

REPRESENTATION PROPERTY OF WEIGHTED HARMONIC BERGMAN FUNCTIONS ON THE UPPER HALF-SPACES

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1. INTRODUCTION

Let $\mathbf{H}$ denote the upper half space $\mathbf{R}^{n-1} \times \mathbf{R}_+$ where $\mathbf{R}_+$ denotes the set of all positive real numbers. We will write points $z \in \mathbf{H}$ as $z = (z', z_n)$ where $z' \in \mathbf{R}^{n-1}$ and $z_n > 0$.

For $\alpha > -1$ and $1 \leq p < \infty$, let $b_\alpha^p = b_\alpha^p(\mathbf{H})$ denote the weighted harmonic Bergman space consisting of all real-valued harmonic functions u on $\mathbf{H}$ such that

$$\|u\|_{L_\alpha^p} := \left(\int_{\mathbf{H}} |u(z)|^p dV_\alpha(z) \right)^{1/p} < \infty$$

where $dV_\alpha(z) = z_n^\alpha dz$ and dz is the Lebesgue measure on $\mathbf{R}^n$. Then we can see easily that the space b_α^p is a Banach space. In particular, b_α^2 is a Hilbert space. Hence, there is a unique Hilbert space orthogonal projection Π_α of L_α^2 onto b_α^2 which is called the weighted harmonic Bergman projection. It is known that this weighted harmonic Bergman projection can be realized as an integral operator against the weighted harmonic Bergman kernel $R_\alpha(z, w)$. See section 2.

The purpose of this paper is to survey [8] concerning the representation property of b_α^p -functions and the interpolation by b_α^p -functions.

In the holomorphic case representation and interpolation properties of Bergman functions have been studied in [5] and [11]. In [5], the representation properties of harmonic Bergman functions, as well as harmonic Bloch functions, were also proved on the unit ball in $\mathbf{R}^n$. See [2] for the interpolation properties of holomorphic (little) Bloch functions. On the setting of the half-space of $\mathbf{R}^n$, Choe and Yi [6] have studied these two properties of harmonic Bergman spaces. In [6], the harmonic (little) Bloch spaces are also considered as limiting spaces of b^p .

2. PRELIMINARIES

First, we introduce the fractional derivative. Let D denote the differentiation with respect to the last component and let $u \in b_\alpha^p$. Then the mean value

property, Jensen's inequality and Cauchy's estimate yield

$$(2.1) \quad |D^k u(z)| \leq c z_n^{-(n+\alpha)/p-k}$$

for each $z \in \mathbf{H}$ and for every nonnegative integer k .

Let $\mathcal{F}_\beta$ be the collection of all functions v on $\mathbf{H}$ satisfying $|v(z)| \leq c z_n^{-\beta}$ for $\beta > 0$ and let $\mathcal{F} = \cup_{\beta>0} \mathcal{F}_\beta$. If $v \in \mathcal{F}$, then $v \in \mathcal{F}_\beta$ for some $\beta > 0$. In this case, we define the fractional derivative of v of order $-s$ by

$$(2.2) \quad \mathcal{D}^{-s} v(z) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} v(z', z_n + t) dt$$

for the range $0 < s < \beta$. (Here, Γ is the Gamma function.)

If $u \in b_\alpha^p$, then for every nonnegative integer k , $D^k u \in \mathcal{F}$ by (2.1). Thus for $s > 0$, we define the fractional derivative of u of order s by

$$(2.3) \quad \mathcal{D}^s u = \mathcal{D}^{-([s]-s)} D^{[s]} u.$$

Here, $[s]$ is the smallest integer greater than or equal to s and $\mathcal{D}^0 = D^0$ is the identity operator. If $s > 0$ is not an integer, then $-1 < [s] - s - 1 < 0$ and $[s] \geq 1$. Thus we know from (2.1) that, for each $z \in \mathbf{H}$ and for every $u \in b_\alpha^p$, the integral

$$\mathcal{D}^s u(z) = \frac{1}{\Gamma([s] - s)} \int_0^\infty t^{[s]-s-1} D^{[s]} u(z', z_n + t) dt$$

always makes sense.

Let $P(z, w)$ be the extended Poisson kernel on $\mathbf{H}$ and put $P_z = P(z, \cdot)$. More explicitly,

$$P_z(w) = P(z, w) = \frac{2}{nV(B)} \frac{z_n + w_n}{|z - \bar{w}|^n}$$

where $z, w \in \mathbf{H}$ and $\bar{w} = (w', -w_n)$ and B is the open unit ball in $\mathbf{R}^n$. It is known that the weighted harmonic Bergman projection Π_α of L_α^2 onto b_α^2 is given by

$$\Pi_\alpha f(z) = \int_{\mathbf{H}} f(w) R_\alpha(z, w) dV_\alpha(w)$$

for all $f \in L_\alpha^2$. Here $R_\alpha(z, w)$ denotes the weighted harmonic Bergman kernel whose explicit formula is given by

$$(2.4) \quad R_\alpha(z, w) = C_\alpha \mathcal{D}^{\alpha+1} P_z(w)$$

where $C_\alpha = (-1)^{[\alpha]+1} 2^{\alpha+1} / \Gamma(\alpha + 1)$. Also, it is known that

$$(2.5) \quad |\mathcal{D}_{z_n}^\beta R_\alpha(z, w)| \leq \frac{C}{|z - \bar{w}|^{n+\alpha+\beta}}$$

for all $z, w \in \mathbf{H}$. Here, $\beta > -n - \alpha$ and the constant C is dependent only on n, α and β . Using (2.5), we know $R_\alpha(z, \cdot) \in b_\alpha^q$ for all $1 < q \leq \infty$. Thus, Π_α

is well defined whenever $f \in L_\alpha^p$ for $1 \leq p < \infty$. Also, for $1 \leq p < \infty$, $u \in b_\alpha^p$, $z \in \mathbf{H}$, we have the reproducing formula

$$(2.6) \quad u(z) = \int_{\mathbf{H}} u(w) R_\beta(z, w) dV_\beta(w)$$

whenever $\beta \geq \alpha$. Furthermore, we have a useful norm equivalence. If $\alpha > -1$, $1 \leq p < \infty$ and $(1 + \alpha)/p + \gamma > 0$, then

$$(2.7) \quad \|u\|_{L_\alpha^p} \approx \|w_n^\gamma \mathcal{D}^\gamma u\|_{L_\alpha^p}$$

as u ranges over b_α^p .

Set $z_0 = (0, 1)$. A harmonic function u on $\mathbf{H}$ is called a Bloch function if

$$\|u\|_{\mathcal{B}} = \sup_{w \in \mathbf{H}} w_n |\nabla u(w)| < \infty,$$

where ∇u denotes the gradient of u . We let $\mathcal{B}$ denote the set of Bloch functions on $\mathbf{H}$ and let $\tilde{\mathcal{B}}$ denote the subspace of functions in $\mathcal{B}$ that vanish at z_0 . Then the space $\tilde{\mathcal{B}}$ is a Banach space under the Bloch norm $\|\cdot\|_{\mathcal{B}}$.

A function $u \in \tilde{\mathcal{B}}$ is called a harmonic little Bloch function if it has the following vanishing condition

$$\lim_{z \rightarrow \partial^\infty \mathbf{H}} z_n |\nabla u(z)| = 0$$

where $\partial^\infty \mathbf{H}$ denotes the union of $\partial \mathbf{H}$ and $\{\infty\}$. Let $\tilde{\mathcal{B}}_0$ denote the set of all harmonic little Bloch functions on $\mathbf{H}$. It is not hard to verify that $\tilde{\mathcal{B}}_0$ is a closed subspace of $\tilde{\mathcal{B}}$. Let $\mathcal{C}_0$ denote the set of all continuous functions on $\mathbf{H}$ vanishing at ∞ .

Because $R_\alpha(z, \cdot)$ is not in L_α^1 , $\Pi_\alpha f$ is not well defined for $f \in L^\infty$. So we need the following modified Bergman kernel. For $z, w \in \mathbf{H}$, define

$$\tilde{R}_\alpha(z, w) = R_\alpha(z, w) - R_\alpha(z_0, w).$$

Then, there is a constant $C = C(n, \alpha)$ such that

$$(2.8) \quad |\tilde{R}_\alpha(z, w)| \leq C \left(\frac{|z - z_0|}{|z - \bar{w}|^{n+\alpha} |z_0 - \bar{w}|} + \frac{|z - z_0|}{|z - \bar{w}| |z_0 - \bar{w}|^{n+\alpha}} \right)$$

for all $z, w \in \mathbf{H}$. Thus, (2.8) implies that $\tilde{R}_\alpha(z, \cdot) \in L_\alpha^1$ for each fixed $z \in \mathbf{H}$ and thus we can define $\tilde{\Pi}_\alpha$ on L^∞ by

$$\tilde{\Pi}_\alpha f(z) = \int_{\mathbf{H}} f(w) \tilde{R}_\alpha(z, w) dV_\alpha(w)$$

for $f \in L^\infty$. It turns out that $\tilde{\Pi}_\alpha$ is a bounded linear map from L^∞ onto $\tilde{\mathcal{B}}$. Also, $\tilde{\Pi}_\alpha$ has the following property: If $\gamma > 0$ and $v \in \tilde{\mathcal{B}}$ then

$$(2.9) \quad \tilde{\Pi}_\alpha(w_n^\gamma \mathcal{D}^\gamma v)(z) = C v(z)$$

where $C = C(\alpha, \gamma)$. The Bloch norm is also equivalent to the normal derivative norm : If $\gamma > 0$, then

$$(2.10) \quad \|u\|_{\mathcal{B}} \approx \|w_n^\gamma \mathcal{D}^\gamma u\|_\infty$$

as u ranges over $\tilde{\mathcal{B}}$. (See [7] for details.)

3. TECHNICAL LEMMAS

We first introduce a distance function on $\mathbf{H}$ which is useful for our purposes. The pseudohyperbolic distance between $z, w \in \mathbf{H}$ is defined by

$$\rho(z, w) = \frac{|z - w|}{|z - \bar{w}|}.$$

This ρ is an actual distance. (See [6].) Note that ρ is horizontal translation invariant and dilation invariant. In particular,

$$(3.1) \quad \rho(z, w) = \rho(\phi_a(z), \phi_a(w))$$

for $z, w \in \mathbf{H}$ where $\phi_a (a \in \mathbf{H})$ denotes the function defined by

$$\phi_a(z) = \left(\frac{z' - a'}{a_n}, \frac{z_n}{a_n} \right)$$

for $z \in \mathbf{H}$. Note that the Jacobian of ϕ_a^{-1} is a_n^n . For $z \in \mathbf{H}$ and $0 < \delta < 1$, let $E_\delta(z)$ denote the pseudohyperbolic ball centered at z with radius δ . Note that $\phi_z(E_\delta(z)) = E_\delta(z_0)$ by the invariance property (3.1). Also, simple calculation shows that

$$(3.2) \quad E_\delta(z) = B \left(\left(z', \frac{1 + \delta^2}{1 - \delta^2} z_n \right), \frac{2\delta}{1 - \delta^2} z_n \right)$$

so that $B(z, \delta z_n) \subset E_\delta(z) \subset B(z, 2\delta(1 - \delta)^{-1} z_n)$ where $B(z, r)$ denotes the Euclidean ball centered at z with radius r . From (3.2), we have two lemmas. For proofs of the following lemmas, see [6].

Lemma 3.1. *Let $z, w \in \mathbf{H}$. Then*

$$\frac{1 - \rho(z, w)}{1 + \rho(z, w)} \leq \frac{z_n}{w_n} \leq \frac{1 + \rho(z, w)}{1 - \rho(z, w)}.$$

This lemma implies the following lemma.

Lemma 3.2. *Let $z, w \in \mathbf{H}$. Then*

$$\frac{1 - \rho(z, w)}{1 + \rho(z, w)} \leq \frac{|z - \bar{s}|}{|w - \bar{s}|} \leq \frac{1 + \rho(z, w)}{1 - \rho(z, w)}$$

for all $s \in \mathbf{H}$.

The following lemma is used to prove the representation theorem. If α is a nonnegative integer, then it is proved in [6].

Lemma 3.3. *Let $\alpha > -1$ and β be real. Then*

$$|z_n^\beta R_\alpha(s, z) - w_n^\beta R_\alpha(s, w)| \leq C \rho(z, w) \frac{z_n^\beta}{|z - \bar{s}|^{n+\alpha}}$$

whenever $\rho(z, w) < 1/2$ and $s \in \mathbf{H}$.

Let $\alpha > -1$ and let $1 \leq p < \infty$. Define Π_β on the weighted Lebesgue space L_α^p by

$$\Pi_\beta f(z) = \int_{\mathbf{H}} f(w) R_\beta(z, w) dV_\beta(w)$$

for $f \in L_\alpha^p$ and $z \in \mathbf{H}$. Then we have the following two lemmas from [7].

Lemma 3.4. *Suppose $\alpha > -1$, $1 \leq p < \infty$ and $\alpha + 1 < (\beta + 1)p$. Then Π_β is bounded projection of L_α^p onto b_α^p .*

Lemma 3.5. *For $b < 0$, $-1 < a + b$, there exists a constant $C = C(a, b)$ such that*

$$\int_{\mathbf{H}} \frac{w_n^{a+b}}{|z - \bar{w}|^{n+a}} dw \leq C z_n^b$$

for every $z, w \in \mathbf{H}$.

Lemma 3.6. *Let $\alpha > -1$, $1 \leq p < \infty$ and let $(1 + \alpha)/p + \gamma > 0$. Suppose $0 < \delta < 1$. Then*

$$z_n^{n+p\gamma} |\mathcal{D}^\gamma u(z)|^p \leq \frac{C}{\delta^{n+pk}} \int_{E_\delta(z)} |u(w)|^p dw$$

for all $z \in \mathbf{H}$ and for every u harmonic on $\mathbf{H}$ where $k = [\gamma]$ if $\gamma > -1$ and $k = 0$ if $\gamma \leq -1$. The constant $C = C(n, p, \gamma)$ is independent of δ .

If γ satisfies the condition of Lemma 3.6, we can show $\mathcal{D}^\gamma u$ is harmonic on $\mathbf{H}$. If γ is a nonnegative integer, then $\mathcal{D}^\gamma u$ is harmonic on $\mathbf{H}$, because it is a partial derivative of a harmonic function. If γ is not a nonnegative integer, we see also $\mathcal{D}^\gamma u$ is harmonic on $\mathbf{H}$ by passing the Laplacian through the integral.

The notation $|E|$ denotes the Lebesgue measure of a Borel subset E of $\mathbf{H}$. Let $|E|_\alpha$ denote $V_\alpha(E)$. The following lemma is proved by using the mean value property and Cauchy's estimates. The notation $d(E, F)$ denotes the euclidean distance between two sets E and F .

Lemma 3.7. *Suppose u is harmonic on some proper open subset Ω of $\mathbf{R}^n$. Let $\alpha > -1$ and let $1 \leq p < \infty$. Then, for a given open ball $E \subset \Omega$,*

$$\int_E |u(z) - u(a)|^p dV_\alpha(z) \leq C \frac{|E|^{p/n} |E|_\alpha}{d(E, \partial\Omega)^{n+p}} \int_\Omega |u(w)|^p dw$$

for all $a \in E$. The constant C depends only on n, α and p .

4. REPRESENTATION THEORY

Let $\{z_m\}$ be a sequence in $\mathbf{H}$ and let $0 < \delta < 1$. We say that $\{z_m\}$ is δ -separated if the balls $E_\delta(z_m)$ are pairwise disjoint or simply say that $\{z_m\}$ is separated if it is δ -separated for some δ . Also, we say that $\{z_m\}$ is a δ -lattice if it is $\delta/2$ -separated and $\mathbf{H} = \bigcup E_\delta(z_m)$. Note that any "maximal" $\delta/2$ -separated sequence is a δ -lattice.

From [4] and [6], we have the following three lemmas.

Lemma 4.1. *Fix a $1/2$ -lattice $\{a_m\}$ and let $0 < \delta < 1/8$. If $\{z_m\}$ is a δ -lattice, then we can find a rearrangement $\{z_{ij} : i = 1, 2, \dots, j = 1, 2, \dots, N_i\}$ of $\{z_m\}$ and a pairwise disjoint covering $\{D_{ij}\}$ of $\mathbf{H}$ with the following properties:*

- (a) $E_{\delta/2}(z_{ij}) \subset D_{ij} \subset E_\delta(z_{ij})$
- (b) $E_{1/4}(a_i) \subset \bigcup_{j=1}^{N_i} D_{ij} \subset E_{5/8}(a_i)$
- (c) $z_{ij} \in E_{1/2}(a_i)$

for all $i = 1, 2, \dots$, and $j = 1, 2, \dots, N_i$.

Lemma 4.2. *Let $r > 0$ and let $0 < r\eta < 1$. If $\{z_m\}$ is an η -separated sequence, then there is a constant $M = M(n, r, \eta)$ such that more than M of the balls $E_{r\eta}(z_m)$ contain no point in common.*

Lemma 4.3. *Let N_i be the sequence defined in Lemma 4.1. Then*

$$\sup_i N_i \leq C\delta^{-n}$$

for some constant C depending only on n .

Analysis similar to that for the proof of Lemma 3.4 shows the following lemma which will be used in the proof of Proposition 4.5.

Lemma 4.4. *Let $\alpha > -1$, $1 \leq p < \infty$ and $\alpha + 1 < (\beta + 1)p$. For $f \in L^p_\alpha$, define*

$$\Phi_\beta f(z) = \int_{\mathbf{H}} f(w) \frac{w_n^\beta}{|z - \bar{w}|^{n+\beta}} dw$$

for $z \in \mathbf{H}$. Then, $\Phi_\beta : L^p_\alpha \rightarrow L^p_\alpha$ is bounded.

Let $\{z_m\}$ be a sequence in $\mathbf{H}$. Let $\alpha > -1$, $1 \leq p < \infty$ and $\alpha + 1 < (\beta + 1)p$. For $(\lambda_m) \in l^p$, let $Q_\beta(\lambda_m)$ denote the series defined by

$$(4.1) \quad Q_\beta(\lambda_m)(z) = \sum \lambda_m z_{mn}^{(n+\beta)(1-1/p)+(\beta-\alpha)/p} R_\beta(z, z_m)$$

for $z \in \mathbf{H}$. For a sequence $\{z_m\}$ good enough, $Q_\beta(\lambda_m)$ will be harmonic on $\mathbf{H}$. We say that $\{z_m\}$ is a b^p_α -representing sequence of order β if $Q_\beta(l^p) = b^p_\alpha$. Lemma 4.4 implies the following proposition which shows $Q_\beta(l^p) \subset b^p_\alpha$ if the underlying sequence is separated.

Proposition 4.5. *Let $\alpha > -1$, $1 \leq p < \infty$ and $\alpha + 1 < (\beta + 1)p$. Suppose $\{z_m\}$ is a δ -separated sequence. Then $Q_\beta : l^p \rightarrow b^p_\alpha$ is bounded.*

The following theorem is the b_α^p -representation result under the lattice density condition.

Theorem 4.6. *Let $\alpha > -1$, $1 \leq p < \infty$ and $\alpha + 1 < (\beta + 1)p$. Then there exists $\delta_0 > 0$ with the following property: Let $\{z_m\}$ be a δ -lattice with $\delta < \delta_0$ and let $Q_\beta : l^p \rightarrow b_\alpha^p$ be the associated linear operator as in (4.1). Then there is a bounded linear operator $\mathcal{P}_\beta : b_\alpha^p \rightarrow l^p$ such that $Q_\beta \mathcal{P}_\beta$ is the identity on b_α^p . In particular, $\{z_m\}$ is a b_α^p -representing sequence of order β .*

Since $\mathcal{D}^\gamma u$ is harmonic and we have (2.7), we can have similar result with Proposition 4.8 of [6].

Proposition 4.7. *Let $\alpha > -1$, $1 \leq p < \infty$ and let $(1 + \alpha)/p + \gamma > 0$. If $\{z_m\}$ is a δ -lattice with δ sufficiently small, then*

$$\|u\|_{L_\alpha^p}^p \approx \sum z_{mn}^{n+\alpha+p\gamma} |\mathcal{D}^\gamma u(z_m)|^p$$

as u ranges over b_α^p .

Let $\{z_m\}$ be a sequence in $\mathbf{H}$ and let $\beta > -1$. For $(\lambda_m) \in l^\infty$, let

$$(4.2) \quad \tilde{Q}_\beta(\lambda_m)(z) = \sum \lambda_m z_{mn}^{n+\beta} \tilde{R}_\beta(z, z_m)$$

for $z \in \mathbf{H}$. We say that $\{z_m\}$ is a $\tilde{\mathcal{B}}$ -representing sequence of order β if $\tilde{Q}_\beta(l^\infty) = \tilde{\mathcal{B}}$. We also say that $\{z_m\}$ is a $\tilde{\mathcal{B}}_0$ -representing sequence of order β if $\tilde{Q}_\beta(\mathcal{C}_0) = \tilde{\mathcal{B}}_0$. Then we have the result which shows that a separated sequence represents a part of the whole space.

Proposition 4.8. *Let $\beta > -1$ and suppose $\{z_m\}$ is a δ -separated sequence. Then, $\tilde{Q}_\beta : l^\infty \rightarrow \tilde{\mathcal{B}}$ is bounded. In addition, $\tilde{Q}_\beta$ maps $\mathcal{C}_0$ into $\tilde{\mathcal{B}}_0$.*

If γ is a positive integer, then the following lemma is proved in [6].

Lemma 4.9. *Let $\gamma > 0$. Then*

$$|z_n^\gamma \mathcal{D}^\gamma u(z) - w_n^\gamma \mathcal{D}^\gamma u(w)| \leq C \rho(z, w) \|u\|_{\mathcal{B}}$$

for all $z, w \in \mathbf{H}$ and $u \in \tilde{\mathcal{B}}$.

The following theorem is the limiting version of the b_α^p -representation theorem.

Theorem 4.10. *Let $\beta > -1$. Then there exists a positive number δ_0 with the following property: Let $\{z_m\}$ be a δ -lattice with $\delta < \delta_0$ and let $\tilde{Q}_\beta : l^\infty \rightarrow \tilde{\mathcal{B}}$ be the associated linear operator as in (4.2). Then there exists a bounded linear operator $\tilde{\mathcal{P}}_\beta : \tilde{\mathcal{B}} \rightarrow l^\infty$ such that $\tilde{Q}_\beta \tilde{\mathcal{P}}_\beta$ is the identity on $\tilde{\mathcal{B}}$. Moreover, $\tilde{\mathcal{P}}_\beta$ maps $\tilde{\mathcal{B}}_0$ into $\mathcal{C}_0$. In particular, $\{z_m\}$ is a both $\tilde{\mathcal{B}}$ -representing and $\tilde{\mathcal{B}}_0$ -representing sequence of order β .*

Lemma 4.9 yields the following result for $\tilde{\mathcal{B}}$ analogous to Proposition 4.7.

Proposition 4.11. *Let $\gamma > 0$. Let $\{z_m\}$ be a δ -lattice with δ sufficiently small. Then*

$$\|u\|_{\mathcal{B}} \approx \sup_m z_{mn}^\gamma |\mathcal{D}^\gamma u(z_m)|$$

as u ranges over $\tilde{\mathcal{B}}$.

5. INTERPOLATION THEORY

Let $\{z_m\}$ be a sequence on $\mathbf{H}$. Let $\alpha > -1$, $1 \leq p < \infty$ and $(1 + \alpha)/p + \gamma > 0$. For $u \in b_\alpha^p$, let $T_\gamma u$ denote the sequence of complex numbers defined by

$$(5.1) \quad T_\gamma u = (z_{mn}^{(n+\alpha)/p+\gamma} \mathcal{D}^\gamma u(z_m)).$$

If $T_\gamma(b_\alpha^p) = l^p$, we say that $\{z_m\}$ is a b_α^p -interpolating sequence of order γ .

The following two lemmas are used to prove that separation is necessary for b_α^p -interpolation.

Lemma 5.1. *Let $\alpha > -1$, $1 \leq p < \infty$ and $(1 + \alpha)/p + \gamma > 0$. Let $\{z_m\}$ be a b_α^p -interpolating sequence of order γ . Then $T_\gamma : b_\alpha^p \rightarrow l^p$ is bounded.*

The following lemma is a b_α^p -version of Lemma 4.9 concerning $\tilde{\mathcal{B}}$ -functions. If γ is a nonnegative integer, then the following lemma is proved in [6].

Lemma 5.2. *Let $\alpha > -1$, $1 \leq p < \infty$ and $(1 + \alpha)/p + \gamma > 0$. Then,*

$$|z_n^{(n+\alpha)/p+\gamma} \mathcal{D}^\gamma u(z) - w_n^{(n+\alpha)/p+\gamma} \mathcal{D}^\gamma u(w)| \leq C \rho(z, w) \|u\|_{L_\alpha^p}$$

for all $z, w \in \mathbf{H}$ and $u \in b_\alpha^p$.

Proposition 5.3. *Let $\alpha > -1$, $1 \leq p < \infty$ and $(1 + \alpha)/p + \gamma > 0$. Every b_α^p -interpolating sequence of order γ is separated.*

For interpolation, we need the sufficient separation condition.

Theorem 5.4. *Let $\alpha > -1$, $1 \leq p < \infty$ and $(1 + \alpha)/p + \gamma > 0$. Then there exists a positive number δ_0 with the following property: Let $\{z_m\}$ be a δ -separated sequence with $\delta > \delta_0$ and let $T_\gamma : b_\alpha^p \rightarrow l^p$ be the associated linear operator as in (5.1). Then there is a bounded linear operator $S_\gamma : l^p \rightarrow b_\alpha^p$ such that $T_\gamma S_\gamma$ is the identity on l^p . In particular, $\{z_m\}$ is a b_α^p -interpolating sequence of order γ .*

Let $\gamma > 0$ and let $\{z_m\}$ be a sequence in $\mathbf{H}$. For $u \in \tilde{\mathcal{B}}$, define

$$(5.2) \quad \tilde{T}_\gamma u = (z_{mn}^\gamma \mathcal{D}^\gamma u(z_m)).$$

Then (2.10) implies the operator

$$\tilde{T}_\gamma : \tilde{\mathcal{B}} \rightarrow l^\infty$$

is bounded. If $\tilde{T}_\gamma(\tilde{\mathcal{B}}) = l^\infty$, $\{z_m\}$ is called a $\tilde{\mathcal{B}}$ -interpolating sequence of order γ . Also, if $\tilde{T}_\gamma(\tilde{\mathcal{B}}_0) = \mathcal{C}_0$, $\{z_m\}$ is called a $\tilde{\mathcal{B}}_0$ -interpolating sequence of order γ .

The following proposition shows that separation is also necessary for $\tilde{\mathcal{B}}_0$ interpolation. Since we have Lemma 4.9, the proof of the following proposition is the same as that of Proposition 5.6 in [6].

Proposition 5.5. *Let $\gamma > 0$. Every $\tilde{\mathcal{B}}$ -interpolating sequence of order γ is separated. Also, every $\tilde{\mathcal{B}}_0$ -interpolating sequence of order γ is separated.*

Theorem 5.6. *Let $\gamma > 0$. Then there exists a positive number δ_0 with the following property: Let $\{z_m\}$ be a δ -separated sequence with $\delta > \delta_0$ and let $\tilde{T}_\gamma : \tilde{\mathcal{B}} \rightarrow l^\infty$ be the associated linear operator as in (5.2). Then there exists a bounded linear operator $\tilde{S}_\gamma : l^\infty \rightarrow \tilde{\mathcal{B}}$ such that $\tilde{T}_\gamma \tilde{S}_\gamma$ is the identity on l^∞ . Moreover, $\tilde{S}_\gamma$ maps $\mathcal{C}_0$ into $\tilde{\mathcal{B}}_0$. In particular, $\{z_m\}$ is a both $\tilde{\mathcal{B}}$ -interpolating and $\tilde{\mathcal{B}}_0$ -interpolating sequence of order γ .*

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