DERIVATIONS IN MATRIX SUBRINGS

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This note is an abstract of the author's papers [1], [2] and [3]. Let $R$ be a ring with identity and let $M_n(R)$ denote the ring of $n \times n$ matrices over $R$. We say that a subring $P$ of $M_n(R)$ is special with the relation $\omega$ if $P$ is of the form

$$P = \{ A \in M_n(R); A_{ij} = 0, \text{ for } (i,j) \notin \omega \},$$

where $\omega$ is a relation (reflexive and transitive) on the set $\{1, \ldots, n\}$.

We describe in this note all derivations, $R$-derivations and higher $R$-derivations of the ring $P$.

1. DERIVATIONS AND HIGHER DERIVATIONS IN A RING

1. Derivations. Let $P$ be a ring with identity. An additive mapping $d: P \longrightarrow P$ is called a derivation (or an ordinary derivation) of $P$ if $d(xy) = d(x)y + xd(y)$, for any $x, y \in P$. We denote by $D(P)$ the set of all derivations of $P$. If $d$ and $d'$ are derivations of $P$ then the mapping $d + d'$ is also a derivation of $P$, so $D(P)$ is an abelian group.

Let $a \in P$ and let $d : P \longrightarrow P$ be a mapping defined by $d_a(x) = ax - xa$, for any $x \in P$. Then $d_a$ is a derivation of $P$.

Let $d \in D(P)$. If there exists an element $a \in P$ such that $d = d_a$ then $d$ is called an inner derivation (with respect to $a$) of $P$. We denote by $ID(P)$ the set of all inner derivations of $P$. $ID(P)$ is a (normal) subgroup of $D(P)$.

We shall say that $P$ is an NS-ring if $ID(P) = D(P)$, that is, a ring $P$ is an NS-ring if and only if every derivation of $P$ is inner.
2. Higher derivations. Let $P$ be a ring with identity and let $S$ be a segment of $N = \{0, 1, \ldots\}$, that is, $S = N$ or $S = \{0, 1, \ldots, s\}$ for some $s \geq 0$. A family $d = (d_m)_{m \in S}$ of mappings $d_m : P \to P$ is called a derivation of order $s$ of $P$ (where $s = \sup(S) \leq \infty$) if the following properties are satisfied:

1. $d_m(x + y) = d_m(x) + d_m(y)$,
2. $d_m(xy) = \sum_{i+j=m} d_i(x)d_j(y)$,
3. $d_0 = \text{id}_P$,

for any $x, y \in P$, $m \in S$.

The set of derivations of order $s$ of $P$, denoted by $D_s(P)$, is the group under the multiplication $\ast$ defined by the formula

$$(d * d')_m = \sum_{i+j=m} d_i \ast d'_j,$$

where $d,d' \in D_s(P)$ and $m \in S$.

Let $\delta : P \to P$ be a mapping. Then $\delta$ is an ordinary derivation of $P$ if and only if $(\text{id}_P, \delta)$ is a derivation of order 1 of $P$. Therefore we may identify: $D(P) = D_1(P)$.

3. Examples of higher derivations. Let $P$ and $S$ be as in Section 2.

Example 3.1. Let $a \in P$, $d_0 = \text{id}_P$, and $d_m(x) = a^m x - a^{m-1} xa$, for $m \geq 1$, $x \in P$. Then $d = (d_m)_{m \in S}$ belongs to $D_s(P)$.

Example 3.2. Let $d \in D_s(P)$, $k \in S - \{0\}$ and let $\delta = (\delta_m)_{m \in S}$ be the family of mappings from $P$ to $P$ defined by

$$\delta = \begin{cases} 0, & \text{if } k \nmid m \\ d_m, & \text{if } m = rk. \end{cases}$$

Then $\delta \in D_s(P)$.

The derivation $d$ (of order $s$) from Example 3.1 will be denoted by $[a, 1]$. The derivation $\delta$ (of order $s$) from Example 3.2, for $d = [a, 1]$
will be denoted by \([a,k]\).

4. Inner derivations. Let \(P\) and \(S\) be as in Section 2 and let \(a = (a_m)\) (where \(m \in S\)) be a sequence in \(P\). Denote by \(\Delta(a)\) the element in \(D_S(P)\) defined by
\[
\Delta(a)_m = ([a_1, 1] \ast [a_2, 2] \ast \cdots \ast [a_m, m])_m,
\]
for any \(m \in S\).

**Definition 4.1.** Let \(d \in D_S(P)\). If there exists a sequence \(a = (a_m)\) of elements of \(P\) such that \(d = \Delta(a)\) then \(d\) is called an **inner derivation of order** \(s\) of \(P\).

Denote by \(ID_S(P)\) the set of inner derivations of order \(s\) of \(P\).

**Proposition 4.2.** \(ID_S(P)\) is a normal subgroup of \(D_S(P)\).

**Proposition 4.3.** The following properties are equivalent

(1) \(P\) is an NS-ring,

(2) \(ID_S(P) = D_S(P)\), for any \(0 < s \leq \infty\),

(3) \(ID_S(P) = D_S(P)\), for some \(0 < s \leq \infty\).

5. \(R\)-derivations. Let \(R \subseteq P\) be rings with identity and let \(S\) be a segment of \(N\). If a derivation (of order \(s\)) \(d \in D_S(P)\) satisfies the condition
\[
d_m(r) = 0,
\]
for all \(m \in S - \{0\}, r \in R\), then \(d\) is called **\(R\)-derivation of order** \(s\) of \(P\), and the set of all such derivations is denoted by \(D^R_S(P)\).

We define similarly an ordinary \(R\)-derivation, an inner \(R\)-derivation, an inner \(R\)-derivation of order \(s\) and also, we define similarly the groups \(D^R(P), ID^R(P)\) and \(ID^R_S(P)\).

The group \(D^R_S(P)\) is a subgroup of \(D_S(P)\) and the group \(ID^R_S(P)\) is a normal subgroup of \(D^R_S(P)\).
We shall say that \( P \) is an \textit{NS-ring over \( R \)} if \( \text{ID}^R(P) = D^R(P) \).

\textbf{Proposition 5.1.} The following properties are equivalent

1. \( P \) is an NS-ring over \( R \),

2. \( \text{ID}^R_s(P) = D^R_s(P) \), for any \( 0 < s \leq \infty \),

3. \( \text{ID}^R_s(P) = D^R_s(P) \), for some \( 0 < s \leq \infty \).

\textbf{II. SPECIAL SUBRINGS OF MATRIX RINGS}

6. \textit{Notices.} Let \( R \) be a ring with identity, \( n \) a fixed natural number and \( \omega \) a reflexive and transitive relation on the set \( I_n = \{1, \ldots, n\} \). We denote by \( M_n(R) \) the ring of \( n \times n \) matrices over \( R \) and by \( Z(R) \) the center of \( R \). Moreover, we use the following conventions:

\( F(R) = \) the set of mappings from \( R \) to \( R \),

\( \tilde{\omega} = \) the smallest equivalence relation on \( I_n \) containing \( \omega \),

\( T_\omega = \) a fixed set of representatives of equivalence classes of \( \tilde{\omega} \),

\( A_{ij} = \) \( ij \)-coefficient of a matrix \( A \),

\( E_{ij} = \) the element of the standard basis of \( M_n(R) \),

\( M_n(R)_\omega = \{ A \in M_n(R); A_{ij} = 0, \text{ for } (i,j) \notin \omega \} \).

The set \( P = M_n(R)_\omega \) is a subring of \( M_n(R) \) called a \textit{special subring with the relation} \( \omega \). Every special subring contains the ring \( R \) (via injection \( r \mapsto \bar{r} \), where \( \bar{r} \) is the diagonal matrix whose all coefficients on the diagonal are equal to \( r \in R \)).

7. \textit{Transitive mappings and regular relations.} Let \( G \) be an abelian group. A mapping \( f: \omega \rightarrow G \) will be called transitive iff

\[ f(a,c) = f(a,b) + f(b,c), \]

for any \( a, b \) and \( b, c \).

If \( f: \omega \rightarrow G \) is a transitive mapping then we denote by \( [f, -] \)
(in the case $G = R$) the mapping from $\omega$ to $F(R)$ defined by

$$[f, _{a,b}](x, y) = f(a, b)x - rf(a, b),$$

for $a \circ b$ and $r \in R$. Clearly, $[f, _{a,b}]$ is transitive too.

We shall say that $f$ is **trivial** if there exists a mapping $\sigma: I_n \rightarrow G$ such that

$$f(a, b) = \sigma(a) - \sigma(b),$$

for any $a \circ b$. Moreover, we shall say that $f$ is **quasi-trivial** (in the case $G = R$) if $[f, _{a,b}]$ is trivial.

Every trivial transitive mapping from $\omega$ to $R$ is quasi-trivial, but the converse is not necessarily true.

**Proposition 7.1.** Let $f: \omega \rightarrow R$ be a quasi-trivial transitive mapping. Then there exists a unique mapping $\tau: I_n \rightarrow F(R)$ such that

1. $[f, 0](i, j) = \tau(i) - \tau(j)$, for all $i \circ j$,
2. $\tau(t) = 0$, for $t \in T_\omega$.

Moreover, $\tau(1), \ldots, \tau(n)$ are inner derivations of $R$.

**Definition 7.2.** The relation $\omega$ is called **regular** over an abelian group $G$ if every transitive mapping from $\omega$ to $G$ is trivial.

**8. The graph $\Gamma(\omega)$ and homology groups.** Let $\equiv$ be the equivalence relation on $I_n$ defined by: $x \equiv y$ iff $x_0 y$ and $y_0 x$. Denote by $[x]$ the equivalence class of $x \in I_n$ with respect to $\equiv$ and let $I_n'$ be the set of all equivalence classes. We define a relation $\omega'$ of partial order on $I_n'$ as follows:

$$[x] \omega' [y] \text{ iff } x_0 y.$$  

We will denote the pair $(I_n', \omega')$ by $\Gamma(\omega)$ and call it the **graph of $\omega$**. Elements of $I_n'$ are called **vertices** of $\Gamma(\omega)$ and pairs $(a, b)$, where $a \omega b$ and $a \neq b$, are **arrows** of $\Gamma(\omega)$.

Let us imbed the set of the vertices of $\Gamma(\omega)$ in an Euclidean space of a sufficiently high dimension so that the vertices will be
linearly independent.

If \(a_0, a_1, \ldots, a_k\) are elements of \(\mathbb{I}_n\) such that \(a_{i+1} \neq a_{i+1}\) for \(i=0,1,\ldots,k-1\), then by \((a_0, a_1, \ldots, a_k)\) we denote the k-dimensional simplex with vertices \(a_0, \ldots, a_k\). The union of all 0, 1, 2 or 3-dimensional such simplicies we will denote also by \(\Gamma(\omega)\). Therefore, \(\Gamma(\omega)\) is a simplicial complex of dimension \(\leq 3\).

Let \(C_k(\omega)\), for \(k=0,1,2,3\), be the free abelian group whose free generators are k-dimensional simplicies of \(\Gamma(\omega)\). We have the following standard complex of abelian groups:

\[
0 \longrightarrow C_3(\omega) \overset{\beta_3}{\longrightarrow} C_2(\omega) \overset{\beta_2}{\longrightarrow} C_1(\omega) \overset{\beta_1}{\longrightarrow} C_0(\omega) \longrightarrow 0
\]

where

\[
\beta_1(a,b) = (b) - (a),
\]

\[
\beta_2(a,b,c) = (b,c) - (a,c) + (a,b),
\]

\[
\beta_3(a,b,c,d) = (b,c,d) - (a,c,d) + (a,b,d) - (a,b,c).
\]

Then \(H_1(\Gamma(\omega)) = \text{Ker} \beta_1 / \text{Im} \beta_2\), \(H_2(\Gamma(\omega)) = \text{Ker} \beta_2 / \text{Im} \beta_3\) and (by the Kunneth formulas)

\[
H^1(\Gamma(\omega), G) = \text{Hom}(H_1(\Gamma(\omega)), G),
\]

for an arbitrary abelian group \(G\).

III DERIVATIONS IN SPECIAL SUBRINGS

9. Examples of derivations. Let \(P = M_n(\mathbb{R})\) be a special subring of \(M_n(\mathbb{R})\).

Example 9.1. Assume that \(f: \omega \longrightarrow \mathbb{R}\) is a quasi-trivial transitive mapping and denote by \(\Delta^f\) the mapping from \(P\) to \(P\) defined by

\[
\Delta^f(B)_{pq} = B_{pq} f(p,q) + \tau_f(p)(B_{pq}),
\]

for \(B \in P\), \(p\omega q\), where \(\tau_f\) is the mapping \(\tau\) from Proposition 7.1.

Then \(\Delta^f\) is a derivation of \(P\). Moreover \(\Delta^f\) is inner if and only if \(f\) is trivial.
Example 9.2. Let $\delta = \{ \delta_t : t \in T_\omega \}$ be a set of derivations of $R$.

Denote by $\Theta_\delta$ the mapping from $P$ to $P$ defined by

$$
\Theta_\delta(B)_{pq} = \delta_t(B)_{pq},
$$

for $B \in P$, $pq$, where $t \in T_\omega$ such that $p \in t$.

Then $\Theta_\delta$ is a derivation of $P$. Moreover, $\Theta_\delta$ is inner if and only if $\delta_t$ is inner for any $t \in T_\omega$.

10. A description of $D(P)$. Let $P = M_n(R)_\omega$ be a special subring of $M_n(R)$. The following theorem describes all derivations of $P$.

Theorem 10.1. Every derivation $d$ of $P$ has a unique representation:

$$
d = d_A + \Delta^f + \Theta_\delta,
$$

where

1. $d_A$ is an inner derivation of $P$ with respect to a matrix $A \in P$ such that $A_{pp} = 0$, for $p=1,\ldots,n$,

2. $f: \omega \longrightarrow R$ is a quasi-trivial transitive mapping and $\Delta^f$ is the derivation from Example 9.1,

3. $\delta = \{ \delta_t : t \in T_\omega \}$ is a set of derivations of $R$ and $\Theta_\delta$ is the derivation from Example 9.2.

The next theorem describes special subrings which are NS-rings.

Theorem 10.2. The following conditions are equivalent

1. $P$ is an NS-ring,

2. $R$ is an SN-ring and the relation $\omega$ is regular over $Z(R)$.

11. $R$-derivations of $M_n(R)_\omega$. Let $P = M_n(R)_\omega$ be a special subring of $M_n(R)$.

Example 11.1. Let $f: \omega \longrightarrow Z(R)$ be a transitive mapping and denote by $\Delta^f$ the mapping from $P$ to $P$ defined by $\Delta^f(B)_{pq} = f(p,q)B_{pq}$, for $B \in P$ and $pq$. Then $\Delta^f$ is an $R$-derivation of $P$. Moreover $\Delta^f$ is
inner if and only if \( f \) is trivial.

**Theorem 11.2.** Any \( R \)-derivation \( d \) of \( P \) has a unique representation

\[
d = d_A + \Delta^f,
\]

where (1) \( d_A \) is an inner derivation of \( P \) with respect to a matrix \( A \in P \) such that \( A_{ij} \in Z(R) \) for \( i,j=1,...,n \), and \( A_{ii} = 0 \) for \( i=1,...,n \).

(2) \( f: \omega \to Z(R) \) is a transitive mapping and \( \Delta^f \) is the derivation from Example 11.1.

**Theorem 11.3.** The following conditions are equivalent

(1) \( P \) is an NS-ring over \( R \),

(2) \( \omega \) is regular over \( Z(R) \).

**Corollary 11.4.** If \( d \) and \( \delta \) are \( R \)-derivations of \( R \) then the derivation \( d\delta - \delta d \) is inner.

**Corollary 11.5.** If \( d \) is an \( R \)-derivation of \( P \) then \( d(Z(R)) = 0 \).

**Corollary 11.6.** If \( d \) is an \( R \)-derivation of \( P \) and \( U \) is an ideal of \( P \) then \( D(U) \subseteq U \).

12. **An example of non-inner \( R \)-derivation.** For \( n \leq 3 \) every relation \( \omega \) (reflexive and transitive) on \( I_n \) is regular over any group. Therefore (by Theorem 11.3), in this case any special subring of \( M_n(R) \) has only inner \( R \)-derivations. For \( n=4 \) it is not true. Let \( \omega_0 \) be the relation on \( I_4 = \{1,2,3,4\} \) defined by the graph

\[
\begin{array}{c}
1 \quad 3 \\
\downarrow \\
4 \leftrightarrow 2,
\end{array}
\]

that is, \( \omega_0 = \{(1,1),(2,2),(3,3),(4,4),(1,3),(1,4),(2,3),(2,4)\} \). Denote by \( S_4(R) \) the special subring of \( M_4(R) \) with the relation \( \omega_0 \).
Then we have

\[
S_4(R) = \begin{bmatrix} R & 0 & R & R \\ 0 & R & 0 & R \\ 0 & 0 & R & 0 \\ 0 & 0 & 0 & R \end{bmatrix}.
\]

Consider the mapping \(d: S_4(R) \to S_4(R)\) defined by

\[
d(\begin{bmatrix} x_{11} & 0 & x_{13} & x_{14} \\ 0 & x_{22} & x_{23} & x_{24} \\ 0 & 0 & x_{33} & 0 \\ 0 & 0 & 0 & x_{44} \end{bmatrix}) = \begin{bmatrix} 0 & 0 & x_{13} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.
\]

Then \(d\) is an \(R\)-derivation of \(S_4(R)\) and \(d\) is not inner.

In [1] there is a description of the group \(D^R(S_4(R))\). Note one of the properties of \(R\)-derivations of the ring \(S_4(R)\).

**Corollary 12.1.** If \(d_1\) and \(d_2\) are \(R\)-derivations of \(S_4(R)\) then the composition \(d_1 d_2\) is also \(R\)-derivation of \(S_4(R)\).

13. **A description of regular relations.** Let \(P = M_n(R)_\omega\) be a special subring of \(M_n(R)\).

We know, by Theorem 10.2, that \(P\) is an NS-ring if and only if \(R\) is an NS-ring and the relation \(\omega\) is regular over \(Z(R)\). We know also, by Theorem 11.3, that \(P\) is an NS-ring over \(R\) if and only if the relation \(\omega\) is regular over \(Z(R)\).

In this section we give some sufficient and necessary conditions for the relation \(\omega\) to be regular over an abelian group.

We may reduce our consideration to the case where \(\omega\) is connected (that is, for any \(a, b \in I_n\) there exist elements \(a_1, \ldots, a_r \in I_n\) such that \(a = a_1, b = a_r\) and \(a_i \omega a_{i+1}\) or \(a_{i+1} \omega a_i\), for \(i=1, \ldots, r-1\)), because it is easy to show the following

**Proposition 13.1.** Let \(G\) be an abelian group. The relation \(\omega\) is regular over \(G\) if and only if every connected component of \(\omega\) is regular over \(G\).
The next proposition says that we may also reduce our consideration to the case where $\omega$ is a partial order.

**Proposition 13.2.** $\omega$ is regular over $G$ if and only if $\omega'$(see Section 8) is regular over $G$.

Now we may give a description of regular relations.

**Theorem 13.3.** Assume that $\omega$ is a connected partial order. The following properties are equivalent:

1. $\omega$ is regular over some non-zero group,
2. $\omega$ is regular over every torsion-free group,
3. $\omega$ is regular over some torsion-free group,
4. $\omega$ is regular over $\mathbb{Z}$,
5. $H_1(\Gamma(\omega))$ is finite,
6. $H^1(\Gamma(\omega), G) = 0$, for any torsion-free group $G$.

**Theorem 13.4.** Assume that $\omega$ is connected partial order. The following properties are equivalent:

1. $\omega$ is regular over any group,
2. $\omega$ is regular over $\mathbb{Q}/\mathbb{Z}$,
3. $H_1(\Gamma(\omega)) = 0$,
4. $H^1(\Gamma(\omega), G) = 0$, for any group $G$.

**Theorem 13.5.** Assume that $\omega$ is connected partial order, such that the order of the group $H_1(\Gamma(\omega))$ is equal to $m > 1$. Let $G$ be an abelian group. The following properties are equivalent:

1. $\omega$ is regular over $G$,
2. $G$ is an $m$-torsion-free group,
3. $H^1(\Gamma(\omega), G) = 0$.

**Corollary 13.6.** Let $P = M_n(R)$ be a special subring of $M_m(R)$. The
following properties are equivalent

(1) Every $R$-derivation of $P$ is inner,
(2) The relation $\omega$ is regular over $Z(R)$,
(3) $H^1(\Gamma(\omega), Z(R)) = 0$.

14. **Examples.** Let $P = M_n(R)$ where
a) $n \leq 3$, or
b) the graph $\Gamma(\omega)$ is a tree, or
c) the graph $\Gamma(\omega)$ is a conne (that is, there exists $b \in I_n$ such that $b_\omega a$ or $a_\omega b$ for any $a \in I_n$), in particular $P = M_n(R)$ is the ring of $n \times n$ matrices over $R$, or $P$ is the ring of triangular $n \times n$ matrices over $R$.

Then every $R$-derivation (or every drivation, if every derivation of $R$ is inner) of $P$ is inner.

By Theorem 13.5 it follows that there exist relations $\omega$ which are regular over some groups and which are not regular over another groups. In the paper [1] there is an example of such a relation $\omega$ (for $n = 17$) that if $R$ is 2-torsion-free ring then $P = M_n(R)$ is an NS-ring over $R$, and if char$(R) = 2$ then $P = M_n(R)$ is not an NS-ring over $R$.

IV HIGHER DERIVATIONS IN SPECIAL SUBRINGS

15. **An example of higher derivations.** Let $P = M_n(R)$ be a special subring of $M_n(R)$, $S$ a segment of $N$, and let $d = \{d(t); t \in T_\omega\}$ be a family of derivations of order $s$ (where $s = \sup(S)$) of the ring $R$.

Denote by $\varphi(d)$ the sequence $(d_m)_{m \in S}$ of mappings from $P$ to $P$ defined by

$$d_m(A)_{ij} = d_m(\nu(1))(A_{ij}),$$

for $m \in S$, $A \in P$, where $\nu : I_n \to T_\omega$ is the mapping: $\nu(p) = t$ iff $p \in T_\omega$.
Then $\Theta(d)$ is a derivation of order $s$ of $P$. If $d \neq 0$ then the derivation $\Theta(d)$ is not an $R$-derivation.

In the next sections of this note we shall interesting only in $R$-derivations of order $s$ of $P$.

16. Transitive mappings of order $s$. A sequence $f = (f_m)_{m \in S}$ of mappings $f_m : \omega \rightarrow Z(R)$ is called a transitive mapping of order $s$ (from $\omega$ to $R$) if the following properties are satisfied:

1. $f_0(p,q) = 1$, for all $p \omega q$,
2. $f_m(p,r) = \sum_{i+j=m} f_i(p,q)f_j(q,r)$, for all $m \in S$ and $p \omega q$ and $q \omega r$.

If $f = (f_m)_{m \in S}$ is a transitive mapping of order $s$ then

$$f_1(p,r) = f_1(p,q) + f_1(q,r),$$

for any $p \omega q \omega r$ so, $f_1 : \omega \rightarrow Z(R)$ is a transitive mapping in the sense of Section 7.

17. $R$-derivations of order $s$. In this section we give a description of the group $D^R_s(P)$.

**Example 17.1.** Let $f = (f_m)_{m \in S}$ be a transitive mapping of order $s$ from $\omega$ to $Z(R)$. Denote by $\Delta^f$ the sequence $(\Delta^f_m)_{m \in S}$ of mappings $\Delta^f_m : P \rightarrow P$ defined by the following formula:

$$\Delta^f_m(A)_{pq} = f_m(p,q)A_{pq},$$

for all $A \in P$ and $p \omega q$.

Then $\Delta^f$ is an $R$-derivation of order $s$ of $P$.

**Theorem 17.2.** Every $R$-derivation $d$ of order $s$ of $P$ has a unique representation:

$$d = \Delta(A) \ast \Delta^f,$$

where
(1) $A = (A^{(m)})_{m \in S - \{0\}}$ is a sequence of matrices $A^{(m)} \in \mathbb{P} \cap \mathbb{M}_n(\mathbb{Z}(\mathbb{R}))$ such that $A^{(m)}_{i1} = 0$, for $i=1,\ldots,n$, and $\Delta(A)$ is the inner derivation of order $s$ with respect to $A$;

(2) $f$ is a transitive mapping of order $s$ from $\omega$ to $\mathbb{R}$ and $\Delta^f$ is the $\mathbb{R}$-derivation from Example 17.1.

**Corollary 17.3.** If $d \in D^R_S(P)$ and $U$ is an ideal of $P$ then $d_m(U) \subseteq U$, for all $m \in S$.

**Corollary 17.4.** If $d \in D^R_S(P)$, then $d_m(\mathbb{Z}(\mathbb{R})) = 0$, for all $m \in S - \{0\}$.

**Corollary 17.5.** Assume that there do not exist three different elements $a,b,c \in I_n$ such that $a \omega b \omega c$. Let $d = (d_m)_{m \in S}$ be a sequence of mappings from $P$ to $P$ such that $d_0 = 1d_P$. Then $d$ is an $\mathbb{R}$-derivation of order $s$ of $P$ if and only if every mapping $d_m$ (for $m \in S - \{0\}$) is an ordinary $\mathbb{R}$-derivation of $P$.

18. **Integrable derivations.** Let $S = \{0,1,\ldots,s\}$, where $s < \infty$. Assume that $S'$ is a segment of $\mathbb{N}$ such that $S \subseteq S'$. We say that an $\mathbb{R}$-derivation $d \in D^R_S(P)$ is $s'$-integrable (where $s' = \sup(S') \leq \infty$) if there exists an $\mathbb{R}$-derivation $d' = (d'_m)_{m \in S'}$ of order $s'$ of $P$ such that $d'_m = d_m$, for all $m \in S$.

In the paper [3] there are some necessary conditions for any $\mathbb{R}$-derivation of order $s$ of $P$ to be $s'$-integrable, and there is an example of non-integrable $\mathbb{R}$-derivation (In this example $n=17$ and $R = \mathbb{Z}_2$).

In this paper there are also proofs of the following two partial results:

**Theorem 18.1.** Let $s < s' \leq \beta$. If $H_2(\Gamma(\omega)) = 0$ and $H_1(\Gamma(\omega))$ is a free abelian group then every $\mathbb{R}$-derivation of order $s$ of $P$ is $s'$-integrable.

**Theorem 18.2.** Assume that the homology group $H_1(\Gamma(\omega))$ is free abelian. Then
(1) Every \( R \)-derivation of order \( s < 3 \) of \( P \) is 3-integrable.

(2) If \( R \) is 2-torsion-free then every \( R \)-derivation of order \( s < 5 \) of \( P \) is 5-integrable.

(3) If \( R \) is 6-torsion-free then every \( R \)-derivation of order \( s < 7 \) of \( P \) is 7-integrable.

References


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