

On asymptotic estimates for coefficients of divergent solutions to second order non-homogeneous linear ordinary differential equations.

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1 Introduction

Consider a second order linear ordinary differential equation

$$\frac{d^2 y}{dx^2} - p(x)y = 0 \quad (1)$$

$$p(x) = x^m + a_1 x^{m-1} + \dots + a_m$$

This equation has a irregular singular point at infinity with irregularity $m + 2$.

It is known that this equation has integral solutions in following four cases:

- (i) $p(x) = x + a_1$ (i.e. $m = 1$)
- (ii) $p(x) = x^2 + a_1 x + a_2$ (i.e. $m = 2$)
- (iii) $p(x) = x^m$
- (iv) $p(x) = x^{2p} + cx^{p-1}$

In these cases, each equation will be transformed into confluent hypergeometric differential equation. Especially, the solution of the equation (1) with coefficient (i) is Airy function and (ii) parabolic cylinder function.

Let $\mathcal{O}$ and $\hat{\mathcal{O}}$ be the ring of convergent power series and the ring of formal power series in x , respectively. Then, for a linear ordinary differential operator P with coefficients in holomorphic function, we see the following isomorphism of linear spaces due to Deligne;

$$H^1(S^1, \text{Ker}(P : \mathcal{A}_0)) \cong \text{Ker}(P; \hat{\mathcal{O}}/\mathcal{O} \rightarrow \hat{\mathcal{O}}/\mathcal{O})$$

where $\mathcal{A}_0$ is sheaf of germs of functions asymptotically developable to the formal power series 0 on the circle S^1 .

To pay attention to this isomorphism, by vanishing theorem due to [3] in asymptotic analysis, we obtain the asymptotic estimates for coefficients of divergent solutions to non-homogeneous linear ordinary differential equations about the differential operators $P = \frac{d^2}{dx^2} - p(x)$.

In this note, we see the numerical calculation about the estimation about the case of (iii), (iv). By these calculation, we can see the estimate and the asymptotic estimate for coefficients of a solution of the non-homogeneous differential equation are almost the same.

2 About the operator $P = \frac{d^2}{dx^2} - x^m$

The differential equation

$$Py = \left(\frac{d^2}{dx^2} - x^m\right)y = 0 \quad (2)$$

has irregular singular point at $x = \infty$. The irregularity of P at $x = \infty$ equals $i_\infty(P) = m + 2$. Put

$$U_k = \left\{ x \in \mathbb{C}; |x| > 0, \frac{2k-3}{m+2}\pi < \arg x < \frac{2k+1}{m+2}\pi \right\}, k = 0, 1, \dots, m+1.$$

Then $\{U_k, k = 0, 1, \dots, m+1\}$ forms an open sectorial covering at $x = \infty$, and put

$$S_k = U_k \cap U_{k+1} = \left\{ x \in \mathbb{C}; |x| > 0, \left| \arg x - \frac{2k}{m+2}\pi \right| < \frac{\pi}{m+2} \right\}, k = 0, 1, \dots, m+1,$$

where $U_{m+2} = U_0$.

We have now a unique entire solution $y = y_k$ of the differential equation (2) with the asymptotic representation:

$$y_k \sim \omega^{\frac{m}{4}k} x^{-\frac{m}{4}} \left(1 + \sum_{N=1}^{\infty} B_N \omega^{\frac{k}{2}N} x^{-\frac{N}{2}} \right) \exp\left\{(-1)^{k+1} \frac{2}{m+2} x^{\frac{m+2}{2}}\right\}$$

as x tends to infinity in any closed subsector of the open sector U_k . Where $\omega = e^{\frac{2}{m+2}\pi i}$ and, for $N = 0, 1, \dots$, the quantities B_N are decided by

$$B_{(m+2)(N+1)} = \prod_{\ell=0}^N \frac{-1}{(m+2)(\ell+1)} \left\{ \frac{m}{4} \left(\frac{m}{4} + 1 \right) + \frac{\ell m(m+2)}{4} + \frac{\ell(m+2)}{2} \left(\frac{\ell(m+2)}{2} + 1 \right) \right\}.$$

On the other hand, we have the solution with above properties:

$$y_{2k} = \frac{\left(\frac{4}{m+2}\right)^{\frac{1}{2} + \frac{1}{m+2}}}{\Gamma\left(\frac{1}{2} - \frac{1}{m+2}\right)} \int_0^{+\infty} e^{-x^{\frac{m+2}{2}} \left(\zeta + \frac{2}{m+2}\right) \zeta^{-\frac{1}{2} - \frac{1}{m+2}} \left(\zeta + \frac{4}{m+2}\right)^{-\frac{1}{2} - \frac{1}{m+2}}} d\zeta,$$

$$y_{2k+1} = \frac{(-1)^{\frac{m}{2}+2} \left(\frac{4}{m+2}\right)^{\frac{1}{2} + \frac{1}{m+2}}}{\Gamma\left(\frac{1}{2} - \frac{1}{m+2}\right)} \int_0^{+\infty} e^{x^{\frac{m+2}{2}} \left(\zeta + \frac{2}{m+2}\right) \zeta^{-\frac{1}{2} - \frac{1}{m+2}} \left(\zeta + \frac{4}{m+2}\right)^{-\frac{1}{2} - \frac{1}{m+2}}} d\zeta.$$

Then each y_k gives subdominant solution in S_k .

2.1 Coefficient and its asymptotic estimation of the solution

By using above representations of solutions, we can choose a basis of $H^1(S^1, \text{Ker}(P : \mathcal{A}_0))$ in the following way:

Put 1-cocycle about $\{U_0, U_1, \dots, U_{m+1}\}$ by

$$\{u_{ij}^{(k)}, (i, j) = (0, 1), (1, 2), \dots, (m, m+1), (m+1, 0)\},$$

$$u_{ij}^{(k)} = \begin{cases} y_i & x \in S_k \quad i = k, \\ 0 & x \in S_k \quad i \neq k, \end{cases}$$

for $k = 0, 1, \dots, m+1$.

In this situation, the pair of cohomology classes of $\{u_{ij}^{(0)}\}, \{u_{ij}^{(1)}\}, \dots, \{u_{ij}^{(m+1)}\}$ forms a basis of $H^1(S^1, \text{Ker}(P : \mathcal{A}_0))$.

By vanishing theorem in asymptotic analysis, we have 0-cochain $\{v_0^{(k)}, v_1^{(k)}, \dots, v_{m+1}^{(k)}\}, k = 0, 1, \dots, m+1$, such that

$$u_{ij}^{(k)} = -v_i^{(k)} + v_j^{(k)}, (i, j) = (0, 1), (1, 2), \dots, (m, m+1), (m+1, 0),$$

where each $v_j^{(k)}$ are defined in U_j for $j = 0, 1, \dots, m+1$ and asymptotically developable to formal power-series $\hat{v}^{(k)}(x) = \sum_{r=0}^{\infty} v(m, k, r)x^{-r}, k = 0, 1, \dots, m+1$, respectively. Then

$$Pu_{ij}^{(k)} = -Pv_i^{(k)} + Pv_j^{(k)}.$$

In S_i , we have

$$Pv_i^{(k)} = Pv_j^{(k)}.$$

Put

$$g^{(k)}(x) = \begin{cases} Pv_i^{(k)} & x \in U_i \\ Pv_j^{(k)} & x \in U_j. \end{cases}$$

Then

$$f^{(k)}(x) = \begin{cases} g^{(k)}(x) & x \in \mathbb{C} \\ \lim_{x \rightarrow \infty} g^{(k)}(x) & x = \infty \end{cases}$$

define holomorphic function, and

$$P\hat{v}_i^{(k)}(x) = f^{(k)}(x).$$

Hence, by the vanishing theorem due to [3] in asymptotic analysis, $\{[\hat{v}^{(0)}], [\hat{v}^{(1)}], \dots, [\hat{v}^{(m+1)}]\}$ forms a basis of $\text{Ker}(\frac{d^2}{dx^2} - x^m; \hat{\mathcal{O}}/\mathcal{O})$.

Moreover, we can have estimates for the coefficients $v(m, k, r)$.

$$\begin{aligned} v(m, 2k+1, r) &= (-1)^{\frac{2r}{m+2}+1} \frac{1}{2\pi i} \Gamma\left(\frac{2r}{m+2}\right) \left(\frac{4}{m+2}\right)^{-\frac{2}{m+2}} \left(\frac{2}{m+2}\right)^{1-\frac{2r}{m+2}} \\ &\times \frac{\Gamma(\frac{1}{2} - \frac{1}{m+2}) \Gamma(\frac{2r+2}{m+2})}{\Gamma(\frac{2r+1}{m+2} + \frac{1}{2})} F\left(\frac{2r}{m+2}, \frac{1}{2} - \frac{1}{m+2}, \frac{2r+1}{m+2} + \frac{1}{2}; -1\right). \end{aligned}$$

$$v(m, 2k, r) = (-1)^{\frac{1}{2} - \frac{1}{m+2}} \frac{1}{2\pi i} \Gamma\left(\frac{2r}{m+2}\right) \left(\frac{4}{m+2}\right)^{-\frac{2}{m+2}} \left(\frac{2}{m+2}\right)^{1 - \frac{2r}{m+2}} \\ \times \frac{\Gamma(\frac{1}{2} - \frac{1}{m+2}) \Gamma(\frac{2r+2}{m+2})}{\Gamma(\frac{2r+1}{m+2} + \frac{1}{2})} F\left(\frac{2r}{m+2}, \frac{1}{2} - \frac{1}{m+2}, \frac{2r+1}{m+2} + \frac{1}{2}; -1\right).$$

And we can have asymptotic estimates for any sufficiently large number r ,

$$v(m, k, r) \sim v_M(m, k, r) \\ = \frac{1}{2\pi i} (-\omega^{r+1+k}) \sum_{N=0}^{M-1} B_{m,N} \left(\frac{m+2}{2}\right)^{\frac{2}{m+2}(r - \frac{m}{4} - \frac{N}{2})-1} \Gamma\left(\frac{2}{m+2}(r - \frac{1}{4}m - \frac{N}{2})\right) \\ + O\left(\Gamma\left(\frac{2}{m+2}(r - \frac{m}{4} - \frac{M}{2})\right)\right),$$

provided $1 \leq M < r$.

2.2 About the operator $P = \frac{d^2}{dx^2} - (x^{2p} + cx^{p-1})$

The differential equation

$$Py = \left(\frac{d^2}{dx^2} - (x^{2p} + cx^{p-1})\right)y = 0 \quad (3)$$

has irregular singular point at $x = \infty$. The irregularity of P at $x = \infty$ equals $i_\infty(P) = 2p + 2$. Put

$$U_k = \left\{ x \in \mathbb{C}; |x| > 0, \frac{2k-3}{2p+2}\pi < \arg x < \frac{2k+1}{2p+2}\pi \right\}, k = 0, 1, \dots, 2p+1.$$

Then $\{U_k, k = 0, 1, \dots, 2p+1\}$ forms an open sectorial covering at $x = \infty$, and put

$$S_k = U_k \cap U_{k+1} = \left\{ x \in \mathbb{C}; |x| > 0, \left| \arg x - \frac{k}{p+1}\pi \right| < \frac{\pi}{2p+2} \right\}, k = 0, 1, \dots, 2p+1,$$

where $U_{2p+2} = U_0$.

We have now a unique entire solution $y = y_k$ of the differential equation (3) with the asymptotic representation:

$$y_k \sim \omega^{\frac{p+1}{2}k} x^{-\frac{p+1}{2}} \left(1 + \sum_{N=1}^{\infty} B_{2N(p+1)}(\omega^{-(p+1)k} c) \omega^{kN} x^{-N}\right) e^{-\frac{1}{p+1}(\omega^{-k} x)^{p+1}}$$

as x tends to infinity in any closed subsector of the open sector U_k . Where $\omega = e^{\frac{1}{p+1}\pi i}$ and, for $N = 0, 1, \dots$, the quantities B_N are decided by

$$B_{2N(p+1)}(c) = \frac{1}{8(p+1)^N} \frac{1}{k!} \prod_{\ell=0}^{N-1} (p+c+2\ell(p+1))(p+c+2\ell(p+1)-1),$$

$$B_\ell(c) = 0, \ell \neq 2N(p+1).$$

On the other hand, we have the solution with above properties:

$$y_{2k} = \frac{\left(\frac{2}{p+1}\right)^{\frac{1}{2}\left(\frac{c+1}{p+1}+1\right)}}{\Gamma\left(\frac{1}{2}\left(\frac{c-1}{p+1}+1\right)\right)} \int_0^{+\infty} e^{-x^{p+1}\left(\zeta+\frac{1}{p+1}\right)} \zeta^{\frac{1}{2}\left(\frac{c-1}{p+1}-1\right)} \left(\zeta+\frac{2}{p+1}\right)^{-\frac{1}{2}\left(\frac{c+1}{p+1}+1\right)} d\zeta,$$

$$y_{2k+1} = \frac{(-1)^{\frac{1}{2}\left(\frac{c+1}{p+1}+1\right)+1} \left(\frac{2}{p+1}\right)^{\frac{1}{2}\left(1-\frac{c-1}{p+1}\right)}}{\Gamma\left(-\frac{1}{2}\left(\frac{c+1}{p+1}+1\right)+1\right)} \int_0^{+\infty} e^{x^{p+1}\left(\zeta+\frac{1}{p+1}\right)} \zeta^{-\frac{1}{2}\left(\frac{c+1}{p+1}+1\right)} \left(\zeta+\frac{2}{p+1}\right)^{\frac{1}{2}\left(\frac{c-1}{p+1}-1\right)} d\zeta.$$

Then each y_k gives subdominant solution in the sector S_k .

2.3 Coefficient and its asymptotic estimation of the solution

By using above representations of solution, we can choose a basis of $H^1(S^1, \text{Ker}(P : \mathcal{A}_0))$ with following way:

Put 1-cocycle about $\{U_0, U_1, \dots, U_{2p+1}\}$ by

$$\{u_{i,j}^{(k)}, (i, j) = (0, 1), (1, 2), \dots, (2p, 2p+1), (2p+1, 0)\},$$

$$u_{i,j}^{(k)} = \begin{cases} y_j & x \in S_j \quad j = k, \\ 0 & x \in S_j \quad j \neq k, \end{cases}$$

for $k = 0, 1, \dots, 2p+1$.

In this situation, the pair of the cohomology class of $\{u_{j,j+1}^{(0)}, \{u_{j,j+1}^{(1)}, \dots, \{u_{j,j+1}^{(2p+1)}\}$ forms a basis of $H^1(S^1, \text{Ker}(P : \mathcal{A}_0))$.

By vanishing theorem in asymptotic analysis, we have 0-cochain $\{v_0^{(k)}, v_1^{(k)}, \dots, v_{2p+1}^{(k)}\}, k = 0, 1, \dots, 2p+1$ such that

$$u_{i,j}^{(k)} = -v_i^{(k)} + v_j^{(k)}, (i, j) = (0, 1), (1, 2), \dots, (2p, 2p+1), (2p+1, 0),$$

where each $v_j^{(k)}$ are defined in U_j , for $j = 0, 1, \dots, 2p+1$ and asymptotically developable to formal power-series $\hat{v}^{(k)}(x) = \sum_{r=0}^{\infty} v(2p+1, c, k, r)x^{-r}, k = 0, 1, \dots, 2p+1$, respectively. Then

$$Pu_{i,j}^{(k)} = -Pv_i^{(k)} + Pv_j^{(k)}.$$

In S_i , we have

$$Pv_i^{(k)} = Pv_j^{(k)}.$$

Put

$$g^{(k)}(x) = \begin{cases} Pv_i^{(k)} & x \in U_i \\ Pv_j^{(k)} & x \in U_j. \end{cases}$$

Then

$$f^{(k)}(x) = \begin{cases} g^{(k)}(x) & x \in \mathbb{C} \\ \lim_{x \rightarrow \infty} g^{(k)}(x) & x = \infty \end{cases}$$

define holomorphic function, and

$$P\hat{v}_i^{(k)}(x) = f^{(k)}(x).$$

Hence, by the vanishing theorem due to [3] in asymptotic analysis, $\langle [\hat{v}^{(0)}], [\hat{v}^{(1)}], \dots, [\hat{v}^{(2p+1)}] \rangle$ forms a basis of $\text{Ker}(\frac{d^2}{dx^2} - (x^{2p} + cx^{p-1}); \hat{\mathcal{O}}/\mathcal{O})$.

Moreover, we can have estimates for the coefficients $v(2p+1, c, k, r)$.

$$\begin{aligned} & v(2p+2, c, 2k+1, r) \\ = & (-1)^{\frac{r}{p+1}+1} \frac{1}{2\pi i} \Gamma\left(\frac{r}{p+1}\right) \left(\frac{2}{p+1}\right)^{-\frac{1}{p+1}} \left(\frac{1}{p+1}\right)^{\frac{r}{p+1}+1} \\ & \times \frac{\Gamma(-\frac{1}{2}(\frac{c+1}{p+1}-1))\Gamma(\frac{-r+1}{p+1})}{\Gamma(\frac{r}{p+1}-\frac{1}{2}(\frac{c-1}{p+1}-1))} F\left(\frac{r}{p+1}, -\frac{1}{2}(\frac{c+1}{p+1}-1), \frac{r}{p+1}-\frac{1}{2}(\frac{c-1}{p+1}-1); -1\right). \end{aligned}$$

$$\begin{aligned} & v(2p+2, c, 2k, r) \\ = & \frac{1}{2\pi i} (-1)^{-\frac{1}{p+1}+1} \Gamma\left(\frac{r}{p+1}\right) (p+1)^{-\frac{r}{p+1}+1} \left(\frac{2}{p+1}\right)^{-\frac{1}{p+1}} \\ & \times \frac{\Gamma(\frac{1}{2}(\frac{c-1}{p+1}+1))\Gamma(\frac{1+r}{p+1})}{\Gamma(\frac{r}{p+1}+\frac{1}{2}(\frac{c+1}{p+1}+1))} F\left(\frac{r}{p+1}, \frac{1}{2}(\frac{c-1}{p+1}+1), \frac{r}{p+1}+\frac{1}{2}(\frac{c+1}{p+1}+1); -1\right). \end{aligned}$$

And we can have asymptotic estimates for any sufficiently large number r ,

$$\begin{aligned} & v(2p+2, c, k, r) \\ \sim & v_M(2p+2, c, k, r) \\ = & -\frac{1}{2\pi i} \omega^{kr} \sum_{N=0}^{M-1} B_{2N(p+1)}(\omega^{-(p+1)k} c) (p+1)^{\frac{1}{p+1}(r-(p+1)(N+\frac{1}{2}))-1} \\ & \times \Gamma\left(\frac{1}{p+1}(r-(p+1)(N+\frac{1}{2}))\right) + O\left(\Gamma\left(\frac{1}{p+1}(r-(p+1)(M+\frac{1}{2}))\right)\right), \end{aligned}$$

provided $1 \leq M < r$.

3 numerical calculation

We shall confirm that the estimate and the asymptotic estimate for a coefficient are almost the same by numerical calculation by using symbolic computation system *Mathematica*.

3.1 About the operator $P = \frac{d^2}{dx^2} - x^m$

3.1.1 $\frac{d^2}{dx^2} - x^5$ and $r = 200$

Put $m = 5$ and $r = 200$.

$$v(19, 2k, 400)/v_{37}(19, 2k, 400) = 0.999999999998969635346939.$$

This shows that $v(19, 2k, 400)$ and $v_{37}(19, 2k, 400)$ are almost the same.

When k is odd number, we can have

$$\begin{aligned} v(19, 2k + 1, 400) &= 6.89228977853905057503399197 \times 10^{78} \\ &\quad - 3.979265358972426994974539153 \times 10^{78} \times i. \end{aligned}$$

If we take M equals 37, we have

$$\begin{aligned} v_{37}(19, 2k + 1, 400) &= 6.892289778539121590751656574 \times 10^{78} \\ &\quad - 3.979265358972467995918249507 \times 10^{78} \times i. \end{aligned}$$

Hence we have the module $v(19, 2k + 1, 400)$ of $v_{37}(19, 2k + 1, 400)$

$$v(19, 2k + 1, 400)/v_{37}(19, 2k + 1, 400) = 0.99999999999998969635346939.$$

This shows that $v(19, 2k + 1, 400)$ and $v_{37}(19, 2k + 1, 400)$ are almost the same.

3.2 $P = \frac{d^2}{dx^2} - (x^{2p} + cx^{p-1})$

3.2.1 $\frac{d^2}{dx^2} - (x^8 + 2x^3)$ and $r = 100$

Put $p = 4$ and $r = 100$.

When k is even number, we can have

$$\begin{aligned} v(10, 2, 2k, 100) &= -3.767003417074743939251665848 \times 10^{28} \\ &\quad - 5.184835397614809175077327119 \times 10^{28} \times i. \end{aligned}$$

If we take M equals 19, we have

$$\begin{aligned} v_{19}(10, 2, 2k, 100) &= -3.767003420044648200708650666 \times 10^{28} \\ &\quad - 5.184835401702531706077014179 \times 10^{28} \times i. \end{aligned}$$

Hence we have the module $v(10, 2, 2k, 100)$ of $v_{19}(10, 2, 2k, 100)$

$$v(10, 2, 2k, 100)/v_{19}(10, 2, 2k, 100) = 0.999999999211600327821899299 + 0.10^{-38} \times i.$$

This shows that $v(10, 2, 2k, 100)$ and $v_{19}(10, 2, 2k, 100)$ are almost the same.

When k is odd number, we can have

$$\begin{aligned} v(10, 2, 2k + 1, 100) &= -2.358251967861247883193952833 \times 10^{29} \\ &\quad + 3.245855372479788578747727395 \times 10^{29} \times i. \end{aligned}$$

If we take M equals 19, we have

$$\begin{aligned} v_{19}(10, 2, 2k + 1, 100) &= -2.3582519540410331435767629207 \times 10^{29} \\ &\quad + 3.245855353457894874109400001 \times 10^{29} \times i. \end{aligned}$$

Hence we have the module $v(10, 2, 2k + 1, 100)$ of $v_{19}(10, 2, 2k + 1, 100)$

$$v(10, 2, 2k + 1, 100)/v_{19}(10, 2, 2k + 1, 100) = 1.000000005860363951330673108.$$

This shows that $v(10, 2, 2k + 1, 100)$ and $v_{19}(10, 2, 2k + 1, 100)$ are almost the same.

3.2.2 $\frac{d^2}{dx^2} - (x^{40} + 10x^{19})$ and $r = 500$

Put $p = 20$ and $r = 500$.

When k is even number, we can have

$$\begin{aligned} v(42, 10, 2k, 500) &= -1.033415566729374674610622875 \times 10^{50} \\ &\quad - 2.358703594391783129486533442 \times 10^{49} \times i. \end{aligned}$$

If we take M equals 19, we have

$$\begin{aligned} v_{19}(42, 10, 2k, 500) &= -1.033415566623428208513677288 \times 10^{50} \\ &\quad - 2.358703594149967234273282553 \times 10^{49} \times i. \end{aligned}$$

Hence we have the module $v(42, 10, 2k, 500)$ of $v_{19}(42, 10, 2k, 500)$

$$v(42, 10, 2k, 500)/v_{19}(42, 10, 2k, 500) = 1.000000000102520679500807233 + 0.10^{-37} \times i.$$

This shows that $v(42, 10, 2k, 500)$ and $v_{19}(42, 10, 2k, 500)$ are almost the same.

When k is odd number, we can have

$$\begin{aligned} v(42, 10, 2k + 1, 500) &= -3.018188293136985920927825471 \times 10^{50} \\ &\quad + 2.00243709397029281822076316 \times 10^{51} \times i. \end{aligned}$$

If we take M equals 19, we have

$$\begin{aligned} v_{19}(42, 10, 2k + 1, 500) &= -3.018188293136015310277848565 \times 10^{50} \\ &\quad + 2.0024370939696488601304956088 \times 10^{51} \times i. \end{aligned}$$

Hence we have the module $v(42, 10, 2k + 1, 500)$ of $v_{19}(42, 10, 2k + 1, 500)$

$$v(42, 10, 2k + 1, 500)/v_{19}(42, 10, 2k + 1, 500) = 1.000000000000321587176050042 + 0.10^{-35} \times i.$$

This shows that $v(42, 10, 2k + 1, 500)$ and $v_{19}(42, 10, 2k + 1, 500)$ are almost the same.

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