数理解析研究所講究録
624巻 1987年 170-193

Heights of simple loops and pseudo-Anosov homeomorphisms

(阪大 理 小林 毅)
Tsuyoshi Kobayashi

1. Introduction

Let H be a 3-dimensional, orientable handlebody of genus g (>1) and ℓ ($\subset \partial H$) be a connected essential simple closed curve. A *simple loop* is the ambient isotopy class of ℓ and we often abbreviate it by denoting ℓ. A *surface* is a connected 2-manifold. Let F be a 2-sided surface properly embedded in a 3-manifold M. F is *essential* if it is incompressible and not parallel to a subsurface of ∂M. Then we define the *height* of ℓ, $h(\ell)$, as follows;

$h(\ell) = \min \{ -\chi(F) \mid (F, \partial F) \ (\subset (H, \partial H - \ell)) \text{ is an essential surface}\}$,

where $\chi(F)$ denotes the Euler characteristic of F.

In section 2 we see that $h(\ell)$ can be defined.

Fix a hyperbolic metric on ∂H. Let L ($\subset \partial H$) be a geodesic lamination. We say that L is of *full type* if there is a system of mutually disjoint incompressible disks $\{D_1, \ldots, D_{3g-3}\}$ in H such that $\partial D_1 \cup \ldots \cup \partial D_{3g-3}$ cuts ∂H into $2g-2$ pants $P_1, \ldots, P_{2g-2}$, which satisfies;

(i) $\cup \partial D_i$ and L intersect transversely and there is no 2-gon B in ∂H such that $\partial B = \alpha \cup \beta$ where α is a subarc of $\partial D_1 \cup \ldots \cup \partial D_{3g-3}$, β is a subarc of L and

(ii) for each P_i we have;

for each pair of boundary components of P_i, there is asubarc of L properly embedded in P_i, which joins the components.

Then the first result of this paper is;

Theorem 1. *Let* $f:\partial H \to \partial H$ *be a pseudo-Anosov homeomorphism and* ℓ $(\subset\partial H)$ *a simple loop. Suppose that the stable lamination of* f *is of full type. Then* $\lim_{p\to\infty} h(f^p(\ell)) = \infty$, i.e. *for each* n (≥ -1) *there exists a constant* N *such that if* $m>N$ *then* $h(f^m(\ell))>n$.

We note that the assumption on the stable lamination of f is essential. In fact Fathi-Laudenbach [4] showed that there exists a pseudo-Anosov homeomorphism $\varphi:\partial H \to \partial H$ which extends to a homeomorphism of H. Then clearly we have $h(\varphi^n(\ell))=h(\ell)$, for each ℓ and n.

Let $f:\partial H \to \partial H$ be a homeomorphism and $D_1\cup\ldots\cup D_g$ a union of mutually disjoint incompressible disks in H such that $D_1\cup\ldots\cup D_g$ cuts H into a 3-cell. Then we get a compact 3-manifold whose boundary is a sphere by attaching 2-handles to H along the simple loops $f(\partial D_1),\ldots,f(\partial D_g)$. We note that the obtained manifold does not depend on the choice of $D_1,\ldots,D_g$ (Lemma 4.1). Hence we denote the manifold by $\bar{M}_f$. Then M_f denotes the manifold obtained from $\bar{M}_f$ by capping off the boundary by a 3-cell. Roughly speaking, M_f is obtained from H by attaching a copy of H by f.

As an application of Theorem 1, we have;

Theorem 2. *Let* $f:\partial H \to \partial H$ *be a pseudo-Anosov homeomorphism. Suppose that the invariant laminations of* f *are of full type. Then, for each* n (≥ 0), *there is a constant* N *such that if* $m>N$, *then* M_{f^m} *does not contain a 2-sided incompressible surface whose genus is less* *than or equal to* n.

2. Preliminaries

For the definitions of the standard terms in the 3-dimensional topology we refer to [6,7]. We assume that the reader is familiar with [1].

Let H, ℓ be as in section 1. First we will show that the height of ℓ can be defined.

Lemma 2.1. *There exists a 2-sided non-separating surface* S *properly embedded in* $(H, \partial H-\ell)$.

Proof. Let M be the 3-manifold obtained from H by attaching a 2-handle $D^2\times I$ along ℓ. Since the genus of H is greater than 1, the first Betti number of M is greater than 0. Hence by [6,Lemma 6.6], M contains a properly embedded, 2-sided, non-separating incompressible surface S'. Then, by moving S' by an ambient isotopy, we may suppose that S' intersects the 2-handle in horizontal disks. Hence $S = S'-\text{Int}\,(S'\cap(D^2\times I))$ is a 2-sided non-separating surface properly embedded in H. Moreover, by moving S by a tiny isotopy, we may suppose that S is properly embedded in $(H, \partial H-\ell)$.

This completes the proof of Lemma 2.1.

Then we have;

Corollary. *There exists an essential surface properly embedded* *in* $(H,\partial H-\ell)$.

Proof. Let S be the surface obtained in Lemma 2.1. If necessary, by applying the loop theorem and performing a surgery on S, we may suppose that S is incompressible in H. Since S is non-separating, S is not parallel to a subsurface in ∂H. Hence S is essential.

Let F be a closed, orientable surface of genus g (>1) with a hyperbolic metric and PL(F) the space of projective measured laminations of F with an appropriate topology. Then PL(F) is homeomorphic to the 6g-7 dimensional sphere S^{6g-7}. A simple loop is the ambient isotopy class of a closed, connected, 1-submanifold of F which is not contractible in F. Then it is known that the set of all simple loops together with the counting measure consist a dense subset of PL(F). Let ℓ be a simple loop. $\mathcal{S}_\ell$ $(\subset PL(F))$ denotes the set of all simple loops which are disjoint from ℓ. Then let $\mathcal{S}(\ell;1) = \mathcal{S}_\ell$, and $\mathcal{S}(\ell;k) = \bigcup_{\ell'\in\mathcal{S}(\ell;k-1)} \mathcal{S}_{\ell'}$, $(k>1)$.

In this section we will prove;

Proposition 2.2. *Let* $f:F\to F$ *be a pseudo-Anosov homeomorphism with an unstable lamination* L^- *and* ℓ *a simple loop. Then there exists a sequence of neighborhoods of* L^- *in* PL(F), $\{U_i\}_{i=1}^{\infty}$, *such that* $U_1 \supset U_2 \supset \ldots$, *and* $U_k\cap\mathcal{S}(\ell;k) = \phi$, *for each* k.

Let τ $(\subset F)$ be a maximal train track ,i.e. the closure of each component of $F-\tau$ is a 3-gon. Then the set of all non-negative weights on τ defines a 6g-7 dimensional ball B_τ in PL(F). And the set of all positive weights on τ corresponds to the interior of B_τ [1,10].

Lemma 2.3. *If* ℓ *is carried by* τ *with all weights positive, then* $\mathcal{S}_\ell \subset B_\tau$.

Proof. Let $N(\tau)$ be a standard neighborhood of τ. Since ℓ

[Figure 1]

is carried by τ, we may suppose that $\ell \subset N(\tau)$ and ℓ is transverse to the fibers. Let $m \in \mathcal{S}_\ell$. Since each component of $F-N(\tau)$ is contractible, we can isotope m into $N(\tau)$ with $m \cap \ell = \phi$. If m is transverse to the fibers of $N(\tau)$, then $m \in B_\tau$. Hence we suppose that m is not transverse to the fibers. We can isotope m so that m is transverse to the fibers except neighborhoods of the switches which are unions of fibers. See Figure 2. Then the neighborhood of

[Figure 2]

a switch will look like as in Figure 3. We note that the subarcs of

[Figure 3]

m as a_1 or a_2 in Figure 3 play a bad role in this situation. Then we will show that we can eliminate such arcs by an isotopy.

Let Δ be the closure of a component of $F - N(\tau)$ and e an edge of the triangle Δ. Then we have;

Assertion. There is a rectangle R in F such that $R \subset N(\tau)$, $\text{Int } R \cap \ell = \phi$, one edge of R is a subarc of ℓ, one edge of R is e, and the rest edges of R are subarcs of two fibers of $N(\tau)$.

We note that the universal cover of F is isometric to the hyperbolic plane $\mathbb{H}^2$. Let $\tilde{\Delta}$ ($\subset \mathbb{H}^2$) be a lift of Δ, $\tilde{e}$ the edge of $\tilde{\Delta}$

[Figure 4]

corresponding to e and $\tilde{N}(\tau)$ ($\subset \mathbb{H}^2$) the lift of $N(\tau)$. Let I_p ($p \in \tilde{e}$) be the fiber of $\tilde{N}(\tau)$ such that $p \in I_p$. Then, by [1,Lemma 5.8], we see that $I_p \cap \tilde{\Delta} = p$. Hence $\bigcup_{p \in \tilde{e}} I_p$ is a disk which is a disjoint union of the intervals I_p. Since ℓ is carried by τ with all weights

[Figure 5]

positive, we have a rectangle $\tilde{R}$ ($\subset \tilde{N}(\tau)$) such that one edge of $\tilde{R}$ is $\tilde{e}$, one edge of $\tilde{R}$ is a subarc of a lift of ℓ, and the rest edges are subarcs of two fibers of $\tilde{N}(\tau)$. Since $\tilde{e}$ projects homeomorphically onto e, we see that $\tilde{R}$ projects to the rectangle as in Assertion.

This completes the proof of Assertion.

Let F' be the component of $F - \ell$ such that $F' \supset m$, F^* the surface obtained from F' by adding a boundary and N ($\subset F^*$) the image of $N(\tau)$. Then we can reduce the subarcs of type a_1 in Figure 3 by an isotopy as in Figure 6. By Assertion, we can reduce

[Figure 6]

the subarcs of type a_2 in Figfure 3 by an ambient isotopy on F^* as in Figure 7.

[Figure 7]

It is easy to see that these operations terminates in finitely many steps and we see that m is carried by τ.

This completes the proof of Lemma 2.3.

Proof of Proposition 2.2. First we will show that U_1 actually exists. Then we will construct U_2, $U_3, \ldots$ inductively.

Assume that U_1 does not exist. Then there is a sequence of elements of $\mathcal{S}_\ell$, $\{m_i\}_{i=1}^{\infty}$, such that $m_i \to L^-$ $(i \to \infty)$. By [10,Chapitre 2, Proposition], there exists a maximal train track τ and a positive integer n such that the stable lamination L^+ is carried by τ with all weights positive and $f^n(\tau)$ is carried by τ. If necessary, by taking $f^{nm}(\tau)$ for τ, we may suppose that $B_\tau \cap L^- = \phi$ [10,Chapitre 2]. Hence there exists a neighborhood of L^- in PL(F) which is disjoint from B_τ. There is an integer N such that $f^N(\ell) \subset \mathrm{Int}\, B_\tau$. On the other hand, since $f: PL(F) \to PL(F)$ is a continuous map, we have $f^N(m_i) \to L^-$ $(i \to \infty)$, contradicting Lemma 2.3. Hence U_1 exists.

Suppose that we have constructed $U_1, \ldots, U_{k-1}$. Let τ, n be as above. Then there exists a positive integer N_1 such that $f^{nN_1}(PL(F)-U_{k-1}) \subset \mathrm{Int}\, B_\tau$. Let $U_k = f^{-nN_1}(PL(F)-B_\tau)$. Since $\mathcal{S}(\ell;k-1) \cap U_{k-1} = \phi$, we have $f^{nN_1}(\mathcal{S}(\ell;k-1)) \cap f^{nN_1}(U_{k-1}) = \mathcal{S}(f^{nN_1}(\ell);k-1)) \cap f^{nN_1}(U_{k-1}) = \phi$. Hence $\mathcal{S}(f^{nN_1}(\ell);k-1) \subset \mathrm{Int}\, B_\tau$. Then, by Lemma 2.3, we have $\mathcal{S}(f^{nN_1}(\ell);k) \subset B_\tau$. Then we have $U_k \cap \mathcal{S}(\ell;k) = \phi$.

This completes the proof of Proposition 2.2.

3. Proof of Theorem 1

In this section we will give a proof of Theorem 1. Throughout this section, let H, ℓ be as in section 1. The key of the proof is Lemma 3.2 which is an easy estimation of $h(\ell)$ and Lemma 3.4.

Lemma 3.1. *Suppose that* $h(\ell) > -1$. *Then there exists an element* ℓ' *of* $\mathcal{S}_{\ell}$ *such that* $h(\ell') < h(\ell)$.

Proof. There exists an essential surface F properly embedded in $(H, \partial H - \ell)$ such that $-\chi(F) = h(\ell)$. Since every incompressible, ∂-incompressible surface in H is a disk, there is a disk D in H such that $\text{Int}\, D \cap F = \phi$, $D \cap \partial H = \partial D \cap \partial H = \alpha$ an arc and $D \cap F = \partial D \cap F = \beta$ an arc such that $\partial\alpha = \partial\beta$, $\alpha \cup \beta = \partial D$. Let F' be the 2-manifold obtained from F by performing a surgery along D and ℓ' a component of ∂F. Clearly $\ell' \in \mathcal{S}_{\ell}$. By moving ℓ' by a tiny isotopy, we may suppose that $\ell' \cap \partial F' = \phi$. Since F is essential, we see that a component of F', say F'', is essential. And $\chi(F'') > \chi(F)$. Hence we have $h(\ell') < h(\ell)$.

This completes the proof of Lemma 3.1.

As an immediate consequence of Lemma 3.1, we have ;

Lemma 3.2. *Assume that* $\mathcal{S}(\ell;n)$ *does not contain a simple loop of height* -1 *, i.e. if* $m \in \mathcal{S}(\ell;n)$, *then* $\partial H - m$ *is incompressible in* H. *Then* $h(\ell) > n$.

Lemma 3.3. *If* ℓ *is of full type, then* $h(\ell) > -1$ *, i.e.* $\partial H - \ell$

is incompressible in H.

For the definition of ℓ being of full type, see section 1.

Proof. Assume that $\partial H-\ell$ is compressible in H. Then there is a compression disk D properly embedded in $(H,\partial H-\ell)$. Let $\{D_1,\ldots,D_{3g-3}\}$ be a system of disks with respect to which ℓ is of full type. Then we may suppose that ∂D_i, ∂D, ℓ are geodesics for a fixed hyperbolic metric on ∂H. Assume that $\partial D = \partial D_i$ for some i. Then ∂D intersects ℓ, a contradiction. Hence we suppose that $\partial D \neq \partial D_i$ for each i. By cut and paste argument, we may suppose that D intersects $\cup D_i$ in transverse arcs. Let Δ be an innermost disk in D ,i.e. $\Delta\cap(\cup D_i) = \partial\Delta\cap(\cup D_i) = \alpha$, an arc and $\Delta\cap\partial H = \beta$, an arc such that $\alpha\cup\beta = \partial\Delta$. Let P be the closure of the component of $H-N(\cup D_i)$ such that $\Delta \cap P \neq \phi$. Then $\beta \cap P$ is an arc and separates two boundary components of the pants $P \cap \partial H$. On the other hand, since ℓ is of full type, there is a subarc γ of ℓ properly embedded in $P \cap \partial H$ such that γ joins the two boundary components. Hence $\beta \cap \gamma \neq \phi$, a contradiction.

This completes the proof of Lemma 3.3.

We say that a train track τ on ∂H is of *full type* if there is a system of disks $D_1,\ldots,D_{3g-3}$ in H and a system of pants $P_1,\ldots,P_{2g-2}$ as in section 1 which satisfies;

(i) τ and $\partial D_1 \cup \ldots \cup \partial D_{3g-3}$ intesect transversely and there is no 2-gon B in ∂H such that $\partial B = a \cup b$, where a is a differentiable arc on τ, and b is a subarc of $\partial D_1 \cup \ldots \cup \partial D_{3g-3}$

and

(ii) for each pair of boundary components of each pants P_i, there is a differentiable arc a on τ such that a is properly embedded in P_i, and a joins the boundary components.

Then, by Lemma 3.3, we have;

Lemma 3.4. *Suppose that the train track* τ $(\subset \partial H)$ *is of full type. If* ℓ *is a simple loop which is carried by* τ *with all weights positive, then* $h(\ell) > -1$.

Proof of Theorem 1. Let L^+, L^- be the stable, and unstable laminations of f. By Proposition 2.2, there is a neighborhood U of L^- in PL(F) such that $U \cap \mathcal{G}(\ell;n) = \phi$. Since L^+ is of full type, there is a train track τ $(\subset \partial H)$ which is of full type such that L^+ is carried by τ with all weights positive ([1,Lemma 5.2]). Let N be a positive integer such that $f^N(PL(F)-U) \subset \mathrm{Int}\ B_\tau$. Then we have $f^N(\mathcal{G}(\ell;n)) = \mathcal{G}(f^N(\ell);n) \subset \mathrm{Int}\ B_\tau$. Hence, by Lemmas 3.2 and 3.4, we see that $h(f^N(\ell)) > n$. Moreover, by [10,Chapitre 2], we may assume that $f(B_\tau) \subset \mathrm{Int}\ B_\tau$. Hence we see that if $m \geq N$, then $h(f^m(\ell)) > n$.

This completes the proof of Theorem 1.

4. Proof of Theorem 2

In this section we will prove Theorem 2. First we will show that M_f is well defined.

Let H be a genus g (>1) handlebody and $f:\partial H \to \partial H$ a homeomorphism. Let $\{D_1,\dots,D_g\}$, $\{D_1',\dots,D_g'\}$ be systems of mutually disjoint incompressible disks in H such that $\cup D_i$ ($\cup D_i'$ resp.) cuts H into a 3-cell. Let M (M' resp.) be the 3-manifold obtained from H by attaching 2-handles along the union of simple loops $\cup f(\partial D_i)$ ($\cup f(\partial D_i')$ resp.).

Then we have;

Lemma 4.1. *M and M' are homeomorphic.*

Proof. First we prepare a terminology. Let B be a rectangle in ∂H such that $\mathrm{Int}\, B \cap (\cup \partial D_i) = \phi$, one edge of B is a subarc α of ∂D_j, another edge of ∂B is a subarc β of ∂D_k, where $j \neq k$, with $B \cap (\cup D_i) = \alpha \cup \beta$. Then $D_j \cup B \cup D_k$ is a disk D in H. By moving D by a tiny isotopy, we may suppose that $D \cap \partial H = \partial D$ and $D \cap (\cup D_i) = \phi$. It is easy to see that the disks $\{D_1,\dots,\hat{D}_j,\dots,D_g,D\}$ cuts H into a 3-cell, where ^ means removing the element. Then we say that $\{D_1,\dots,\hat{D}_j,\dots,D_g,D\}$ is obtained from $\{D_1,\dots,D_g\}$ by a *band move*. It is known that $\bigcup_{i=1}^{g} D_i'$ is ambient isotopic to the union of disks which is obtained from $\{D_1,\dots,D_g\}$ by a sequence of band moves ([9,Korrolar 2]). Hence we may suppose that $\{D_1',\dots,D_g'\}$ is obtained from $\{D_1,\dots,D_g\}$ by a band move. We may suppose that $D_i = D_i'$ $(2 \le i \le g)$, and $D_1 \cap D_1' = \phi$.

Then $\mathring{M}$ and $\mathring{M}'$ are homeomorphic to the manifold obtained from H by attaching 2-handles along the union of simple loops $\partial D_1 \cup \partial D_1' \cup \partial D_2 \cup \ldots \cup \partial D_g$, where $\mathring{M}$ denotes the manifold obtained from M by removing a 3-cell from its interior. This shows that M and M' are homeomorphic.

This completes the proof of Lemma 4.1.

By Lemma 4.1, we may denote the manifold obtained as above $\bar{M}_f$. Then M_f denotes the closed 3-manifold obtained from $\bar{M}_f$ by capping off the bondary by a 3-cell. It is easy to see that $H' = c\ell(M_f - H)$ is a genus g handlebody. Hence (H,H';F), where $F = \partial H = \partial H'$ ($\subset M_f$) is a Heegaard splitting of M_f. (H,H';F) is called a *cannonical Heegaard splitting* of M_f.

Then we will introduce the concept of rectangle condition of a Heegaard splitting by Casson-Gordon [2].

Let S be the genus g orientable surface and P_i (i=1,2) be a pants (:disk with two holes) embedded in S, with $\partial P_i = \ell_1{}^i \cup \ell_2{}^i \cup \ell_3{}^i$. We suppose that ∂P_1 and ∂P_2 intersect transversely. Let $Q_1, \ldots, Q_m$ be the compact surfaces obtained from $S-(\partial P_1 \cup \partial P_2)$ by adding a boundary. We say that P_1 and P_2 are *tight* if;

(i) each Q_i is not a 2-gon,

(ii) for each pair of, pair of boundary components, $((\ell_s{}^1, \ell_t{}^1), (\ell_p{}^2, \ell_q{}^2))$ of P_1 and P_2 with $s \neq t$, $p \neq q$, there exists a rectangle Q_u embedded in P_1 and P_2 such that $\text{Int } Q_u \cap (P_1 \cup P_2) = \phi$, and the edges of Q_u are subarcs of $\ell_s{}^1$, $\ell_t{}^1$, $\ell_p{}^2$, and $\ell_q{}^2$.

Let $\ell_1, \ldots, \ell_{3g-3}$ ($\ell_1', \ldots, \ell_{3g-3}'$ resp.) be a system of mutually disjoint simple loops such that the closure of the complement of a regular neighborhood of $\cup \ell_i$ ($\cup \ell_i'$ resp.) is a union of 2g-2 pants $P_1,\dots,P_{2g-2}$ ($P_1',\dots,P_{2g-2}'$ resp.). We say that $\{\ell_1,\dots,\ell_{3g-3}\}$ and $\{\ell_1',\dots,\ell_{3g-3}'\}$ are *tight* if, for each pair (i,j), P_i and P_j are tight.

Let (H,H';F) be a Heegaard splitting of a 3-manifold M. We say that (H,H';F) satisfies a *rectangle condition* if there are two systems of mutually disjoint simple loops $\{\ell_1,\dots,\ell_{3g-3}\}$ and $\{\ell_1',\dots,\ell_{3g-3}'\}$ which are tight and each ℓ_i (ℓ_i' resp.) is a boundary of a disk properly embedded in H (H' resp.).

We say that a Heegaard splitting (H,H';F) is *strongly irreducible* if there does not exist incompressible disks $D\subset H$, $D'\subset H'$ such that $\partial D\cap\partial D'=\phi$.

Then Casson-Gordon proved;

Theorem [2]. *If a Heegaard splitting* (H,H';F) *satisfies a rectangle condition, then* (H,H';F) *is strongly irreducible.*

In this paper we will show;

Theorem 4.2. *Let* H *be as in section 1 and* $f:\partial H\to\partial H$ *be a pseudo-Anosov homeomorphism. Suppose that the invariant laminations of* f *are of full type. Then there is an integer* N_1 *such that if* $m>N_1$, *then the cannonical Heegaard splitting of* M_{f^m} *satisfies a rectangle condition.*

Proof. Let L^+ (L^- resp.) be the stable (unstable resp.) lamination of f. We suppose that L^+ (L^- resp.) is of full type with respect to a system of disks $\{D_1,\ldots,D_{3g-3}\}$ ($\{D_1',\ldots,D_{3g-3}'\}$ resp.) in H. Then we may suppose that L^+, L^-, $\cup\partial D_i$, $\cup\partial D_i'$ are unions of geodesics on ∂H. The universal cover of ∂H is isometric to the hyperbolic plane $\mathbb{H}^2$. Let $\tilde{f}:\mathbb{H}^2 \to \mathbb{H}^2$ be a lift of f, and S^1_∞ be the circle at ∞ of $\mathbb{H}^2$. Then it is known that $\tilde{f}$ has a unique continuous extension to $\mathbb{H}^2 \cup S^1_\infty = D^2$ ([1,Lemma 3.7]). Moreover, there is a positive integer n such that if $g:D^2 \to D^2$ is the extension of a lift of f^n then $g|_{S^1_\infty}$ has finitely many fixed pionts $A_1,\ldots,A_{2n}$ $(n\geq 1)$ on S^1_∞, alternately contracting and expanding. We suppose that $A_1,\ldots,A_{2n}$ are on S^1_∞ in this order, $A_1,\ldots,A_{2n-1}$ are contracting and $A_2,\ldots,A_{2n}$ are expanding. Let I_j $(j=1,\ldots,2n-1)$ be a closed interval in S^1_∞ bounded by A_j and A_{j+1} such that $\mathrm{Int}\, I_j \cap (\cup A_i) = \phi$, and $I_{2n} = c\ell(S^1_\infty - \bigcup_{j=1}^{2n-1} I_j)$. Let $\tilde{L}^+$ ($\tilde{L}^-$ resp.) ($\subset\mathbb{H}^2$) be the lift of L^+ (L^- resp.), and γ_1 (δ_1 resp.) be the geodesics of $\tilde{L}^+$ ($\tilde{L}^-$ resp.) which joins A_1 and A_3 (A_2 and A_4 resp.).

Then we will show;

Assertion. For each ∂D_i ($\partial D_i'$ resp.), there exists a lift e ($\subset\mathbb{H}^2$) of it such that the endpoints of e are contained in I_1 and I_2 (I_1 and I_{2n} resp.).

Proof. Since each leaf of L^+ is dense in L^+ ([1,Theorem 4.8]), there is a sequence of lifts of the points of

$L^+ \cap \partial D_i$, $(a_i)_{i=1}^{\infty}$ such that $a_i \in \delta_1$, and $a_i \to A_2$ $(i \to \infty)$. Let $(\varepsilon_i)_{i=1}^{\infty}$ be the sequence of the lifts of ∂D_i such that $\varepsilon_i \cap \delta_1 = a_i$. Since each ε_i is a geodesic, we see that at least one endpoint of ε_i is contained in I_1 or I_2 for sufficiently large i. Assume that the other endpoints of ε_i are not contained in I_2 or I_1. Then we have $\lim_{n\to\infty} \arg(\varepsilon_i, \delta_1) = 0$, where $\arg(\varepsilon_i, \delta_1)$ denotes the angle between the two geodesics. Since L^+ is a closed set, we have $\partial D_i \subset L^+$, a contradisction.

This completes the proof of Assertion.

Let $P_1, \ldots, P_{2g-2}$ ($P_1', \ldots, P_{2g-2}'$ resp.) be a system of pants obtained from ∂H by cutting along $\cup \partial D_i$ ($\cup \partial D_i'$ resp.). Let ∂D_α, ∂D_β and $\partial D_\gamma'$, $\partial D_\delta'$ be pairs of boundary components of a pair (P_i, P_j'). By Assertion, there are lifts $\tilde{\alpha}$, $\tilde{\beta}$ ($\tilde{\gamma}$, $\tilde{\delta}$ resp.) of ∂D_α, ∂D_β ($\partial D_\gamma'$, $\partial D_\delta'$ resp.) in $\mathbb{H}^2$ such that $\tilde{\alpha} \cap \gamma_1 \neq \phi$, $\tilde{\beta} \cap \gamma_1 \neq \phi$, $\tilde{\gamma} \cap \delta_1 \neq \phi$, $\tilde{\delta} \cap \delta_1 \neq \phi$, the endpoints of $\tilde{\alpha}$, $\tilde{\beta}$ ($\tilde{\gamma}$, $\tilde{\delta}$ resp.) are contained in I_1 and I_2 (I_1 and I_{2n-1} resp.), and the subarc of γ_1 (δ_1 resp.) bounded by $\tilde{\alpha} \cap \gamma_1$, $\tilde{\beta} \cap \gamma_1$ ($\tilde{\gamma} \cap \delta_1$, $\tilde{\delta} \cap \delta_1$ resp.) projects to an arc properly embedded in P_i (P_j' resp.). Then there is a constant $N_{i,j,\alpha,\beta,\gamma,\delta}$ such that if $m > N_{i,j,\alpha,\beta,\gamma,\delta}$, then the endpoints of $\tilde{f}^m(\tilde{\alpha})$ and $\tilde{f}^m(\tilde{\beta})$ are separated by the endpoints of $\tilde{\gamma}$ and $\tilde{\delta}$. Let $N_1 = \max\{N_{i,j,\alpha,\beta,\gamma,\delta} \mid 1 \le i, j \le 2g-2, \alpha, \beta, \gamma, \delta\}$. Then N_1 satisfies the conclusion of Theorem 4.2.

This completes the proof of Theorem 4.2.

Lemma 4.3. *Suppose that a Heegaard splitting* (H,H';F) *of* M *is strongly irreducibe and that* M *contains a 2-sided incompressible* *surface* S. *Then there is a surface* S' *which is ambient isotopic to* S *and satisfies;*

(i) S' *intersects* F *in transverse loops,*

(ii) *every component of* $S' \cap H$ *is not a disk, and*

(iii) *just one component of* $S' \cap H'$ *is a disk.*

Proof. This lemma is proved by using the argument of [6,Chapter II], which was used to prove the Haken's theorem. We assume that the reader is familiar with the proof. We note that Jaco considered the case of S being the 2-sphere, but the argument works in this situation. We also note that if (H,H';F) is strongly irreducible, then M is irreducible ([3,Theorem 2.1]).

We may suppose that each component of $S \cap H$ is a disk and the number of the component is minimal among all surfaces which are ambient isotopic to S. By the minimality, we see that $S \cap H'$ is an incompressible surface in H'. Hence we have a hierarchy for $S \cap H'$, $(S_2^{(0)}, \alpha_0)$, $(S_2^{(1)}, \alpha_1), \dots, (S_2^{(p)}, \alpha_p)$, and a sequence of isotopies of type A in M which realizes the hierarchy. Let $S^{(q)}$ $(0 \leq q \leq p+1)$ be the image of S after the q-th isotopy of type A ,i.e. $S^{(q)} \cap H' = S_2^{(q)}$. Let k be the integer such that $S_2^{(k)}$ does not contain a disk and $S_2^{(k+1)}$ contains a disk. We note that $S_2^{(k+1)}$ contains just one disk. Then, by Theorem 4.2 and a theorem of Casson-Gordon, we see that each component of $S^{(k+1)} \cap H$ is not a disk. Hence $S' = S^{(k+1)}$ satisfies the the conclusions of Lemma 4.3.

This completes the proof of Lemma 4.3.

Lemma 4.4. *Let* $f: \partial H \to \partial H$ *be a pseudo-Anosov homeomorphism.*

Suppose that the stable lamination L^+ *of* f *is of full type. Then, for each* n (≥ -1), *there exists a neighborhood* U *of* L^+ *in* $PL(\partial H)$ *such that the height of every simple loop contained in* U *is greater than* n.

Proof. Assume that the conclusion of Lemma 4.3 does not hold. Then there exists an integer n (≥ -1), and a sequence of simple loops $\{\ell_i\}_{i=1}^{\infty}$ on ∂H such that $\ell_i \to L^+$ ($i \to \infty$), and $h(\ell_i) \leq n$. By [10], there is a maximal train track τ on ∂H and a positive integer m such that L^+ is carried by τ with all weights positive and $f^m(\tau)$ is carried by τ. Set $B_i = B_{f^{mi}(\tau)}$ ($i \geq 1$). Then, by [10], we see that $B_{p+1} \subset \text{Int } B_p$, and B_p converges to L^+. If necessary, by taking a subsequence of $\{\ell_i\}$, we may suppose that $\ell_i \subset \text{Int } B_i$ ($i \geq 1$). By Lemmas 3.1 and 2.2, we have a sequence of simple loops $\{\ell_i'\}_{i=1}^{\infty}$ such that $h(\ell_i') \leq n-1$, $\ell_i' \in B_i$, hence, $\ell_i' \to L^+$ ($i \to \infty$). By applying this argument finitely many times, we get a sequence of simple loops $\{\ell_i^*\}_{i=1}^{\infty}$ such that $h(\ell_i^*) = -1$ and $\ell_i^* \to L^+$ ($i \to \infty$). But this contradicts Lemma 3.4.

This completes the proof of Lemma 4.4.

Proof of Theorem 2. By Lemma 4.4, there is a neighborhood U^+ (U^- resp.) of L^+ (L^- resp.) such that the height of every simple loop in U^+ (U^- resp.) is greater than $2n+1$ (1 resp.). Let N' be an integer such that $f^m(PL(\partial H)-U^-) \subset U^+$ for each m ($>N'$). And let $N = \max\{N', N_1\}$, where N_1 is the constant obtained in Theorem 4.2. Then we will see that this N satisfies the conclusion of Theorem 2.

Assume that there is an integer m (>N) such that M_{f^m} contains an incompressible surface S whose genus is less than or equal to n, i.e. $\chi(S) \geq 2-2n$. We may suppose that S satisfies the conclusions of Lemma 4.2. Let $S_1 = S \cap H$ and $S_2 = S \cap H'$. Then $\chi(S) = \chi(S_1) + \chi(S_2) \geq 2-2n$. Let D be the disk component of S_2. Then D is the only component which makes a positive contribution in $\chi(S_1) + \chi(S_2)$. Hence we have $\chi(S_1) + \chi(S_2) \leq \chi(S_2) + 1$. Then $2-2n \leq \chi(S_2) + 1$ and $\chi(S_2) \geq 1-2n$. Let S^* be a component of S_2. Then $\chi(S^*) \geq 1-2n$. Hence $f^m(\partial D) \subsetneq U^+$, contradicting the fact that ∂D does not contained in U^-.

This completes the proof of Theorem 2.

5. Examples

In this section we will give ways of constructing pseudo-Anosov homeomorphisms which satisfies the assumptions of Theorems 1 and 2. Throughout this section let H be as in section 1 and T_ℓ the Dehn twist along the simple loop ℓ in ∂H.

Example 1. Let ℓ $(\subset \partial H)$ be a simple loop which is of full type and m $(\subset \partial H)$ a simple loop such that $\ell \cup m$ fills up ∂H, i.e. the number of intersection of ℓ and m is minimal among all simple loops which are isotopic to m and each component of $\partial H - (\ell \cup m)$ is an open disk. Then, by [4,Exposé 13], we see that $T_m \circ T_\ell^{-n}$ is isotopic to a pseudo-Anosov homeomorphism for each positive integer n and the stable lamination tends to ℓ if n grows larger. Hence $T_m \circ T_\ell^{-n}$ satisfies the assumption of Theorem 1 if n is sufficiently large.

Example 2. Let φ be a pseudo-Anosov homeomorphism such that φ^{-1} satisfies the assumption of Theorem 1, i.e. the unstable lamination L^- of φ is of full type. Let D be a component of the system of disks with respect to which L^- is of full type. Then, as observed in the proof of [9,Lemma 2.5], $\varphi^{-k} \circ T_{\partial D} \circ \varphi^k \circ T_{\partial D}^{-1}$ is isotopic to a pseudo-Anosov homeomorphism for a sufficiently large k, and the invariant laminations tend to L^- and $T_{\partial D}(L^-)$ if k grows larger. Clearly $T_{\partial D}(L^-)$ is of full type. Hence $\varphi^{-k} \circ T_{\partial D} \circ \varphi^k \circ T_{\partial D}^{-1}$ satisfies the assuption of Theorem 2 if k is sufficiently large.

References

[1] A. J. Casson, Automorphisms of surfaces after Nielsen and Thurston, Lecture note, University of Texas at Austin

[2] A. J. Casson, and C. McA Gordon, Manifolds with irreducible Heegaard splittings of arbitrarily high genus, to appear

[3] ________________________________, Reducing Heegaard splittings, preprint

[4] A. Fathi, and F. Laudenbach, Diffeomorphismes pseudo-Anosov et dècompositions de Heegaard, C. R. Acad. Sc. Paris, 291(1980) A423

[5] A. Fathi, F. Laudenbach, and V. Poenaru, Travaux de Thurston sur les surfaces, Astèrisque, 66-67(1979)

[6] J. Hempel, 3-manifolds, Ann. of Math. Studies 86(1976), Princeton University Press, Princeton N.J.

[7] W. Jaco, Lectures on three-manifold topology, CBMS Regional Conference Series in Math., 43(1980)

[8] T. Kobayashi, Structures of full Haken manifolds, Osaka J. Math. (to appear)

[9] D. D. Long, A note on the normal subgroups of mapping class groups, Math. Proc. Camb. Phil. Soc. 99(1986), 79-87

[10] A. Papadopoulos, Reseaux ferroviaires, diffeomorphismes pseudo-Anosov et automorphismes symplectiques de l'homologie d'une surface, publications mathematiques d'Orsay (1983)

[11] H. Zieschang, Über einfache Kurven auf Vollbrezeln, Abh. Math. Sem. Univ. Hamburg 25(1962), 231-250

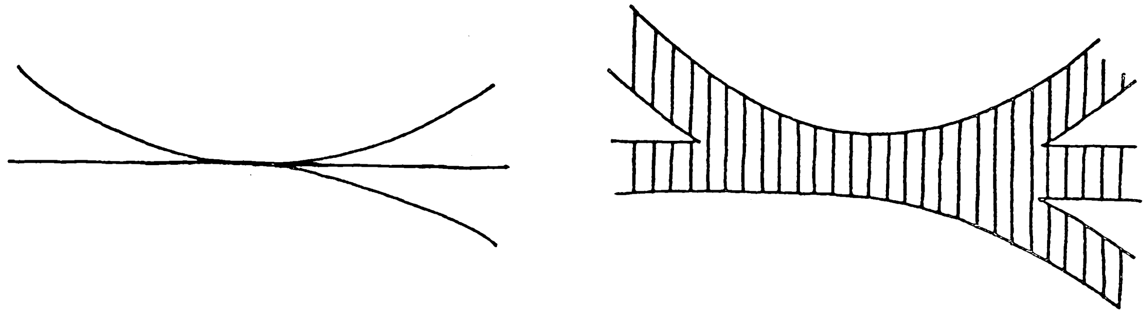

Figure 1

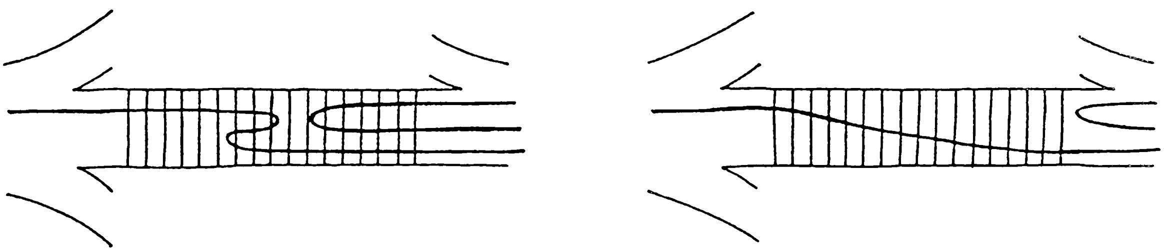

Figure 2

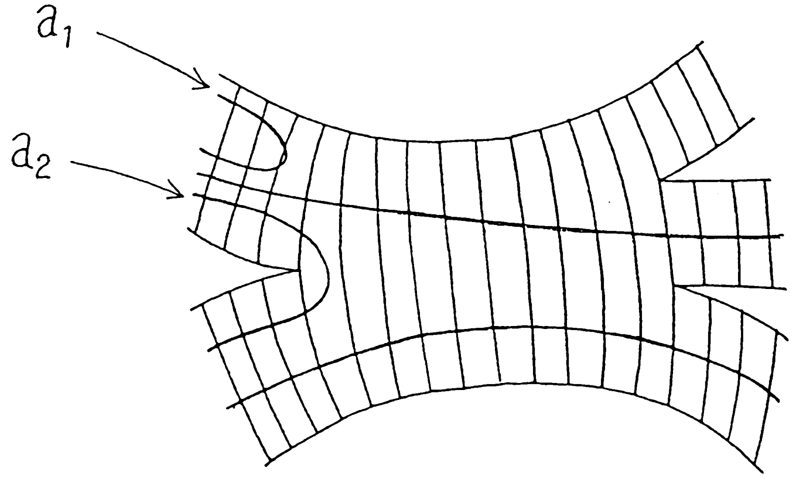

Figure 3

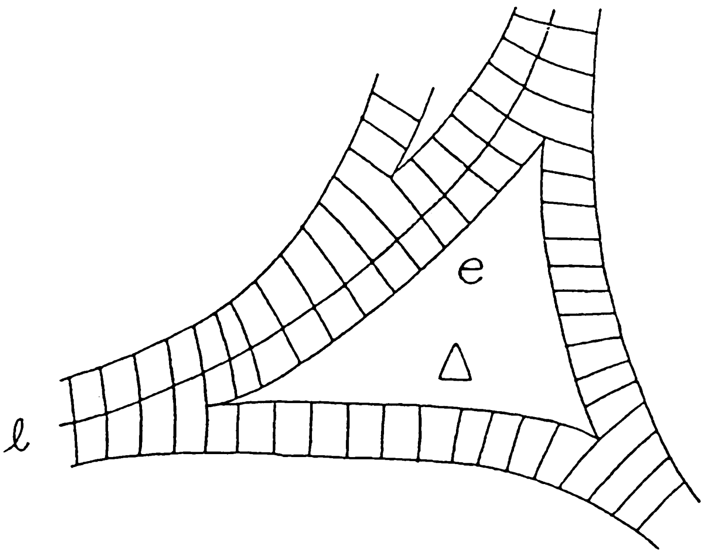

Figure 4

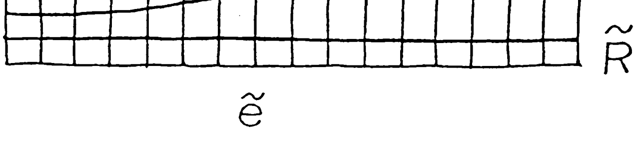

Figure 5

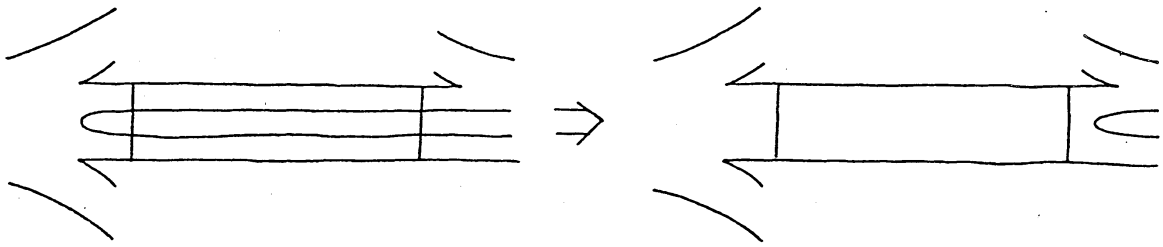

Figure 6

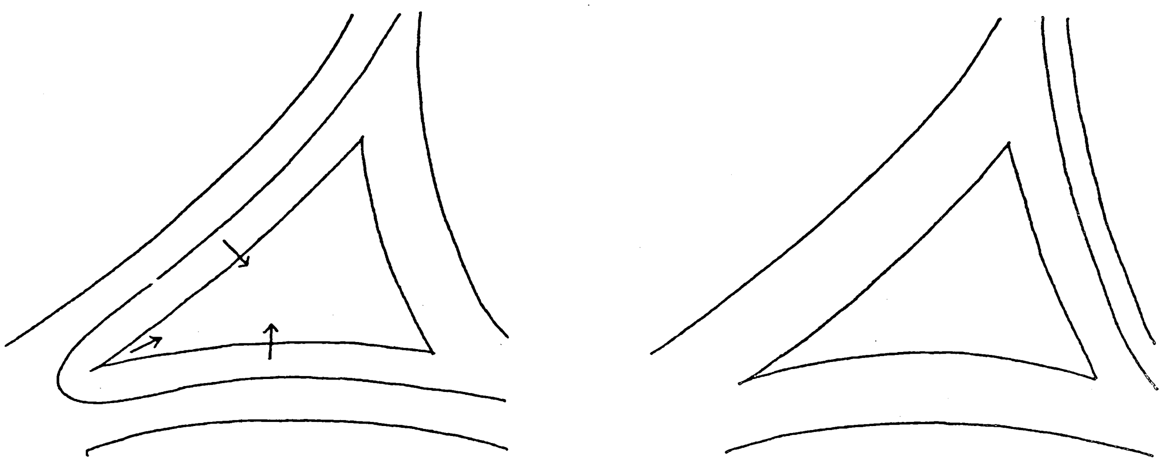

Figure 7