By the Thom-Gysin's theorem, the group generated by all G-vector bundles over $\tilde{\nu}_X^*\oplus\rho^*(\nu^*)$ is isomorphic to that by those over

$\tilde{\nu}^*_{X\times M}\oplus\pi^*(\nu^*)$ under the Gysin'smapping $\mathfrak{J}!$ of $\mathfrak{J}$. Suppose ξ is a G-vector bundle over $\tilde{\nu}^*_X\oplus\rho^*(\nu^*)$ and put $\tilde{\xi} = \mathfrak{J}!(\xi)$. Then it is a G-vector bundle over $\tilde{\nu}^*_{X\times M}\oplus\pi^*(\nu^*)$ which is G-isomorphic to ξ. Let τ τ_m be the cotangent bundle $T^*(\pi^{-1}(m))$ of $\pi^{-1}(m)$ $(m\in M)$. Since π is a submersion, the G-space $\tilde{\nu}^*_{X\times M}\oplus\pi^*(\nu^*)$ is the total space of the bundle over τ under the canonical projection $\tilde{\pi}$ whose fibers are $\nu^*\oplus\nu^*$. Therefore, $\tilde{\xi}$ can be considered as a G-bundle over τ under the Gysin mapping $\tilde{\pi}!$ of $\tilde{\pi}$. Let $\tilde{\xi}_m = \tilde{\xi}\,\|_{\tau_m}$ be the restriction of $\tilde{\xi}$ over τ_m. By the definition of $\tilde{\xi}_m$, there exist elliptic differential operators D_m on $\pi^{-1}(m)$ such that $\tilde{\xi}_m$ is the symbol $\sigma(D_m)$ of D_m. Let D be the G-equivariant field of $D_m, m\in M$. Then it is a G-invariant differetial operator on $X\times M$ such that (i) D_m are elliptic on $\pi^{-1}(m)$, and (ii) $\tilde{\xi}$ is the symbol $\sigma(D)$ of D. We now take the K-theoretic index ind D of D in $K_a(M,F)$ as follows:

$$\text{ind } D = [\text{Ker } D] - [\text{Coker } D]$$

where $[\cdot]$ means a $C^*_r(M,F)$-module generated by $\cdot$. Put $\mu(X,\xi) = \text{ind } D$. Then μ depends only on the equivalence class of (X,ξ). Therefore it determines a homomorphism from $K_g(M,F)$ to $K_a(M,F)$. We now state the first Baum-Connes conjecture as follows:

Baum-Connes conjecture I. Given any foliated manifold (M,F), the K-index mapping μ is an isomorphism from $K_g(M,F)$ to $K_a(M,F)$.

On the other hand, suppose (M,G,α) is a differential dynamical system where α is free. Then the family F consisting of all G-orbits becomes a foliation of M, and its C^*-algebra $C^*_r(M,F)$ is nothing but the C^*-crossed product $C(M)\times_\alpha G$ of $C(M)$ by α. Thus it follows that $K_a(M,F) = K(C(M)\times_\alpha G)$. Moreover, $K_g(M,F)$ is isomorphic to the abelian group $K_g(M,G)$ defined in the following manner: Let X be a proper G-manifold and π a G-mapping from X to M. Consider the set $\Gamma(M,G)$ of all triples (X,ξ,π) for G-vector bundles ξ over $T^*(X)\oplus\pi^*(T^*(M))$.

Then it has an equivalence relation as before. In other words, $(X_1,\xi_1,\pi_1) \sim (X_2,\xi_2,\pi_2)$ if and only if there exist a proper G-manifold X and G-mappings π, ρ_j such that

$$\text{(i)} \quad \pi_j = \pi \cdot \rho_j \quad \text{and} \quad \text{(ii)} \quad \rho_1!(\xi_1) = \rho_2!(\xi_2) \quad ,$$

where $\rho_j!$ are the Gysin mappings from the groups generated by all G- vector bundles over $T^*(X_j)\oplus\pi_j^*(T^*(M))$ to the group generated by those over $T^*(X)\oplus\pi^*(T^*(M))$. Denote by $K_g(M,G)$ the set of all equivalence classes in $\Gamma(M,G)$ with respect to $\sim$. Then it is an abelian group by the canonical sum. According to the conjecture I, the next one is also due to Baum-Connes[1]:

Baum-Connes conjecture II. Given any differential dynamical system (M,G,α), the K-index mapping μ is an isomorphism from $K_g(M,G)$ to $K_a(M,G)$ where the latter is defined as $K(C(M)\times_\alpha G)$.

Remark. Let BG be the classifying space of G and EG the total space of the universal principal G-bundle over BG. Let us denote by τ the vector bundle over BG whose fibers are $T^*(M)$. If we define $K^\tau((EG\times M)/G)$ by the K-group $K(B\tau/S\tau)$ of the quotient space $B\tau/S\tau$ of the ball bundle $B\tau$ of τ by its sphere bundle $S\tau$, then there exists a homomorphism δ from $K^\tau((EG\times M)/G)$ to $K_g(M,G)$ such that $\mu\cdot\delta$ is the Kasparov β-mapping if M is one point. Moreover, if G is discrete, then δ is $\mathbb{Q}$-injective. If G is torsion-free, then δ is bijective.

If the conjectures I and II are affirmative, then so are those due to Novikov, Gromov-Lawson-Rosenberg and Kadison in topology, differential geometry and C^*-algebra theory respectively. We shall explain them succeedingly:

Let M be a closed oriented manifold, and let p_j be the rational j-Pontrjagin class of M in $H^{4j}(M,\mathbb{Q})$. Namely, $p_j = (-1)^j c_{2j}$ where c_j is the rational j-Chern class of $T(M)\otimes_{\mathbb{R}}\mathbb{C}$. As a known fact, it is a topological invariant due to Novikov whereas the integral class is no longer topologically invariant by Milnor. Moreover, p_j are without homotopy invariance by Tamura, Shimada and Thom though they are homotopy invariant for ambient manifolds with nonpositive curvature. Let π be the fundamental group of M, and let us consider the total Hirzebruch L-class defined by

$$L(M) = \Sigma_{k\geq 0} L_k(M) = 1 + p_1/3 + 1/45(7p_2 - p_1^2) + \cdots .$$

By definition, the <u>higher</u> <u>signature</u> $\sigma_x(M)$ of M for $x \in H^*(B\pi,\mathbb{Q})$ is formulted as

$$\sigma_x(M) = \langle L(M)vf^*(x), [M] \rangle$$

where f is the classifying mapping from M to $B\pi$, f^* is the lift of f from $H^*(B\pi,\mathbb{Q})$ to $H^*(M,\mathbb{Q})$ and [M] is the fundamental homology class of M. We then state the Novikov conjecture in the following:

<u>Novikov</u> <u>conjecture</u>. Given any oriented closed manifold M and $x \in H^*(B\pi,\mathbb{Q})$, the higher signature $\sigma_x(M)$ is a homotopy invariant of M.

In fact, if the Baum-Connes conjecture II is affirmative for M = pt, so is the Novikov conjecture. We shall see it briefly in what follows. It suffices to show that $f_*(L(M)^\wedge)$ in $H_*(B\pi,\mathbb{Q})$ is a homotopy invaruant of M, where $L(M)^\wedge$ is the Poincare dual $L(M)\frown[M]$ of L(M) in $H_*(M,\mathbb{Q})$. We may assume that dim M is even if necessary replacing M by $M \times S^1$. Let $\Lambda^*(M)$ be the Grassmann algebra of $T^*(M)$. For any $[\xi]$ in $K^0(M)$, consider the signature operator $D_\xi = d_\xi + d_\xi^*$ on the tensor bundle $\Lambda^*(M)\otimes \xi$ where d_ξ is the tensor product $d \otimes 1$ of the exterior derivative d of M and the trivial mapping 1 of ξ. Since D_ξ is elliptic, we can define the analytic index $\mathrm{ind}_a D_\xi$ of D_ξ, which is nothing more than the Kasparov product $[\xi]\otimes_M[D] \in \mathbb{Z}$ of $[\xi]$ and [D] for the signature operator D on $\Lambda^*(M)$, where the latter is defined as follows: let $L^2(\Lambda^*(M))$ be the Hilbert space consisting of all L^2-sections of $\Lambda^*(M)$ and λ the canonical representation of C(M) on $L^2(\Lambda^*(M))$, then $[D] = [(L^2(\Lambda^*(M)), \lambda, D(1+D^2)^{-1/2})]$ in $KK(M,\mathbb{C})$. Denote by $\mathrm{ind}_g D_\xi$ the geometric index of D_ξ. Then it is equal to $\langle L(M)\vee ch([\xi]), [M] \rangle$ where ch is the Chern character from $K^0(M)$ to $H^{ev}(M,\mathbb{Q})$. By Atiyah-Singer index theorem, it implies that

$$\mathrm{ind}_a D_\xi = \mathrm{ind}_g D_\xi$$

which means that

$$[\xi]\otimes_M[D] = \langle L(M)\vee ch([\xi]), [M] \rangle$$

for all $[\xi]$ in $K^0(M)$. Since $ch_{\mathbb{Q}}$ is an isomorphism from $K^0(M)\otimes_{\mathbb{Z}} \mathbb{Q}$ to $H^{ev}(M,\mathbb{Q})$, it follows that

$$ch_{\mathbb{Q}}^{-1}(f^*(x)) \otimes_M [D] = \langle f^*(x), L(M)^\wedge \rangle$$

for all x in $H^{ev}(B\pi,\mathbb{Q})$. As a well-known fact, it follows that

$$ch_{\mathbb{Q}}^{-1}\cdot f^* = f^* \cdot ch_{\mathbb{Q}}^{-1} \quad \text{and} \quad f^*(a)\otimes_Q b = a \otimes_R f_*(b)$$

for a in $KK(P,R)$ and b in $KK(Q,R)$ where f is a continuous mapping from Q to R and f^*, f_* are the lifts of f from $KK(P,R)$, $KK(Q,R)$ to $KK(P,Q)$, $KK(R,R)$ respectively. We then see that

$$ch_{\mathbb{Q}}^{-1}(x) \otimes_{B\pi} f_*([D]) = \langle x , f_*(L(M)^\wedge) \rangle$$

for all x in $H^*(B\pi,\mathbb{Q})$. Thus, the homotopy invariance of $f_*(L(M)^\wedge)$ is equivalent to that of $f_*([D])$ in $K_0(B\pi)\otimes_{\mathbb{Z}} \mathbb{Q} = \underline{\lim}_{X\subset B\pi} K_0(X)\otimes_{\mathbb{Z}} \mathbb{C}$.

Let us now define the Kasparov homomorphism β from $K_*(B\pi)$ to $K_*(C_r^*(\pi))$ by the following way: Given a compact subset X of $B\pi$, put $\tilde{X} = i_X^*(E\pi)$ for the natural imbedding i_X from X to $B\pi$. Then it is a regular covering space with the property that $X = \tilde{X}/\pi$. Let E_X be the set of all continuous mappings f from $\tilde{X}$ to $C_r^*(\pi)$ such that $f(gx) = \lambda(g)f(x)$ for all g in π and x in $\tilde{X}$. It becomes a Hilbert $C(X)\otimes C_r^*(\pi)$-module equipped with

$$(fa)(\tilde{x}) = f(\tilde{x})a\cdot p(\tilde{x}) \quad \text{and} \quad \langle f_1 \| f_2\rangle\cdot p(\tilde{x}) = f_1(\tilde{x})^* f_2(\tilde{x})$$

for all f, $f_j \in E_X$, $a \in C(X)\otimes C_r^*(\pi)$ and $\tilde{x} \in \tilde{X}$, where p means the projection from $\tilde{X}$ to X. We then denote by $[E_X]$ the homotopy class of $(E_X,0)$ which belongs to $KK(\mathbb{C},C(X)\otimes C_r^*(\pi)) = K_0(C(X)\otimes C_r^*(\pi))$. Let

$\beta_X(\xi) = [E_X] \otimes_X \xi$ for ξ in $K_0(X)$. Then it is a homomorphism from $K_0(X)$ to $K^0(C^*_r(\pi))$. Put $\beta = \underline{\lim}_{X\subset B\pi}\, \beta_X$. Then it is a homomorphism from $K_0(B\pi)$ to $K_0(C^*_r(\pi))$ such that $\beta_X = \beta \mid_{K_0(X)}$. By Mischenko and Kasparov[6], the image $\beta_{\mathbb{Q}}(f_*([D]))$ of $f_*([D])$ under $\beta_{\mathbb{Q}} = \beta \otimes 1_{\mathbb{Q}}$ is a homotopy invariant of M in $K_0(B\pi)\otimes_{\mathbb{Z}} \mathbb{Q}$. Therefore, if $\beta_{\mathbb{Q}}$is a monomorphism from $K_0(B\pi)\otimes_{\mathbb{Z}} \mathbb{Q}$ to $K_0(C^*_r(\pi))\otimes_{\mathbb{Z}} \mathbb{Q}$, then $f_*([D])$ is also homotopically invariant of M. Remembering the definition of β,δ and μ, one can see that $\beta = \mu\cdot\delta$. Henceforth, if the conjecture II is affirmative or $\mu_{\mathbb{Q}}$ is injective in more general, then so is $\beta_{\mathbb{Q}}$. This implies that Novikov conjecture is affirmative (cf:[6],[9]~[11]).

We shall next state the Gromov-Lawson-Rosenberg in differential topology in connection with the Baum-Connes conjecture II. Let M be an oriented closed spin manifold and π its fundamental group. Given the classifying mapping f from M to $B\pi$, consider the lift f^* of f from $H^*(B\pi,\mathbb{Q})$ to $H^*(M,\mathbb{Q})$. Let us define the Hirzebruch $\hat{A}$-class $\hat{A}(M)$ of M by

$$\hat{A}(M) = 1 - p_1/24 - 1/32\cdot 45(p_2 - 7/4\, p_1^2) - \cdots.$$

where p_j are the rational Pontrjagin classes of M. We now consider the <u>higher</u> <u>$\hat{A}$-genus</u> $\rho_x(M)$ of M for any $x \in H^*(B\pi,\mathbb{Q})$ as follows:

$$\rho_x(M) = \langle \hat{A}(M)\vee f^*(x),\ [M] \rangle$$

where [M] is the fundamental homology class of M. It is of cource differentially invariant of M. Let $\kappa_m(M)$ be the scalar curvature of M at $m \in M$, in other words

$$\kappa_m(M) = \Sigma_{i,j} \langle R(X_i,X_j)\, X_j \mid X_i \rangle_m$$

where $\{X_j\}$ is an orthonormal basis of $T_m(M)$ and R is the curvature tensor of M with respect to a Riemannian metric. The conjecture is given by the following fashion:

<u>Gromov</u>-<u>Lawson</u>-<u>Rosenberg</u> <u>conjecture</u>. Let M be a closed spin manifold. Suppose there exists a Riemannian metric of M for which

the scalar curvature κ is nonnegative and κ_m is positive for some m in M, the higher $\hat{A}$-genus $\rho_x(M)$ of M vanishes for all x in $H^*(B\pi,\mathbb{Q})$.

This conjecture is affirmative if the Kasparov mapping $\beta_{\mathbb{Q}}$ is injective, which is satisfied if the Baum-Connes conjectue II holds. In fact, let ξ be the flat $C_r^*(\pi)$-bundle over M. In other words, $\xi = \tilde{M} \times_\pi C_r^*(\pi)$ where $\tilde{M}$ is the universal covering space of M. One may assume that dim M is even by the same reason as before. Since M has a spin structure S, there exist half spinor bundles S^+, S^- of S. Let $C^\infty(S^+\otimes \xi)$, $C^\infty(S^-\otimes \xi)$ be the sets of all C^∞-sections of $S^+\otimes \xi$, $S^-\otimes \xi$ respectively. Denote by D^+ the Dirac operator from $C^\infty(S^+\otimes \xi)$ to $C^\infty(S^-\otimes \xi)$ with respect to the flat connection of ξ. Then there exists the conjugate operator D^- of D^+ from $C^\infty(S^-\otimes \xi)$ to $C^\infty(S^+\otimes \xi)$. We explain the Chern character ch(ξ) of ξ due to Miscenko-Solov'ev. Given a $C_r^*(\pi)$-bundle ξ over M, its fibers have the structure of finitely generated projective left $C_r^*(\pi)$-modules. Then the classes $[\xi]$ of ξ by stabe equivalence generate the K-group $K_0(C(M)\otimes C_r^*(\pi))$ of the C^*-tensor product $C(M)\otimes C_r^*(\pi)$ of C(M) and $C_r^*(\pi)$. Using the ordinally Chern character and the Kunneth formula, one obtains the Chern character ch($[\xi]$) of $[\xi]$ as a homomorphism from $K_0(C(M)\otimes C_r^*(\pi))$ to $H^{ev}(M,\mathbb{Q})\otimes K_0(C_r^*(\pi)) \oplus H^{od}(M,\mathbb{Q})\otimes K_1(C_r^*(\pi))$, which is actually an isomorphism modulo torsion. Since $S^+\otimes \xi$ and $S^-\otimes \xi$ are smooth $C_r^*(\pi)$-vector bundles over M, and D^+ is an elliptic bounded $C_r^*(\pi)$-operator from a Sobolev $C_r^*(\pi)$-module $H^*(S^+\otimes \xi)$ of $S^+\otimes \xi$ to that $H^*(S^-\otimes \xi)$ of $S^-\otimes \xi$, there exists a $C_r^*(\pi)$-compact operator C from $H^*(S^+\otimes \xi)$ to $H^*(S^-\otimes \xi)$ so that both $[\mathrm{Ker}(D^+ + C)]$ and $[\mathrm{Coker}(D^+ + C)]$ are finitely generated projective $C_r^*(\pi)$-modules. Therefore, one can define the $C_r^*(\pi)$-index $\mathrm{ind}_{C_r^*(\pi)} D^+$ of D^+ by

$$\mathrm{ind}_{C_r^*(\pi)} D^+ = [\mathrm{Ker}(D^+ + C)] - [\mathrm{Coker}(D^- + C)] .$$

It follows from Miscenko-Fomenko[14] that

$$\mathrm{ind}_{C_r^*(\pi)}\, D^+ = \langle\, ch\cdot\sigma(D^+)\vee Td(M), [T^*(M)]\,\rangle \quad \text{in} \quad K_*(C_r^*(\pi))\otimes_{\mathbb{Z}} \mathbb{Q},$$

where $Td(M)$ is the Todd class of M and $[T^*(M)]$ is the fundamental class of $T^*(M)$. Since M has a spin structure, there exists the Thom isomorphism Th from $H^*(M,\mathbb{Q})$ to $H_r^*(T^*(M),\mathbb{Q})$. It then follows that

$$ch([\xi])\vee\hat{A}(M) \;=\; Th^{-1}(ch\cdot\sigma(D^+)\vee Td(M)) \quad .$$

Therefore, one has that

$$\mathrm{ind}_{C_r^*(\pi)}\, D^+ \;=\; \langle\, ch([\xi])\vee\hat{A}(M),\, [M]\,\rangle \quad .$$

On the Sobolev $C_r^*(\pi)$-module $H^*(S^+\otimes \xi)$, the operator D^-D^+ satisfies the generalized Bochner-Weizenbeck formula:

$$D^-D^+ \;=\; \nabla^*\nabla \;+\; \kappa/4 \quad ,$$

where ∇ is the canonical flat connection of $S^+\otimes \xi$. Similarly, D^+D^- has the following equality:

$$D^+D^- \;=\; \nabla\nabla^* \;+\; \kappa/4 \quad .$$

By the assumption of κ, it implies due to Kazdan-Warner that there exist a Riemannian metric of M and a $c > 0$ such that $\kappa_m(M) \geq c1$ for all m in M. Thus, D^-D^+ and D^+D^- have bounded inverse operators, which means that $\mathrm{ind}_{C_r^*(\pi)}\, D^+ = 0$. Henceforth, one obtains that

$$\langle\, ch([\xi])\vee\hat{A}(M),\, [M]\,\rangle \;=\; 0 \quad .$$

By the definition of β and $\mathrm{ind}_a\, D_{\xi} = \mathrm{ind}_g\, D_{\xi}$, it follows that

$$\begin{aligned}
\beta_{\mathbb{Q}}(ch_{\mathbb{Q}}^{-1}\cdot f_*(\hat{A}(M)\wedge[M])) &= [\tilde{\xi}]\otimes_{B\pi} ch_{\mathbb{Q}}^{-1}\cdot f_*(\hat{A}(M)\wedge[M]) \\
&= \langle\, ch([\tilde{\xi}])\;,\; f_*(\hat{A}(M)\wedge[M]\,\rangle \\
&= \langle\, f^*\cdot ch([\tilde{\xi}])\vee\hat{A}(M)\;,\;[M]\,\rangle \quad ,
\end{aligned}$$

where $\tilde{\xi}$ is the universal $C_r^*(\pi)$-bundle over $B\pi$. Since ξ is the flat $C_r^*(\pi)$-bundle over M, it is the pull back $f^*(\tilde{\xi})$ of $\tilde{\xi}$ with respect to f. Therefore, it implies that

$$\begin{aligned}
\beta_{\mathbb{Q}}(ch_{\mathbb{Q}}^{-1}\cdot f_*(\hat{A}(M)\wedge[M])) &= \langle\, ch([\xi])\vee\hat{A}(M)\;,\;[M]\,\rangle \\
&= 0 \quad .
\end{aligned}$$

Suppose $\beta_{\mathbb{Q}}$ is injective, then one has that

$$ch_{\mathbb{Q}}^{-1}\cdot f_*(\hat{A}(M)\wedge[M]) \;=\; 0 \quad .$$

Since $ch_{\mathbb{Q}}$ is an isomorphism from $K_*(B\pi)\otimes_{\mathbb{Z}}\mathbb{Q}$ to $H_*(B\pi,\mathbb{Q})$, it implies that $f_*(\hat{A}(M)\frown[M]) = 0$. By the definition of $\rho_x(M)$, one concludes that

$$\begin{aligned}\rho_x(M) &= \langle \hat{A}(M)\vee f^*(x), [M] \rangle \\ &= \langle x , f_*(\hat{A}(M)\frown[M]) \rangle = 0\end{aligned}$$

for all x in $H^*(B\pi,\mathbb{Q})$.

Especially, if the Baum-Connes conjecture II is affirmative, then so is the Gromov-Lawson-Rosenberg conjecture. For instance, as $\rho_1(K^4) = 2$ for the K3-surface K^4, there is no Riemannian metric of K^4 which induces a positive scalar curvature.

As an application toward C^*-algebras, we shall state the generalized Kadison conjecture concerning the existance of nontrivial projections in group C^*-algebras:

<u>Generalized Kadison conjecture</u>. Suppose G is a torsion free discrete group, the reduced group C^*-algebra $C^*_r(G)$ of G has no nontrivial projections.

In fact, let us consider the geometric K-theory $K_g(\cdot,G)$ for M = pt. By the definition of the K-index mapping μ, given a $[(X,\xi)]$ in $K_g(\cdot,G)$, there exists a G-invariant elliptic differential operator D_ξ on X such that

$$\mu(X,\xi) = \mathrm{ind}_a\, D_\xi \quad \text{and} \quad \sigma(D_\xi) = \xi \quad .$$

As G is torsion free, it acts on X freely. By Atiyah, it follows that

$$tr_*(\mathrm{ind}_a\, D_\xi) \in \mathbb{Z}$$

where tr_* is the lift of the canonical normalized trace of $C^*_r(G)$ to $K_*(C^*_r(G))$. Suppose μ is onto, it implies that

$$tr_*(K_0(C^*_r(G)) \subset \mathbb{Z} \quad .$$

Therefore, $C^*_r(G)$ has no nontrivial projections.

Summing up the argument discused until here, we have the following observation:

Observation. Suppose the Baum-Connes conjecture II holds for one point manifold, then affirmative are all the conjectures of Novikov, Gromov-Lawson-Rosenberg and Kadison.

Remark. At the present stage, Kadison conjecture is solved affirmatively only for the free groups with finite generators due to Pimsner-Voiculescu.

§3 Miscellaneous results Let (A,G,α) be a C^*-dynamical system where G is simply connected solvable. By Iwasawa, G is the multi semidirect products of $\mathbb{R}$. Using the duality for C^*-crossed products, Connes has shown that $K_j(A\times_\alpha\mathbb{R})$ is isomorphic to $K_{j+1}(A)$ under the Thom isomorphism. Since crossed products are compatible with semidirect products, one obtains the following theorem:

Theorem 1. Let (A,G,α) be a C^*-dynamical system where G is simply connected solvable. Then $K_j(A\times G)$ is isomorphic to $K_{j+\dim G}(A)$ under the Thom isomorphism.

Given a differential dynamical system (M,G,ϕ) where G is simply connected solvable, it follows from Theorem 1 that $K_a(M,G)$ is equal to $K_{\dim G}(M)$ via the Thom isomorphism. On the other hand, since G has no torsion, one can show that $K_g(M,G)$ is isomorphic to $K(B\tau/S\tau)$ where τ is the $T^*(M)$-bundle over $(EG\times M)/G$, and $B\tau, S\tau$ are the ball, sphere bundles of τ respectively (cf;[1]). By the assumption of G, BG is homotopic to $\mathbb{R}^{\dim G}$. Therefore, it follows from Bott periodicity that $K_g(M,G) = K^{\dim G}(M)$. Combining it with the previous argument, one has the following proposition:

Proposition 2. Let (M,G,ϕ) be a differential dynamical system where G is simply connected solvable. Then the conjecture II holds for the triplet.

Suppose G is a compact Lie group, the conjecture II is naturally affirmative by virtue of Atiyah-Singer index theory:

<u>Proposition</u> 3. Let (M,G,ϕ) be a differential dynamical system where G is compact. Then the same conclusion holds as Proposition 2.

Due to the above propositions, we may restricts our interest to the case where G is a noncompact semisimple Lie group in the next stage. Let G be as above and let K be the maximal compact subgroup of G. Suppose G/K has a G-invariant spin^c-structure,it follows from Baum-Connes[1] that $K_g(M,G) = K_g^{\dim G/K}(M,K)$. By Proposition 3, it implies that $K_g(M,K) = K_a(M,K)$ up to the K-index mapping. Thus, it suffices to show that $K_a(M,G) = K_a^{\dim G/K}(M,K)$. The next result is one example having the equality:

<u>Proposition</u> 4. Let G be a connected Lie group and let K be the maximal compact subgroup of G with the property that G/K has a G-invariant spin^c-structure. If there exists an amenable normal subgroup H of G such that G/H is locally isomorphic to the finite product of $SO_0(n,1)$ and compact groups, then $K_a(M,G) = K_a^{\dim G/K}(M,K)$ (cf;[7]).

Especially, suppose M is one point, the conjecture II is proved affirmatively for more wider classes of G:

<u>Proposition</u> 5. Let G be the connected reductive Lie group and K, G/K as in Proposition 4. Then $K_a(\cdot,G) = K_a^{\dim G/K}(\cdot,K)$.

When G is a discrete group, there is no theorem to support the conjecture II affirmatively at the present stage. The only example that one knows is the following case:

<u>Observation</u> 6. $K_g(\cdot,SL(2,\mathbb{Z})) = K_a(\cdot,SL(2,\mathbb{Z}))$ ([15]).

In fact, the above result is deduced from the fact that $SL(2,\mathbb{Z})$ is the amalgamated product of $\mathbb{Z}_4$ and $\mathbb{Z}_6$ with respect to $\mathbb{Z}_2$. Since $SL(n,\mathbb{Z}), n\geq 3$ has no such fashion, one may ask the following question:

<u>Question</u> 1. Is it true that $K_g(\cdot, SL(n,\mathbb{Z})) = K_a(\cdot, SL(n,\mathbb{Z}))$ for all $n \geq 3$? More generally, suppose G is a discrete subgroup of a connected Lie group, can one show that $K_g(\cdot, G) = K_a(\cdot, G)$?

Concerning the conjecture I, we list up several examples to satisfy it affirmatively in what follows.

<u>Proposition</u> 7. The conjecture I holds affirmatively for the Reeb foliations of 2-torus or 3-sphere ([16]).

<u>Proposition</u> 8. The conjecture I holds affirmatively for the Anosov foliations of infra-homogeneous manifolds ([12]).

Suppose a manifold has an Anosov foliation, its rank is one automatically. The next example is the case where the conjecture I holds for a foliated manifold with an arbitrary rank:

<u>Proposition</u> 9. Given any $n \in \mathbb{N}$, there exists a foliated manifold (M_n, F_n) such that

(i) rank $M_n = n$ and (ii) $K_g(M_n, F_n) = K_a(M_n, F_n)$

(cf;Section 4).

The foliations cited above have nontrivial holonomy in general since the Godbillon-Vey invariant is nonzero in general. However, the next two cases are without holonomy:

<u>Proposition</u> 10. The conjecture I is true for all foliations of codimension one without hohonomy on any smooth manifold ([15]).

<u>Observation</u> 11. The K-index mapping is injective for Anosov foliations derived from topologically transitive diffeomorphisms of any compact smooth manifold.

In order to verify the conjecture I, the next question is quite fundamental:

<u>Question</u> 2. Given a K-oriented foliation whose leaves are contractible, does the conjecture I hold affirmatively ?

§4 <u>Anosov actions</u> In this section, we shall check the Baum-Connes conjecture I for generalized Anosov foliations on a infra-homogeneous manifold. Let (M,G,ϕ) be a differentiable dynamical system. The action ϕ is called <u>Anosov</u> if there exist an element g in G and subbundles E^s, E^u, E^c of $T(M)$ such that

(i) $T(M) = E^s \oplus E^u \oplus E^c$, $d\phi_g(E^j) = E^j$,

(ii) E^j are all integrable, $E^c = T(\phi(G))$ $(j=s,u,c)$ and

(iii) $\| d\phi_g^n(\xi)\| < \lambda\|\xi\|$ $(\xi\in E^s)$, $\mu\|\xi\| < \| d\phi_g^n(\xi)\|$ $(\xi\in E^u)$,

$\lambda\|\xi\| < \| d\phi_g^n(\xi)\| < \mu\|\xi\|$ $(\xi\in E^c)$ for some $0<\lambda<1<\mu$.

Then there exist foliations F^s, F^u, F^c of M such that $T(F^j) = E^j$ for $j=s,u,c$. Each leaf W_m^j in F^j $(j=s,u,c)$ is given by the following fashion:

$$W_m^s = \{ x \in M \mid d(\phi_g^n(x),\phi_g^n(m))\lambda^{-n} \longrightarrow 0 \ (n \longrightarrow \infty)\} ,$$
$$W_m^u = \{ x \in M \mid d(\phi_g^{-n}(x),\phi_g^{-n}(m))\mu^{n} \longrightarrow 0 \ (n \longrightarrow \infty)\} ,$$
$$W_m^c = \phi(G)m .$$

Let us now take a noncompact semisimple Lie group G with finite center and K its maximal compact subgroup. We denote by G' the Lie algebra of G. Let $G' = K' + P'$ be a Cartan decomposition of G' and A' a maximal abelian subspace of P'. Suppose Λ is the root system with respect to A', then we have the root space decomposition of G' in the following manner:

$$G' = M' \oplus A' \oplus \textstyle\sum_{\lambda\in\Lambda} G'_\lambda ,$$

where M' be the centralizer of A' in K' and G'_λ the λ-eigen subspace of G'. Given a regular element $a \in A = \exp A'$, define two subsets Λ_a^+ , Λ_a^- as follows:

$$\Lambda_a^+ = \{ \lambda \in \Lambda \mid \lambda(\log a) > 0 \} , \ \Lambda_a^- = \{ \lambda \in \Lambda \mid \lambda(\log a) < 0 \} ,$$

where $\log a$ is the element of A' such that $\exp(\log a) = a$. Let us define N'_+ , N'_- as the direct sum of G'_λ $(\lambda \in \Lambda_a^+ , \Lambda_a^-)$ respectively.

Concerning the diffeomorphism ϕ_a of G/M ($M = \exp M'$) such that $\phi_a(gM) = gaM$ ($g \in G$), one can see that

$$d\phi_a(\xi) = \Sigma_{\lambda\in\Lambda_a^j} e^{-\lambda(\log a)} \xi_\lambda \quad (\xi = \Sigma_{\lambda\in\Lambda_a^j} \xi_\lambda \in N'_j \ , \ j=+,-) ,$$
$$d\phi_a(\xi) = \xi \quad (\xi \in A') .$$

Therefore there exists a constant $c > 0$ such that

$$\| d\phi_a(\xi)\| \leq e^{-c} \|\xi\| \ (\xi \in N'_+) , \quad \| d\phi_a(\xi)\| \geq e^{c} \|\xi\| \ (\xi \in N'_-) .$$

As the tangent space $T_M(G/M)$ of G/M at M is $N'_- \oplus A' \oplus N'_+$, it implies that ϕ is an Anosov action of A on G/M.

<u>Remark</u>. If $a \in A$ is singular, then the decomposition of G'/M' with respect to a is obtained in the following way:

$$G'/M' = N'_- \oplus A' \oplus N'_+ \oplus \Sigma_{\lambda(\log a)=0} G'_\lambda .$$

Hence, $d\phi_a$ is without Anosov condition.

Let Γ be a torsion free uniform lattice of G. We define an action ϕ of A on $\Gamma\backslash G/M$ by $\phi_a(\Gamma gM) = \Gamma\phi_a(gM) = \Gamma gaM$ ($a \in A, g \in G$). Then we have the following lemma:

<u>Lemma</u> 1 The action ϕ is an Anosov action of A on $\Gamma\backslash G/M$.

Except the foliations F^j of $\Gamma\backslash G/M$ with respect to ϕ ($j=s,u,c$), there exist other foliations F^j ($j=cs,cu$) such that

$$T(F^{cs}) = E^s \oplus E^c , \quad T(F^{cu}) = E^u \oplus E^c .$$

Each leaf $W^j_m \in F^j$ ($j=cs,cu$) has the following form:

$$W^{cs}_m = \cup_{x\in\phi(G)m} W^s_x , \quad W^{cu}_m = \cup_{x\in\phi(G)m} W^u_x .$$

We now check the structure of leaves in F^j ($j=s,cs$) on $\Gamma\backslash G/M$. For any $gM \in W^s_M$, there exists a smooth curve $g(t)$ in G such that

$$g(0) = e , \ g(1)M = gM \ \text{ and } \ d/dt(g(t)M) \in E^s_{g(t)M} .$$

It follows that putting

$$X(t) = d/ds(g(t)^{-1}g(s)M)|_{s=t} \in N'_+ \quad (t \in \mathbb{R}) ,$$
$$d/dt(g(t)M) = d\phi_{g(t)}(X(t)) , \quad g(0)M = M .$$

If one defines one parameter family $h(t)$ of $N^+ = \exp N'_+$ by

$$h(t) = \exp \int_0^t X(t)\, dt ,$$

then one easily checks that $g(t)M = h(t)M$ $(t \in \mathbb{R})$. Thus it means that

$$(g(t)K, g(t)P) = (h(t)K, h(t)P) = (h(t), P) \in (N^+K/K) \times \{P\}$$

for all $t \in \mathbb{R}$ where $P = MAN^+$, which implies that

$$W_M^s \subset (N^+K/K) \times \{P\} \quad .$$

Similarly, one obtains that

$$W_M^{cs} \subset (G/K) \times \{P\} \quad .$$

Conversely, given a $gM \in G/M$ $(g \in P)$, there are $a \in A$ and $n \in N$ such that $gaM = nM$. Thus, $gaM \in W_M^s$ implies $gM \in W_M^{cs}$. The leaves W_M^u and W_M^{cu} are also determined by the same way as W_M^s and W_M^{cs} concerning $G = N^-AK$, $P^- = N^-AM$ where $N^- = \exp N'_-$. Let π_Γ be the canonical projection from G/M to $\Gamma\backslash G/M$. Identifying G/M with $(G/K)\times(G/P)$ G-equivariantly by taking the mapping $gM \longrightarrow (gK, gP)$, we obtain the following lemma:

<u>Lemma</u> 2 The Anosov dynamical system $(\Gamma\backslash G/M, A, \phi)$ gifts five foliations F^j of $\Gamma\backslash G/M$ $(j=s,u,c,cs,cu)$ whose leaves $W^j_{\Gamma gM}$ are given by

$$W^s_{\Gamma gM} = \pi_\Gamma((N^+K/K)\times\{gP\}) = \Gamma\backslash((N^+K/K)\times\{gP\}) \ ,$$
$$W^u_{\Gamma gM} = \Gamma\backslash((N^-K/K)\times\{gP^-\} \quad , \quad W^c_{\Gamma gM} = \phi(A)\Gamma gM \quad ,$$
$$W^{cs}_{\Gamma gM} = \Gamma\backslash((G/K)\times\{gP\}) \quad , \quad W^{cu}_{\Gamma gM} = \Gamma\backslash((G/K)\times\{gP^-\}) \ .$$

<u>Remark</u>. The following observation means a geometric approach to the above lemma. According to Oshima[17], there exists a real analytic closed manifold $\widetilde{G/K}$ containing G/K as an open submanifold and G/P as the boundary of G/K. For the decomposition $G = N^-AK$, one knows that $N^-\times \mathbb{R}^l$ is embedded in $\widetilde{G/K}$ and $N^-\times \mathbb{R}^l_+$ is isomorphic to G/K by the mapping $(n^-, \exp-\lambda_1(\log a), \cdots, \exp-\lambda_l(\log a)) \longrightarrow n^-aK$ where $l = \mathrm{rank}_{\mathbb{R}} G$ and $\{\lambda_j\}_{j=1}^l$ is a restricted positive simple root system of Λ . Moreover, G/P can be identified with $N^-\times \{0\}^l$. Using the fact that $g\exp(t\log a)K \longrightarrow gP$ as $t \longrightarrow \infty$, the geodesic half lines

$\{g \exp(t\log a)K\}_{t\geq 0}$ and $\{h \exp(t\log a)K\}_{t\geq 0}$ are asymptotically approaching to get $hM \in W^{cs}_{gM}$. On the other hand, W^{s}_{gM} is interpreted as a horosphere whose boundary passes through gP . The leaves W^{cu}, W^{u} are similarly translated as W^{cs}, W^{s} .

We now study the foliations F^{j} of $\Gamma\backslash G/M$ ($j=s,u,cs,cu$) in more detail. Since $G/M = G/K \times G/P$, we see that $\Gamma\backslash G/M$ is a G/P -bundle over $\Gamma\backslash G/K$. Applying Lemma 2, the following lemma holds:

<u>Lemma</u> 3 The foliated manifolds $(\Gamma\backslash G/M, F^{cs})$ is the foliated G/P -bundle over $\Gamma\backslash G/K$ whose holonomy group is the image of the left translation action of Γ on G/P . The same is true for $(\Gamma\backslash G/M, F^{cu})$ replacing P by P^{-} .

Let us consider the principal M-bundle $\Gamma\backslash G$ over $\Gamma\backslash G/M$ and π_M the natural projection from $\Gamma\backslash G$ to $\Gamma\backslash G/M$. Then the following lemma is also verified:

<u>Lemma</u> 4 The pull back foliations $\pi^{*}_{M}(F^{s}),\pi^{*}_{M}(F^{u})$ of F^{s},F^{u} by π_M are $MN,N^{-}M$ - orbital with respect to the right translation action ρ of G on $\Gamma\backslash G$.

Since Hausdorff are the holonomy groupoids of F^{j} ($j=s,u,cs,cu$), we have the following lemma combining Lemma 3 and Natsume-Takai's result for foliated bundles:

<u>Lemma</u> 5 Concerning $(\Gamma\backslash G/M, F^{j})$ ($j=cs,cu$), one obtains that

$$C^{*}_{r}(\Gamma\backslash G/M, F^{cs}) = (C(G/P)\times_{\lambda}\Gamma)_{r}\otimes BC(L^{2}(\Gamma\backslash G/K)) ,$$

$$C^{*}_{r}(\Gamma\backslash G/M, F^{cu}) = (C(G/P^{-})\times_{\lambda}\Gamma)_{r}\otimes BC(L^{2}(\Gamma\backslash G/K)) ,$$

up to isomorphism where $(\cdot\times\cdot)_{r}$ means the reduced crossed product and $BC(H)$ is the C^{*}-algebra of all compact operators on H .

By Rieffel's work on Morita equivalence, $(C(G/P)\times_{\lambda}\Gamma)_{r}$ is stably isomorphic to $(C(\Gamma\backslash G)\times_{\rho}P)_{r}$, which is equal to $C(\Gamma\backslash G)\times_{\rho}P$. Since $N^{-} = \theta(N^{+})$ for the Cartan involution θ of G, $C^{*}_{r}(\Gamma\backslash G/M, F^{cs})$ is stably isomorphic to $C^{*}_{r}(\Gamma\backslash G/M, F^{cu})$. By Lemma 4, one has the following lemma:

<u>Lemma</u> 6 Concerning $(\Gamma\backslash G,\pi_M^*(F^j))$ $(j=s,u)$, one obtains that

$$C_r^*(\Gamma\backslash G,\pi_M^*(F^s)) = C(\Gamma\backslash G)\times_\rho NM \ , \quad C_r^*(\Gamma\backslash G,\pi_M^*(F^u)) = C(\Gamma\backslash G)\times_\rho N^-M$$

up to isomorphism.

Let (M,F) be a foliated manifold and ξ a bundle over M whose fibers are a compact manifold X. Consider the pull back $\pi^*(F)$ of F by the natural projection π from ξ to M. Then one has the following lemma:

<u>Lemma</u> 7 Concerning $(\xi,\pi^*(F))$, one obtains that

$$C_r^*(\xi,\pi^*(F)) = C_r^*(M,F) \otimes BC(L^2(X))$$

up to isomorphism.

Combining Lemma 6 and 7, the next one is autmatically deduced:

<u>Lemma</u> 8 Concerning $(\Gamma\backslash G/M,F^j)$ $(j=s,u)$, one obtains that

$$C_r^*(\Gamma\backslash G/M,F^s) \otimes BC(L^2(M)) = C(\Gamma\backslash G)\times_\rho N^+M \ ,$$

$$C_r^*(\Gamma\backslash G/M,F^u) \otimes BC(L^2(M)) = C(\Gamma\backslash G)\times_\rho N^-M \ .$$

We now compute the analytic K-theory $K_a(\Gamma\backslash G/M,F^j)$ of $(\Gamma\backslash G/M,F^j)$ $(j=s,u,cs,cu)$ using Lemma 5~8. It certainly follows that

$$K_a(\Gamma\backslash G/M,F^s) = K_a(\Gamma\backslash G,N^+M) \ , \quad K_a(\Gamma\backslash G/M,F^u) = K_a(\Gamma\backslash G,N^-M) \ ,$$

$$K_a(\Gamma\backslash G/M,F^{cs}) = K_a(G/P,\Gamma) \ , \quad K_a(\Gamma\backslash G/M,F^{cu}) = K_a(G/P^-,\Gamma) \ .$$

Since $(C(G/P)\times_\lambda\Gamma)_r$, $(C(G/P^-)\times_\lambda\Gamma)_r$ are stably isomorphic to $C(\Gamma\backslash G)\times_\rho P$, $C(\Gamma\backslash G)\times_\rho P^-$ respectively, it then follows that

$$K_a(G/P,\Gamma) = K_a(\Gamma\backslash G,P) \ , \quad K_a(G/P^-,\Gamma) = K_a(\Gamma\backslash G,P^-) \ .$$

To analyze the right hand side of the above equality, one prepares a generalized Thom isomorphism business due to Connes and Julg:

<u>Lemma</u> 9 Let (A,G,α) be a C^*-dynamical system where G is the semidirect product $\mathbb{R}^n\times_s C$ of $\mathbb{R}^n$ by a compact group C. Then there is an C-equivariant Thom isomorphism between $K_a(A,G)$ and $K_{a,C}^n(A)$.

<u>Remark</u>. In the above lemma, if C is one point, it is due to Connes. If $n = 0$, it is thanks to Julg.

Since P is the semidirect product of N^+ of MA , it follows from Lemma 9 that

$$K_a(\Gamma\backslash G,P) = K_a(C(\Gamma\backslash G)\times_\rho MN^+,A) = K_a^{\dim A}(\Gamma\backslash G,MN^+)$$
$$= K_{a,M}^{\dim AN^+}(\Gamma\backslash G,\cdot)$$

As Γ is torsion free, it has no intersection with M. Thus, ρ is a free action of M on $\Gamma\backslash G$. Therefore, we deduce from Segal that

$$K_{a,M}^{\dim AN^+}(\Gamma\backslash G,\cdot) = K^{\dim AN^+}(\Gamma\backslash G/M) .$$

Consequently, it follows that

$$K_a(\Gamma\backslash G/M,F^{cs}) = K^{\dim AN^+}(\Gamma\backslash G/M) .$$

We shall next compute the geometric K-theory $K_g(\Gamma\backslash G/M,F^{cs})$ of $(\Gamma\backslash G/M,F^{cs})$. Let us look at the leave structure of F^{cs}. Since $W^{cs}_{\Gamma gM} = \pi_\Gamma(G/K \times \{gP\})$ and G/K is contractible, it implies that $W^{cs}_{\Gamma gM}$ are all $K(\Gamma,1)$-spaces. Since Γ is torsion free, so is $\tilde{G} = \mathrm{Hol}(F^{cs})$. It follows from Baum-Connes[1] that

$$K_g(\Gamma\backslash G/M,F^{cs}) = K^\tau(B\tilde{G})$$

where τ is the vector bundle over $B\tilde{G}$ via ν_F^*cs. By definition, $\tilde{G}$ is isomorphic to $(G/P \times_\lambda \Gamma) \times (B\Gamma \times B\Gamma)$ as a Borel groupoid by Natsume-Takai. Let us study this correspondence more closely. Consider the mapping Φ from $\tilde{G}$ to $B\Gamma \times B\Gamma$ by taking $\Phi(\gamma) = (\pi_\Gamma(s(\gamma)),\pi_\Gamma(r(\gamma)))$. Then the groupoids $\Phi^{-1}(x,y)$ $(x,y\in B\Gamma)$ are isomorphic to the principal groupoid $G/P \times_\lambda \Gamma$, namely one has that

$$G/P \times_\lambda \Gamma \xrightarrow{\iota} \tilde{G} \xrightarrow{\Phi} B\Gamma \times B\Gamma .$$

Taking the classifying space of the above spaces, it follows that

$$B(G/P \times_\lambda \Gamma) \xrightarrow{B\iota} B\tilde{G} \xrightarrow{B\Phi} B(B\Gamma \times B\Gamma) .$$

Since $B(B\Gamma \times B\Gamma)$ is homotopic to one point, one obtains that $B\tilde{G}$ is homotopic to $B(G/P \times_\lambda \Gamma)$ by $B\iota$. Since one knows that ι is an isomorphism from $G/P \times_\lambda \Gamma$ into $\tilde{G}$ as a topological groupoid, it follows that $\tilde{G}$ is a Hausdorff space. Let us consider the back $\sigma = (B\iota)^*(\tau)$

of τ by $B\iota$. Then it is a vector bundle over $B(G/P \times_\lambda \Gamma)$ whose fibers come from $\nu_F^* cs$. By the above discussion, one obtains the following:

Lemma 10 $\quad K^\sigma(B(G/P \times_\lambda \Gamma)) = K^\tau(B\tilde{G})$ via $(B\iota)!$.

By definition, $\nu_F^* cs$ is tangential to $T^*(G/P)$. Since Γ is torsion free, it implies from Baum-Connes[1] that

$$K_g(G/P,\Gamma) = K^\delta(E\Gamma \times_\Gamma G/P)$$

where $\delta = E\Gamma \times_\Gamma T^*(G/P)$. Since $E\Gamma \times_\Gamma G/P$ is the base space of a principal $(G/P \times_\lambda \Gamma)$-bundle, there exists a classifying mapping f of $E\Gamma \times_\Gamma G/P$ into $B(G/P \times_\lambda \Gamma)$. Let us take the pull back bundle $f^*(\sigma)$ of σ by f. Then it is actually isomorphic to δ as a vector bundle. Therefore, one has the following lemma:

Lemma 11 $\quad K^\delta(E\Gamma \times_\Gamma G/P) = K^{f^*(\sigma)}(E\Gamma \times_\Gamma G/P)$.

The next lemma seems to be quite crucial to determine the geometric K-theory of $(\Gamma\backslash G/M, F^{cs})$:

Lemma 12 $\quad K^{f^*(\sigma)}(E\Gamma \times_\Gamma G/P) = K^\sigma(B(G/P \times_\lambda \Gamma))$ via $f!$.

Combining Lemma 10~12, one obtains the following:

Lemma 13 $\quad K_g(\Gamma\backslash G/M, F^{cs}) = K_g(G/P,\Gamma)$.

Let H_j be two closed subgroups of G $(j=1,2)$. One compares the two geometric K-groups $K_g(G/H_1,H_2)$ and $K_g(H_2\backslash G,H_1)$. By the same phenomenon as the analytic K-theory, one can verify the following:

Lemma 14 $\quad K_g(G/H_1,H_2) = K_g(H_2\backslash G,H_1)$.

Applying the above lemma to $H_1 = P$ and $H_2 = \Gamma$, it implies that

$$K_g(G/P,\Gamma) = K_g(\Gamma\backslash G,P) \quad .$$

One finally check the following lemma:

Lemma 15 $\quad K_g(\Gamma\backslash G,P) = K_M^{\dim AN^+}(\Gamma\backslash G)$.

Summing up the argument discussed above, we obtain the following main theorem:

Theorem 16 $\quad$ The Baum-Connes conjecture I is affirmative for the foliated manifolds $(\Gamma\backslash G/M, F^j)$ $(j=s,u,c,cs,cu)$.

In fact, the similar method takes place to show the conjecture even in the case of j=s,u,c,cu.

References

[1] P.Baum-A.Connes, Geometric K-theory for Lie groups and foliations, Preprint (1982).

[2] P.Baum-A.Connes, Chern character for discrete groups, Preprint (1987).

[3] A.Connes, A survey of foliations and operator algebras, Proc. Symp. Pure Math.,38(1982) Part 1, 521-628.

[4] A.Connes, Noncommutative differential geometry, chapter I: The Chern character in K-homology, chapter II: De Rham homology and noncommutative algebras, Publ.Math.IHES 62(1986),257-360.

[5] A.Connes, Cyclic cohomology and the transverse fundamental class of a foliation, Pitman Res.Notes.Math.,Ser 123 (1986),52-144.

[6] G.G.Kasparov, Group C^*-algebras and higher signatures, Preprint, Chernogolovka (1981).

[7] G.G.Kasparov, The index of invariant elliptic operators, K-theory and Lie group representations, Preprint, Chernogolovka (1982).

[8] T.Natsume, On $K_*(C^*(SL_2(\mathbb{Z})))$, J.Operator Theory 13 (1985), 103-118.

[9] J.Rosenberg, C^*-algebras, positive scalar curvature and the Novikov conjecture, Publ.Math. IHES., 58 (1983), 197-212.

[10] J.Rosenberg, C^*-algebras, positive scalar curvature and the Novikov conjecture II, Res.Notes Math.Ser., Pitman 123 (1986), 341-374.

[11] J.Rosenberg, C*-algebras, positive scalar curvature and the Novikov conjecture III, Topology 25 (1986), 319-336.

[12] H.Takai, C*-algebras of Anosov foliations, Lec. Notes Math., Springer 1132 (1985), 509-516.

[13] H.Takai, KK-theory for the C*-algebras of Anosov foliations, Res.Notes Math.Ser., Pitman 123 (1986), 387-399.

[14] A.S.Miscenko-T.Fomenko, The index of elliptic operators over C*-algebras, Izv.Akad.Nauk SSSR,Ser.Math.43 (1979), 831-859.

[15] T.Natsume, Topological K-theory for codimension one foliations without holonomy, Adv.Stud.Pure Math.,5 (1985),15-27.

[16] A.M.Terp, K-theory for the leaf space of foliations by Reeb components, Jour.Func.Anal., 61 (1985),15-71.

[17] T.Oshima, A realization of Riemannian symmetric spaces, Jour. Math.Soc. Japan 30 (1978),117-132.