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On Primitive Ideal Spaces of C^*-Algebras over Certain Locally Compact Groupoids

by

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Introduction. Let Γ be a locally compact Hausdorff second countable groupoid with a left Haar system $\{\nu^x\}_{x\in X}$ in the sense of [9](X = the unit space of Γ). By analogy with Fell's algebraic bundles over groups, we define the notion of C^*-algebras over Γ and , given a C^*-algebra A over Γ , we can form a C^*-algebra $C^*(\Gamma,A)$ as the completion of the cross sectional algebra of A. In this note, under some stringent assumptions on Γ, we present a concrete realization of the primitive ideal space of $C^*(\Gamma,A)$. This is a C^*-version of [12]. For the proofs, we refer to [13].

§1.Assumptions.

We impose a series of conditions on Γ. First we assume that

(A1) Γ is minimal,i.e., for all $x\in X$, $[x]=\{y\in X$; there exists $\gamma\in\Gamma$ such that $s(\gamma)=x$, $t(\gamma)=y\}$ is dense in X.

(A2) Stabilizers of Γ are discrete and uniformly abelian,i.e., there are a discrete abelian group G and a family of topological isomorphisms $\{\iota_x:G\to\Gamma_x^x\}_{x\in X}$ such that for $g\in G$, $X\ni x\to\iota_x(g)\in\Gamma$ is continuous and $\iota_{t(\gamma)}(g)\gamma=\gamma\iota_{s(\gamma)}(g)$ holds for $\gamma\in\Gamma$.

(A3) Γ is amenable, i.e., for any quasi-invariant measure μ in X there is a net $(f_i)_{i\in I}$ of functions in $C_c(\Gamma)$ such that

(i) for all $x\in X$ and all $i\in I$, we have $\int \nu^x(d\gamma)|f_i(\gamma)|^2 = 1$

(ii) the function $\gamma \longmapsto \int \nu^{s(\gamma)}(d\gamma')\, f_i(\gamma')\overline{f_i(\gamma\gamma')}$ converges to 1 as $i\to\infty$, relative to the weak* topology of $L^\infty(\Gamma,\mu\circ\nu)$.

<u>Comment</u>. The amenability in (A3) is due to [9]. (A2) means that Γ is a central extension of a principal groupoid by an abelian group. We write $\iota_{t(\gamma)}(g)\gamma$ and $\gamma\iota_{s(\gamma)}(g)$ as $g\gamma$ and γg in the following.

<u>Definition 1</u>. Let $A = (A_\gamma)_{\gamma\in\Gamma}$ be a separable continuous field of Banach spaces over Γ. A is called a C^*-algebra over Γ if the following structures are specified:

(i) For $(\gamma_1,\gamma_2)\in\Gamma$, a multiplication $m: A_{\gamma_1}\times A_{\gamma_2} \ni (a_1,a_2)\longmapsto a_1a_2 \in A_{\gamma_1\gamma_2}$ is defined. It is associative and satisfies $\|a_1a_2\|\le\|a_1\|\|a_2\|$.

(ii) For $\gamma\in\Gamma$, a conjugate linear map $*: A_\gamma \longrightarrow A_{\gamma^{-1}}$ is defined and satisfies $(a^*)^* = a$, $\|a^*a\| = \|a\|^2$, and $(a_1a_2)^* = a_2{}^*a_1{}^*$.

(iii) If φ_1, φ_2 are continuous sections of A, then

$$\Gamma^{(2)} \ni (\gamma_1,\gamma_2) \longmapsto \varphi_1(\gamma_1)\varphi_2(\gamma_2) \in A_{\gamma_1\gamma_2}$$

is a continuous section of $m*(A)$, where $m*(A)$ is the pull back of A under the map $m:\Gamma^{(2)}\longrightarrow\Gamma$.

(iv) If φ is a continuous section of A, $\varphi^*(\gamma) = \varphi(\gamma^{-1})^*$ is also a continuous section of A.

Let $C_c(\Gamma, A)$ be the set of continuous sections of A with compact supports. We make $C_c(\Gamma, A)$ into a $*$-algebra in the following way:

$$(\varphi_1\varphi_2)(\gamma) = \int \nu^{t(\gamma)}(d\gamma')\ \varphi_1(\gamma')\varphi_2(\gamma'^{-1}\gamma)$$

$$\varphi^*(\gamma) = \varphi(\gamma^{-1})^* \quad .$$

$C_c(\Gamma, A)$ is completed to the C^*-algebra $C^*(\Gamma, A)$ according to the maximal continuous C^*-norm (the topology of $C_c(\Gamma, A)$ is that of uniform convergence on compact subsets.). Analogously as in [3], we define the multiplier bundle $\{M(A_\gamma)\}_{\gamma\in\Gamma}$. We assume

(A4) for all $\gamma \in \Gamma$, $U(A_\gamma) = \{u \in M(A_\gamma);\ b^*b = 1_{s(\gamma)}\ ,\ bb^* = 1_{t(\gamma)}\}$ is not empty and there is a Borel section of $\{U(A_\gamma)\}_{\gamma\in\Gamma}$ in the sense that we can find a section $u(\gamma) \in U(A_\gamma)$ such that for any continuous section a of A, $\Gamma \ni \gamma \longmapsto u(\gamma)^*a(\gamma) \in A_{s(\gamma)}$ is a Borel section.

Set $\Gamma(X) = \cup_{x\in X} \Gamma_x^x$. $\Gamma(X)$ is a closed subgroupoid of Γ and the restriction of A to $\Gamma(X)$ is a C^*-algebra over $\Gamma(X)$, so we can form the cross sectional C^*-algebra as in the above definition, which is denoted by $C^*(\Gamma(X), A)$. Similarly, from the restriction of A to Γ_x^x , we have a C^*-algebra $C^*(\Gamma_x^x, A)$. Let $G^*(x) = \mathrm{Prim}\ C^*(\Gamma_x^x, A)$ and set $G^* = \cup_{x\in X} G^*(x)$ (disjoint union).

Lemma 1(cf. [4]).

Prim $C^*(\Gamma(X), A) = G^*$.

The topology of G^* is defined by this identification. Let $u \in U(A_\gamma)$ and define an isomorphism from $C^*(\Gamma_x^x, A)$ onto $C^*(\Gamma_y^y, A)$ ($x = s(\gamma)$, $y = t(\gamma)$) by $C_c(\Gamma_x^x, A) \ni \varphi \longmapsto u\varphi \in C_c(\Gamma_y^y, A)$ where $(u\varphi)(\gamma') = u\varphi(\gamma^{-1}\gamma'\gamma)u^*$.

Now we define an equivalence relation $\simeq$ in G^* by $\omega_1 \simeq \omega_2$ $\Longleftrightarrow$ there exist $\gamma \in \Gamma$ and $u \in U(A_\gamma)$ such that $\omega_1 \in G^*(s(\gamma))$, $\omega_2 \in G^*(t(\gamma))$, and $\omega_2 = u\omega_1$. Here we assume that

(A5) $\simeq$ is an open equivalence relation in G^*.

Comment. This is automatically satisfied if Γ (and A) is arising from a group action.

Definition 2 (cf. [1]). We define an equivalence relation $\sim$ in G^* by

$$\omega_1 \sim \omega_2 \Longleftrightarrow \overline{[\omega_1]} = \overline{[\omega_2]} .$$

Here $[\omega]$ denotes the equivalence class of $\simeq$ containing ω and $\overline{[\omega]}$ is the closure in G^*.

Given $\omega \in G^*$, we can construct the induced primitive ideal ind ω of $C^*(\Gamma,A)$ (see , for example , [2]). Since ind ω depends only on the equivalence class of $\sim$, we have obtained a map from $G^*/\sim$ into $\mathrm{Prim}(C^*(\Gamma,A))$, which is also denoted by ind.

Now the following is proved by the method of [5], [6], [10] (cf. [2], [4]).

Lemma 2.

$$\mathrm{ind} : G^*/\sim \longrightarrow \mathrm{Prim}(C^*(\Gamma,A))$$

is a homeomorphism.

Our next problem is the concrete description of $\widehat{G/\sim}^*$. For that purpose, we need one more assumption. Through the isomorphism $\iota_x : G \longrightarrow \Gamma_x^x$, $A|_{\Gamma_x^x}$ will be regarded as a C^*-algebraic bundle over G and then, by Theorem 9.1 in [3], we have an action of G on $\mathrm{Prim}(A_x)$. Now suppose that

(A6) for each $x \in X$, all G-orbits in $\mathrm{Prim}(A_x)$ is dense in $\mathrm{Prim}(A_x)$.

For example, if A_x is simple, this condition is trivially satisfied.

§2. Topological decomposition of $C^*(\Gamma, A)$.

For $g \in G$, we denote by A^g the pull-back bundle of A under the map $X \ni x \longmapsto \iota_x(g) \in \Gamma$. Let $M(A^g) = \{M(A_{\iota_x(g)})\}_{x \in X}$ be the multiplier bundle of A^g. A section φ of $M(A^g)$ is called strictly continuous if for each continuous section f of A^g, $X \ni x \longmapsto \varphi(x)f(x) \in A_{\iota_x(g)}$ is a continuous section of A^g. Let C_g be the set of bounded strictly continuous sections, say φ, of $M(A^g)$ satisfying

$$\varphi(t(\gamma))a = a\varphi(s(\gamma)) \quad \text{for all } \gamma \in \Gamma \text{ and all } a \in A_\gamma.$$

Lemma 3.

(i) $\dim C_g \leq 1$.

(ii) If $C_g \neq 0$, we can find $\varphi \in C_g$ such that $\varphi(x)$ is a unitary element in $M(A_{\iota_x(g)})$ for all $x \in X$.

Set $S = \{g \in G;\ C_g \neq 0\}$.

Lemma 4. S is a subgroup of G.

Set $\Omega = \{\omega : S \ni g \longmapsto \omega_g \in C_g;$ for all $x \in X$, $\omega_g(x) \in U(A_{\iota_x(g)})$ and $\omega_{g_1}(x)\omega_{g_2}(x) = \omega_{g_1 g_2}(x)$ $(g, g_1, g_2 \in G)\}$. Then $S^\wedge$(= dual group of S) acts on Ω as a transformation group:

$$(\sigma\omega)_g = \langle \sigma, g \rangle \omega_g, \ \sigma \in S^\wedge, \ \omega \in \Omega, \ g \in S.$$

Lemma 5. Ω is a principal homogeneous space under the action of $S^\wedge$(in particular, $\Omega \neq \phi$).

By this lemma, we can transform the topology of $S^\wedge$ into Ω and Ω becomes a compact Hausdorff space.

Let $\omega \in \Omega$ and define an action of S on A by

$$ga = \omega_g(t(\gamma))a, \ g \in S, \ a \in A_\gamma.$$

Taking quotient, we obtain a C^*-algebra A^ω over Γ/S. Let $\varphi \in C_c(\Gamma, A)$ and set

$$\varphi_\omega(\gamma) = \sum_{g \in S} \omega_g(t(\gamma))\varphi(g^{-1}\gamma), \ \gamma \in \Gamma.$$

Then φ_ω gives rise to an element in $C_c(\Gamma/S, A^\omega)$ and we have

Lemma 6. $\{(\varphi_\omega)_{\omega \in \Omega}; \varphi \in C_c(\Gamma, A)\}$ defines a continuous field structure for the family of C^*-algebras $\{C^*(\Gamma/S, A^\omega)\}_{\omega \in \Omega}$.

Theorem. Let Γ be a locally compact Hausdoff second countable groupoid satisfying (A1)~(A3) and A be a C^*-algebra over Γ satisfying (A4)~(A6). Then

(i) for all $\omega \in \Omega$, $C^*(\Gamma/S, A^\omega)$ is simple,

(ii) $C_c(\Gamma,A) \ni \varphi \longmapsto \{\varphi_\omega\}_{\omega \in \Omega} \in \{C^*(\Gamma/S,A^\omega)\}_{\omega \in \Omega}$ is extended to the isomorphism between $C^*(\Gamma,A)$ and the cross section C^*-algebra of $\{C^*(\Gamma/S,A^\omega)\}_{\omega \in \Omega}$.

<u>Sketch of the proof</u>. We define an action of $G^\wedge$ on $C^*(\Gamma(X),A)$ by

$$(\sigma\varphi)(g,x) = \langle\sigma,g\rangle\varphi(g,x).$$

Then the associated action of $G^\wedge$ on G^* preserves the equivalence relation $\sim$ and we can show that this action is transitive. So $G^*/\sim$ is identified with $G^\wedge/S'^\perp$, where S' is a subgroup of G and $S'^\perp = \{\sigma \in G^\wedge; \sigma|_{S'}=1\}$. Now imbed $C(G^*/\sim)$ into $C_b(G^*)$ and regard it as a subalgebra of the center Z of the multiplier algebra $M(C^*(\Gamma(X),A))$ (by Dauns-Hofmann Theorem). Then we can explicitly write down the condition when an element in Z belongs to $C(G^*/\sim)$, and this shows that $S' = S$ and $S^\wedge = G^\wedge/S'^\perp = \Omega$. The topological decomposition follows from [11].

<u>Corollary 1</u>. $\Omega = \mathrm{Prim}(C^*(\Gamma,A))$.

<u>Corollary 2</u>. $C^*(\Gamma,A)$ is simple if and only if $S = \{e\}$.

Let G be a locally compact group and N be an open normal subgroup of G with G/N abelian. Let A be a separable C^*-algebra and $\alpha:G \longrightarrow \mathrm{Aut}(A)$, $\rho:N \longrightarrow U(A)$ be homomorphisms, which form a twisted covariance system in the sense of [6]. Set $S = \{g \in G$; there exists $u \in U(A)$ such that $\alpha_{h^{-1}}(u)a = \rho(h^{-1}g^{-1}hg)\alpha_{g^{-1}}(a)u$ for all $h \in G$ and all $a \in A\}$. Due to §9 of [3], we can construct a C^*-algebraic bundle over G/N. Since algebraic bundles are special cases of C^*-algebras over groupoids, we have

Corollary 3. S is a subgroup of G containing N and, if A is G-simple the primitive ideal space of the twisted covariance algebra $A\times_N G$ is homeomorphic to $(S/N)^\wedge$.

This is a supplementary result to [7],[8].

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