

(α, δ) -neighborhood defining by a new operator for certain analytic functions

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Abstract

For analytic functions $f(z)$ in the open unit disk $\mathbb{U}$, a new operator $D^j f(z)$ for any integer j which is the generalization of Sălăgean differential operator and Alexander integral operator is introduced. The object of the present paper is to discuss some properties for (α, δ) -neighborhood defining by a new operator $D^j f(z)$ and to apply Miller-Mocanu lemma (J. Math. Anal. Appl. **65**(1978)) for (α, δ) -neighborhood.

1 Introduction and definitions

Let $\mathcal{A}$ be the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

that are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. For $f(z) \in \mathcal{A}$, Sălăgean [3] has introduced the following operator $D^j f(z)$ which is called Sălăgean differential operator.

$$D^0 f(z) = f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

$$D^1 f(z) = Df(z) = z f'(z) = z + \sum_{n=2}^{\infty} n a_n z^n$$

and

$$D^j f(z) = D(D^{j-1} f(z)) = z + \sum_{n=2}^{\infty} n^j a_n z^n \quad (j = 1, 2, 3, \dots).$$

Also, Alexander [1] has defined the following Alexander integral operator

$$D^{-1} f(z) = \int_0^z \frac{f(\zeta)}{\zeta} d\zeta = z + \sum_{n=2}^{\infty} n^{-1} a_n z^n.$$

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Futher, we introduce

$$D^{-j}f(z) = D^{-1}(D^{-(j-1)}f(z)) = z + \sum_{n=2}^{\infty} n^{-j} a_n z^n \quad (j = 1, 2, 3, \dots)$$

which is the generalization integral operator of Alexander integral operator. Therefore, combining Sălăgean differential operator and Alexander integral operator, we introduce the operator $D^j f(z)$ by

$$D^j f(z) = z + \sum_{n=2}^{\infty} n^j a_n z^n$$

for any integer j . Applying the above operator, we consider the subclass $(\alpha_1, \alpha_2, \dots, \alpha_p; \delta) - N_{m+1}^{j+1}(g_1, g_2, \dots, g_p)$ of $\mathcal{A}$ as follows. A function $f(z) \in \mathcal{A}$ is said to be in the class $(\alpha_1, \alpha_2, \dots, \alpha_p; \delta) - N_{m+1}^{j+1}(g_1, g_2, \dots, g_p)$ if it satisfies

$$\left| \frac{D^{j+1}f(z)}{z} - \sum_{k=1}^p \alpha_k \frac{D^{m+1}g_k(z)}{z} \right| < \delta \quad (z \in \mathbb{U})$$

for some $\delta > \sqrt{1 - 2 \sum_{k=1}^p |\alpha_k| \cos \beta + \left(\sum_{k=1}^p |\alpha_k| \right)^2}$, where $\beta = \arg \alpha_k$ for all k with $-\pi \leq \beta \leq \pi$, and for some $g_k(z) \in \mathcal{A}$ ($k = 1, 2, \dots, p$). Let us define $(\alpha_1, \alpha_2, \dots, \alpha_p; \delta) - N_{m+1}^{j+1}(g_1, g_2, \dots, g_p)$ by

$$(\alpha, \delta) - N_{m+1}^{j+1}(g) \equiv (\alpha_1, \alpha_2, \dots, \alpha_p; \delta) - N_{m+1}^{j+1}(g_1, g_2, \dots, g_p)$$

through this paper.

2 Main theorem

Let us define $g_k(z) \in \mathcal{A}$ ($k = 1, 2, \dots, p$) by

$$g_k(z) = z + \sum_{n=2}^{\infty} b_{n,k} z^n$$

through this paper. Our first result of $f(z)$ for $(\alpha, \delta) - N_{m+1}^{j+1}(g)$ is contained in

Theorem 2.1 *If $f(z) \in \mathcal{A}$ satisfies*

$$\sum_{n=2}^{\infty} n \left| n^j a_n - n^m \sum_{k=1}^p \alpha_k b_{n,k} \right| \leq \delta - \sqrt{1 - 2 \sum_{k=1}^p |\alpha_k| \cos \beta + \left(\sum_{k=1}^p |\alpha_k| \right)^2}$$

for some $\delta > \sqrt{1 - 2 \sum_{k=1}^p |\alpha_k| \cos \beta + \left(\sum_{k=1}^p |\alpha_k| \right)^2}$, where $\beta = \arg \alpha_k$ for all k with $-\pi \leq \beta \leq \pi$, and for some $g_k(z) \in \mathcal{A}$ ($k = 1, 2, \dots, p$), then $f(z) \in (\alpha, \delta) - N_{m+1}^{j+1}(g)$.

Proof. Note that

$$\begin{aligned}
 \left| \frac{D^{j+1}f(z)}{z} - \sum_{k=1}^p \alpha_k \frac{D^{m+1}g_k(z)}{z} \right| &= \left| 1 + \sum_{n=2}^{\infty} n^{j+1} a_n z^{n-1} - \sum_{k=1}^p \alpha_k \left(1 + \sum_{n=2}^{\infty} n^{m+1} b_{n,k} z^{n-1} \right) \right| \\
 &= \left| 1 - \sum_{k=1}^p \alpha_k + \sum_{n=2}^{\infty} n \left(n^j a_n - n^m \sum_{k=1}^p \alpha_k b_{n,k} \right) z^{n-1} \right| \\
 &\leq \left| 1 - \sum_{k=1}^p \alpha_k \right| + \sum_{n=2}^{\infty} n \left| n^j a_n - n^m \sum_{k=1}^p \alpha_k b_{n,k} \right| |z|^{n-1} \\
 &< \sqrt{1 - 2 \sum_{k=1}^p |\alpha_k| \cos \beta + \left(\sum_{k=1}^p |\alpha_k| \right)^2} + \sum_{n=2}^{\infty} n \left| n^j a_n - n^m \sum_{k=1}^p \alpha_k b_{n,k} \right|.
 \end{aligned}$$

If

$$\sum_{n=2}^{\infty} n \left| n^j a_n - n^m \sum_{k=1}^p \alpha_k b_{n,k} \right| \leq \delta - \sqrt{1 - 2 \sum_{k=1}^p |\alpha_k| \cos \beta + \left(\sum_{k=1}^p |\alpha_k| \right)^2},$$

then we see that

$$\left| \frac{D^{j+1}f(z)}{z} - \sum_{k=1}^p \alpha_k \frac{D^{m+1}g_k(z)}{z} \right| < \delta \quad (z \in \mathbb{U}).$$

This gives us that $f(z) \in (\alpha, \delta) - N_{m+1}^{j+1}(g)$. □

Example 2.2 For given $g_k(z) = z + \sum_{n=2}^{\infty} b_{n,k} z^n \in \mathcal{A}$, we consider $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A}$ with

$$a_n = \frac{\delta - \sqrt{1 - 2 \sum_{k=1}^p |\alpha_k| \cos \beta + \left(\sum_{k=1}^p |\alpha_k| \right)^2}}{n^{j+2}(n-1)} e^{i\gamma} + n^{m-j} \sum_{k=1}^p \alpha_k b_{n,k} \quad (n = 2, 3, 4, \dots).$$

Then, we have that

$$\begin{aligned}
 \sum_{n=2}^{\infty} n \left| n^j a_n - n^m \sum_{k=1}^p \alpha_k b_{n,k} \right| &= \sum_{n=2}^{\infty} n \left| \frac{\delta - \sqrt{1 - 2 \sum_{k=1}^p |\alpha_k| \cos \beta + \left(\sum_{k=1}^p |\alpha_k| \right)^2}}{n^{j+2}(n-1)} e^{i\gamma} \right| \\
 &= \left(\delta - \sqrt{1 - 2 \sum_{k=1}^p |\alpha_k| \cos \beta + \left(\sum_{k=1}^p |\alpha_k| \right)^2} \right) \left(\sum_{n=2}^{\infty} \frac{1}{n(n-1)} \right) \\
 &= \left(\delta - \sqrt{1 - 2 \sum_{k=1}^p |\alpha_k| \cos \beta + \left(\sum_{k=1}^p |\alpha_k| \right)^2} \right) \left\{ \sum_{n=2}^{\infty} \left(\frac{1}{n-1} - \frac{1}{n} \right) \right\} \\
 &= \delta - \sqrt{1 - 2 \sum_{k=1}^p |\alpha_k| \cos \beta + \left(\sum_{k=1}^p |\alpha_k| \right)^2}.
 \end{aligned}$$

Therefore, $f(z) \in (\alpha, \delta) - N_{m+1}^{j+1}(g)$.

In view of Theorem 2.1, we have the following corollary.

Corollary 2.3 *Let $f(z) \in \mathcal{A}$ satisfy*

$$\sum_{n=2}^{\infty} n \left| n^j |a_n| - n^m \sum_{k=1}^p |\alpha_k| |b_{n,k}| \right| \leq \delta - \sqrt{1 - 2 \sum_{k=1}^p |\alpha_k| \cos \beta + \left(\sum_{k=1}^p |\alpha_k| \right)^2}$$

for some $\delta > \sqrt{1 - 2 \sum_{k=1}^p |\alpha_k| \cos \beta + \left(\sum_{k=1}^p |\alpha_k| \right)^2}$, where $\beta = \arg \alpha_k$ for all k with $-\pi \leq \beta \leq \pi$, and for some $g_k(z) \in \mathcal{A}$ ($k = 1, 2, \dots, p$) with $\arg a_n - \arg b_{n,k} = \beta$ ($n = 2, 3, 4, \dots$) for all k , then $f(z) \in (\alpha, \delta) - N_{m+1}^{j+1}(g)$.

Proof. By Theorem 2.1, we have that if $f(z) \in \mathcal{A}$ satisfies

$$\sum_{n=2}^{\infty} n \left| n^j a_n - n^m \sum_{k=1}^p \alpha_k b_{n,k} \right| \leq \delta - \sqrt{1 - 2 \sum_{k=1}^p |\alpha_k| \cos \beta + \left(\sum_{k=1}^p |\alpha_k| \right)^2},$$

then $f(z) \in (\alpha, \delta) - N_{m+1}^{j+1}(g)$. Since $\arg a_n - \arg b_{n,k} = \beta$, if $\arg a_n = \varphi_n$, then $\arg b_{n,k} = \varphi_n - \beta$. Therefore, we see that

$$\begin{aligned} n^j a_n - n^m \sum_{k=1}^p \alpha_k b_{n,k} &= n^j |a_n| e^{i\varphi_n} - n^m \sum_{k=1}^p |\alpha_k| e^{i\beta} |b_{n,k}| e^{i(\varphi_n - \beta)} \\ &= \left(n^j |a_n| - n^m \sum_{k=1}^p |\alpha_k| |b_{n,k}| \right) e^{i\varphi_n}, \end{aligned}$$

that is, that

$$\left| n^j a_n - n^m \sum_{k=1}^p \alpha_k b_{n,k} \right| = \left| n^j |a_n| - n^m \sum_{k=1}^p |\alpha_k| |b_{n,k}| \right|.$$

This completes the proof of the corollary. $\square$

Next, we discuss the necessary conditions for neighborhoods.

Theorem 2.4 *If $f(z) \in (\alpha, \delta) - N_{m+1}^{j+1}(g)$ with*

$$\arg \left(n^j a_n - n^m \sum_{k=1}^p \alpha_k b_{n,k} \right) = (n-1)\varphi \quad (\varphi \in \mathbb{R}),$$

for $n = 2, 3, 4, \dots$, then,

$$\sum_{n=2}^{\infty} n \left| n^j a_n - n^m \sum_{k=1}^p \alpha_k b_{n,k} \right| \leq -1 + \sum_{k=1}^p |\alpha_k| \cos \beta + \sqrt{\delta^2 - \left(\sum_{k=1}^p |\alpha_k| \sin \beta \right)^2}.$$

Proof. For $f(z) \in (\alpha, \delta) - N_{m+1}^{j+1}(g)$, if we consider a point $z \in \mathbb{U}$ such that $\arg z = -\varphi$, then

$$z^{n-1} = |z|^{n-1} e^{-i(n-1)\varphi},$$

and hence we have

$$\begin{aligned} \left| \frac{D^{j+1}f(z)}{z} - \sum_{k=1}^p \alpha_k \frac{D^{m+1}g_k(z)}{z} \right| &= \left| 1 - \sum_{k=1}^p \alpha_k + \sum_{n=2}^{\infty} n \left(n^j a_n - n^m \sum_{k=1}^p \alpha_k b_{n,k} \right) z^{n-1} \right| \\ &= \left| 1 - \sum_{k=1}^p \alpha_k + \sum_{n=2}^{\infty} n \left| n^j a_n - n^m \sum_{k=1}^p \alpha_k b_{n,k} \right| |z|^{n-1} \right| < \delta. \end{aligned}$$

Letting $|z| \rightarrow 1^-$ we have

$$\begin{aligned} &\left| 1 - \sum_{k=1}^p \alpha_k + \sum_{n=2}^{\infty} n \left| n^j a_n - n^m \sum_{k=1}^p \alpha_k b_{n,k} \right| \right| \\ &= \left\{ \left(1 - \sum_{k=1}^p |\alpha_k| \cos \beta + \sum_{n=2}^{\infty} n \left| n^j a_n - n^m \sum_{k=1}^p \alpha_k b_{n,k} \right| \right)^2 + \left(\sum_{k=1}^p |\alpha_k| \sin \beta \right)^2 \right\}^{\frac{1}{2}} \leq \delta, \end{aligned}$$

which implies that

$$\begin{aligned} &\left(\sum_{n=2}^{\infty} n \left| n^j a_n - n^m \sum_{k=1}^p \alpha_k b_{n,k} \right| \right)^2 + 2 \left(1 - \sum_{k=1}^p |\alpha_k| \cos \beta \right) \left(\sum_{n=2}^{\infty} n \left| n^j a_n - n^m \sum_{k=1}^p \alpha_k b_{n,k} \right| \right) \\ &\quad + 1 + \left(\sum_{k=1}^p |\alpha_k| \right)^2 - 2 \sum_{k=1}^p |\alpha_k| \cos \beta - \delta^2 \leq 0. \end{aligned}$$

Therefore, it is easy to see that

$$\sum_{n=2}^{\infty} n \left| n^j a_n - n^m \sum_{k=1}^p \alpha_k b_{n,k} \right| \leq -1 + \sum_{k=1}^p |\alpha_k| \cos \beta + \sqrt{\delta^2 - \left(\sum_{k=1}^p |\alpha_k| \sin \beta \right)^2}.$$

□

3 Applications of Miller-Mocanu lemma

In this section, we will give a certain implication for the class $(\alpha, \delta) - N_{m+1}^{j+1}(g)$. To considering our problem, we need the following lemma given by Miller and Mocanu [2].

Lemma 3.1 *Let n be a positive integer, and let $F(z)$ be analytic in $\mathbb{U}$ with $F^{(k)}(0) = 0$ ($k = 1, 2, \dots, n-1$), $F(0) = a$ and $F(z) \neq a$ for a complex number a . If there exists a point $z_0 \in \mathbb{U}$ such that*

$$\max_{|z| \leq |z_0|} |F(z)| = |F(z_0)|,$$

then

$$\frac{z_0 F'(z_0)}{F(z_0)} = m,$$

where m is real and

$$m \geq n \frac{|F(z_0) - a|^2}{|F(z_0)|^2 - |a|^2} \geq n \frac{|F(z_0)| - |a|}{|F(z_0)| + |a|}.$$

Applying Lemma 3.1, we derive

Theorem 3.2 *If $f(z) \in \mathcal{A}$ satisfies*

$$\left| \frac{D^{j+1}f(z)}{z} - \sum_{k=1}^p \alpha_k \frac{D^{m+1}g_k(z)}{z} \right| < \frac{2\delta^2}{\delta + \sqrt{1 - 2 \sum_{k=1}^p |\alpha_k| \cos \beta + \left(\sum_{k=1}^p |\alpha_k| \right)^2}} \quad (z \in \mathbb{U})$$

for some $\delta > \sqrt{1 - 2 \sum_{k=1}^p |\alpha_k| \cos \beta + \left(\sum_{k=1}^p |\alpha_k| \right)^2}$, where $\beta = \arg \alpha_k$ for all k with $-\pi \leq \beta \leq \pi$, and for some $g_k(z) \in \mathcal{A}$ ($k = 1, 2, \dots, p$), then

$$\left| \frac{D^j f(z)}{z} - \sum_{k=1}^p \alpha_k \frac{D^m g_k(z)}{z} \right| < \delta \quad (z \in \mathbb{U}),$$

which implies that $f(z) \in (\alpha, \delta) - N_{m+1}^{j+1}(g)$.

Proof. We define the function $F(z)$ by

$$F(z) = \frac{D^j f(z)}{z} - \sum_{k=1}^p \alpha_k \frac{D^m g_k(z)}{z} \quad (z \in \mathbb{U}).$$

Then,

$$\begin{aligned} \frac{zF'(z)}{F(z)} &= \frac{\frac{D^{j+1}f(z)}{z} - \frac{D^j f(z)}{z} - \sum_{k=1}^p \alpha_k \left(\frac{D^{m+1}g_k(z)}{z} - \frac{D^m g_k(z)}{z} \right)}{\frac{D^j f(z)}{z} - \sum_{k=1}^p \alpha_k \frac{D^m g_k(z)}{z}} \\ &= \frac{1}{F(z)} \left(\frac{D^{j+1}f(z)}{z} - \sum_{k=1}^p \alpha_k \frac{D^{m+1}g_k(z)}{z} \right) - 1. \end{aligned}$$

Therefore,

$$\left| \frac{D^{j+1}f(z)}{z} - \sum_{k=1}^p \alpha_k \frac{D^{m+1}g_k(z)}{z} \right| = \left(1 + \frac{zF'(z)}{F(z)} \right) F(z).$$

Then $F(z)$ is analytic in $\mathbb{U}$ with $F(0) = 1 - \sum_{k=1}^p \alpha_k$ and $|F(0)| < \delta$. In view of the condition, let us suppose that there is a point $z_0 \in \mathbb{U}$ such that $\max_{|z| \leq |z_0|} |F(z)| = |F(z_0)| = \delta$. Then, by Lemma 3.1, we can write that

$$F(z_0) = \delta e^{i\theta}, \quad \frac{z_0 F'(z_0)}{F(z_0)} = m \quad \text{and} \quad m \geq \frac{\left| \delta e^{i\theta} - \left(1 - \sum_{k=1}^p \alpha_k \right) \right|^2}{\delta^2 - \left| 1 - \sum_{k=1}^p \alpha_k \right|^2}.$$

Therefore, we see that

$$\begin{aligned}
\left| \frac{D^{j+1}f(z_0)}{z_0} - \sum_{k=1}^p \alpha_k \frac{D^{m+1}g_k(z_0)}{z_0} \right| &= |1+m||F(z_0)| \\
&= \delta(1+m) \\
&\geq \delta + \delta \frac{\left| \delta e^{i\theta} - \left(1 - \sum_{k=1}^p \alpha_k\right) \right|^2}{\delta^2 - \left|1 - \sum_{k=1}^p \alpha_k\right|^2} \\
&\geq \delta + \delta \frac{\delta - \left|1 - \sum_{k=1}^p \alpha_k\right|}{\delta + \left|1 - \sum_{k=1}^p \alpha_k\right|} \\
&= \frac{2\delta^2}{\delta + \sqrt{1 - 2 \sum_{k=1}^p |\alpha_k| \cos \beta + \left(\sum_{k=1}^p |\alpha_k|\right)^2}}.
\end{aligned}$$

This contradicts our condition in Theorem 3.2. Thus, there is no point $z_0 \in \mathbb{U}$ such that $|F(z_0)| = \delta$. This means that $|F(z)| < \delta$ for all $z \in \mathbb{U}$. Therefore, we have that

$$\left| \frac{D^j f(z)}{z} - \sum_{k=1}^p \alpha_k \frac{D^m g_k(z)}{z} \right| < \delta \quad (z \in \mathbb{U}).$$

□

Taking $p = 1$ in Theorem 3.2, and letting

$$\alpha_1 = e^{i\alpha} \quad \text{and} \quad g_1(z) = g(z),$$

we find the following corollary.

Corollary 3.3 *If $f(z) \in \mathcal{A}$ satisfies*

$$\left| \frac{D^{j+1}f(z)}{z} - e^{i\alpha} \frac{D^{m+1}g(z)}{z} \right| < \frac{2\delta^2}{\delta + \sqrt{2(1 - \cos \alpha)}} \quad (z \in \mathbb{U})$$

for some $-\pi \leq \alpha \leq \pi$, $\delta > \sqrt{2(1 - \cos \alpha)}$ and for some $g(z) \in \mathcal{A}$, then

$$\left| \frac{D^j f(z)}{z} - e^{i\alpha} \frac{D^m g(z)}{z} \right| < \delta \quad (z \in \mathbb{U}).$$

In particular, by putting $\delta = \tilde{\delta} + \sqrt{2(1 - \cos \alpha)}$ for some $-\pi \leq \alpha \leq \pi$ and $\tilde{\delta} > 0$, we can obtain the assertion as follows.

Corollary 3.4 *If $f(z) \in \mathcal{A}$ satisfies*

$$(3.1) \quad \left| \frac{D^{j+1}f(z)}{z} - e^{i\alpha} \frac{D^{m+1}g(z)}{z} \right| < 2\tilde{\delta} + \frac{4(1 - \cos \alpha)}{\tilde{\delta} + 2\sqrt{2(1 - \cos \alpha)}} \quad (z \in \mathbb{U})$$

for some $-\pi \leq \alpha \leq \pi$, $\tilde{\delta} > 0$ and for some $g(z) \in \mathcal{A}$, then

$$(3.2) \quad \left| \frac{D^j f(z)}{z} - e^{i\alpha} \frac{D^m g(z)}{z} \right| < \tilde{\delta} + \sqrt{2(1 - \cos \alpha)} \quad (z \in \mathbb{U}).$$

Remark 3.5 Recently, in the paper by Kugita, Kuroki and Owa [4], we obtained the implication that

$$(3.3) \quad \left| \frac{D^{j+1} f(z)}{z} - e^{i\alpha} \frac{D^{m+1} g(z)}{z} \right| < 2\tilde{\delta} - \sqrt{2(1 - \cos \alpha)} \quad (z \in \mathbb{U})$$

implies the inequality (3.2), where $\tilde{\delta} > \sqrt{2(1 - \cos \alpha)}$. Here, a simple check gives us that if $f(z) \in \mathcal{A}$ satisfies the inequality (3.3), then $f(z)$ satisfies the inequality (3.1). Hence, it follows this fact that if $f(z) \in \mathcal{A}$ satisfies the assertion of Corollary 3.4, then the implication which were proven by Kugita, Kuroki and Owa [4] holds.

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