

Ikeda's conjecture on the period of the Ikeda lift

by Hidenori KATSURADA* and Hisa-aki KAWAMURA†

Abstract

As an affirmative answer to the Duke-Imamoğlu conjecture, Ikeda constructed a certain lifting of classical cusp forms on the special linear group SL_2 towards Siegel cusp forms, namely cuspidal automorphic forms on the symplectic group Sp_{2n} of general even genus $2n$. Afterwards he also proposed a certain conjecture concerning the periods (Petersson norms squared) of such forms. In this paper, we would like to explain a brief sketch of a proof of the conjecture. Details will appear elsewhere.

1 Introduction

For each positive integer $n \in \mathbb{Z}$, the symplectic modular group $Sp_{2n}(\mathbb{Z})$ of genus $2n$ is defined to be

$$Sp_{2n}(\mathbb{Z}) = \left\{ \gamma \in GL_{2n}(\mathbb{Z}) \mid {}^t \gamma J \gamma = J, J = \begin{pmatrix} 0_n & 1_n \\ -1_n & 0_n \end{pmatrix} \right\}.$$

For either an integer or a half-integer $\kappa \in \frac{1}{2}\mathbb{Z}$, we denote the complex vector space consisting of all Siegel cusp forms of weight κ with respect to a suitable congruence subgroup Γ of $Sp_{2n}(\mathbb{Z})$ by $S_\kappa(\Gamma)$. Then for each $F, G \in S_\kappa(\Gamma)$, we define the Petersson scalar product $\langle F, G \rangle$ by

$$\langle F, G \rangle := [Sp_{2n}(\mathbb{Z}) : \Gamma \cdot \{\pm 1_{2n}\}]^{-1} \int_{\Gamma \backslash \mathfrak{H}_n} F(Z) \overline{G(Z)} \det(\operatorname{Im}(Z))^\kappa dZ^*,$$

*Muroran Institute of Technology

†Department of Mathematics, Hokkaido University

where $Z = X + \sqrt{-1}Y \in \mathfrak{H}_n = \{ Z \in \text{Mat}_{n \times n}(\mathbb{C}) \mid {}^t Z = Z, \text{Im}(Z) > 0 \}$ and $dZ^* = \det Y^{-(n+1)} dX dY$ is a finite volume element on $\text{Sp}_{2n}(\mathbb{Z}) \backslash \mathfrak{H}_n$. As is well-known, this defines a Hermitian scalar product on the space $S_\kappa(\Gamma)$ and hence we can introduce the norm $\|F\|^2 := \langle F, F \rangle$ for each $F \in S_\kappa(\Gamma)$. We note that if F is a Hecke eigenform, that is, a common eigenfunction of all Hecke operators, then the Petersson norm squared $\|F\|^2$ plays an important role within the framework of studying critical values of the standard L -function $L(s, F, \text{st})$ attached to F (cf. [1]).

On the other hand, for a couple of positive even integers n and k such that $k > n + 1$, let $f \in S_{2k-n}(\text{Sp}_2(\mathbb{Z})) = S_{2k-n}(\text{SL}_2(\mathbb{Z}))$ be a normalized Hecke eigenform. Then we can consider the lift of f towards the space $S_k(\text{Sp}_{2n}(\mathbb{Z}))$ as follows. Namely, Ikeda ([9]) showed that there exists a Hecke eigenform $F_f \in S_k(\text{Sp}_n(\mathbb{Z}))$ such that

$$L(s, F_f, \text{st}) = \zeta(s) \prod_{i=1}^n L(s + k - i, f),$$

where $\zeta(s)$ and $L(s, f)$ are the Riemann zeta function and the Hecke L -function associated with f , respectively. We note that the above lifting coincides with the Saito-Kurokawa lifting in case $n = 2$, and the existence of the lifting was firstly conjectured by Duke and Imamoglu in case $n > 2$ (cf. [2]). More precisely, Ikeda explicitly constructed F_f by Fourier expansions of f and a Hecke eigenform $g \in S_{k-n/2+1/2}(\Gamma_0^{(2)}(4))$ corresponding to f under the Shimura correspondence, where $\Gamma_0^{(2)}(4) = \{ \gamma \in \text{SL}_2(\mathbb{Z}) \mid \gamma \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod{4} \}$. In this paper, we simply call F_f the Ikeda lift of f .

As will be explained precisely in the subsequent part, Ikeda also conjectured in [10] that the ratio $\|F_f\|^2 / \|g\|^2$ should be expressed in terms of special values of certain L -functions attached to f . The purpose of this paper is to explain a proof of the conjecture. We note that F_f could not necessarily be realized as a theta lift except for the case $n = 2$. Thus we cannot use a general method for evaluating Petersson scalar products of theta lifts due to Rallis (cf. [24]). The method we use is to give explicit formulae for several kinds of Dirichlet series of Rankin-Selberg type attached to Siegel modular forms and then to compare their residues.

We note that we can consider an application of the main result to a problem concerning congruences between Ikeda lifts and some genuine Siegel modular forms. This has been announced in [13, 16], and the details will be discussed in [14].

2 Main results

Throughout this section, we fix a pair of positive even integers $n, k \in \mathbb{Z}$ such that $k > n + 1$.

2.1 Construction of the Ikeda lift

Let $\text{Sym}_n^*(\mathbb{Z})_+$ be the set of all positive definite half-integral symmetric matrices of size n . For each $B \in \text{Sym}_n^*(\mathbb{Z})_+$ and a rational prime p , we put

$$b_p(B; s) := \sum_{R \in \text{Sym}_n(\mathbb{Z}[p^{-1}]) / \text{Sym}_n(\mathbb{Z})} \mathbf{e}(\text{tr}(BR)) p^{-s \cdot \mu_p(R)},$$

where $\mathbf{e}(x) = \exp(2\pi\sqrt{-1}x)$ for $x \in \mathbb{C}$, and $\mu_p(R) = [\mathbb{Z}_p^n R + \mathbb{Z}_p^n : \mathbb{Z}_p^n]$. As is known by Kitaoka ([18]), we have that there exists a unique polynomial $F_p(B; X) \in \mathbb{Z}[X]$ such that

$$b_p(B; s) = F_p(B; p^{-s}) \times \frac{(1 - p^{-s}) \prod_{i=1}^{n/2} (1 - p^{2i-2s})}{1 - \chi_B(p) p^{n/2-s}},$$

where $\chi_B : \mathbb{Z} \rightarrow \{\pm 1, 0\}$ denotes the Kronecker character corresponding to the quadratic field extension $\mathbb{Q}(\sqrt{\mathfrak{D}_B})/\mathbb{Q}$ with $\mathfrak{D}_B := (-1)^{n/2} \det(2B)$. In addition, we can write $\mathfrak{D}_B = \mathfrak{d}_B \mathfrak{f}_B^2$ in terms of a fundamental discriminant $\mathfrak{d}_B$, that is, the discriminant of $\mathbb{Q}(\sqrt{\mathfrak{D}_B})/\mathbb{Q}$ and $\mathfrak{f}_B = \sqrt{\mathfrak{D}_B/\mathfrak{d}_B} \in \mathbb{Z}$. Then it is also known that the Laurent polynomial $\tilde{F}_p(B; X) := X^{-\text{ord}_p(\mathfrak{f}_B)} F_p(B; p^{-(n+1)/2} X)$ is invariant under $X \mapsto X^{-1}$ (cf. [12]).

On the other hand, let

$$f(\tau) = \sum_{m \geq 1} a_f(m) \mathbf{e}(m\tau) \in S_{2k-n}(\text{SL}_2(\mathbb{Z})) \quad (\tau \in \mathfrak{H}_1)$$

be a Hecke eigenform normalized as $a_f(1) = 1$. Then we can associate f with a Hecke eigenform

$$g(\tau) = \sum_{\substack{m \geq 1, \\ (-1)^{k-n/2} m \equiv 0, 1 \pmod{4}}} c_g(m) \mathbf{e}(m\tau) \quad (\tau \in \mathfrak{H}_1)$$

in Kohnen's plus space $S_{k-n/2+1/2}^+(\Gamma_0^{(2)}(4))$ of half-integral weight $k - n/2 + 1/2$, that is, a subspace of $S_{k-n/2+1/2}(\Gamma_0^{(2)}(4))$ characterized by the Shimura's Hecke-equivariant isomorphism

$$S_{k-(n-1)/2}^+(\Gamma_0^{(2)}(4)) \xrightarrow{\simeq} S_{2k-n}(\text{SL}_2(\mathbb{Z}))$$

(cf. [20]). Then Ikeda's lifting theorem is stated as follows:

Theorem I (cf. [9]). *For each $B \in \text{Sym}_n^*(\mathbb{Z})_+$, we put*

$$C_{F_f}(B) := c_g(|\mathfrak{d}_B|) \mathfrak{f}_B^{k-n/2-1/2} \prod_{p \nmid \mathfrak{f}_B} \tilde{F}_p(B; \alpha_p),$$

where $\alpha_p + \alpha_p^{-1} = p^{-k+n/2+1/2} a_f(p)$. Then

$$F_f(Z) = \sum_{B \in \text{Sym}_n^*(\mathbb{Z})_+} C_{F_f}(B) \mathbf{e}(\text{tr}(BZ)) \quad (Z \in \mathfrak{H}_n)$$

belongs to the space $S_k(\text{Sp}_{2n}(\mathbb{Z}))$, and forms a Hecke eigenform such that

$$L(s, F_f, \text{st}) = \zeta(s) \prod_{i=1}^n L(s + k - i, f).$$

We do not consider Eisenstein series here. However, one can formally look at the Ikeda lift as an analogy to the association between Siegel Eisenstein series $E_k^{(2n)}$ of weight k with respect to $\text{Sp}_{2n}(\mathbb{Z})$ and Eisenstein series $E_{2k-n}^{(2)}$ of weight $2k - n$ with respect to $\text{SL}_2(\mathbb{Z})$. Namely, we have

$$L(s, E_k^{(2n)}, \text{st}) = \zeta(s) \prod_{i=1}^n L(s + k - i, E_{2k-n}^{(2)}).$$

2.2 Ikeda's conjecture and the main theorem

In order to state Ikeda's conjecture precisely, we introduce some notations of L -functions as follows. For a given normalized Hecke eigenform $f \in S_{2k-n}(\text{SL}_2(\mathbb{Z}))$ as in the previous section, we put

$$\begin{cases} \tilde{\xi}(s) := \Gamma_{\mathbb{C}}(s) \zeta(s), \\ \Lambda(s, f) := \Gamma_{\mathbb{C}}(s) L(s, f), \\ \tilde{\Lambda}(s, f, \text{ad}) := \Gamma_{\mathbb{C}}(s) \Gamma_{\mathbb{C}}(s + 2k - n - 1) L(s, f, \text{ad}), \end{cases}$$

where $\Gamma_{\mathbb{C}}(s) := 2(2\pi)^{-s} \Gamma(s)$ and $L(s, f, \text{ad})$ denotes the adjoint L -function of f defined by

$$L(s, f, \text{ad}) = \prod_p \{(1 - p^{-s})(1 - \alpha_p^2 p^{-s})(1 - \alpha_p^{-2} p^{-s})\}^{-1}.$$

As is well-known, we have $\tilde{\xi}(2i) = |B_{2i}|/2i \in \mathbb{Q}^\times$ for each positive $i \in \mathbb{Z}$, where B_{2i} is the $2i$ -th Bernoulli number. It is also known that $\tilde{\Lambda}(2i -$

$1, f, \text{Ad})/\|f\|^2$ is an algebraic number for each $1 \leq i < k - n/2$. In particular, we have $\tilde{\Lambda}(1, f, \text{Ad}) = 2^{2k-n}\|f\|^2$ (cf. [26]). Then Ikeda proposed the following:

Conjecture I (cf. [10]). *Under the same situation as in Theorem I, there exists $\alpha(n, k) \in \mathbb{Z}$ such that*

$$\frac{\|F_f\|^2}{\|g\|^2} = 2^{\alpha(n,k)} \Lambda(k, f) \tilde{\xi}(n) \prod_{i=1}^{n/2-1} \tilde{\xi}(2i) \tilde{\Lambda}(2i+1, f, \text{ad}).$$

When $n = 2$, it has been already known by Kohnen and Skoruppa that the above conjecture holds true (cf. [21], see also [23]). Then the main theorem in this paper is stated as follows.

Theorem 2.1. *Conjecture I holds true for any positive even n .*

In the subsequent sections, we will explain a proof of Theorem 2.1 by using a three step-wise approach.

3 Rankin-Selberg method for the Fourier-Jacobi expansion of the Ikeda lift

For the moment, let us review the theory of Fourier-Jacobi expansions of Siegel modular forms of genus $2n \geq 4$ and its application towards the evaluation of Petersson norm squared.

For each positive $k \in \mathbb{Z}$, let $F \in S_k(\text{Sp}_{2n}(\mathbb{Z}))$ possess the Fourier expansion

$$F(Z) = \sum_{B \in \text{Sym}_n^*(\mathbb{Z})_+} C_F(B) \mathbf{e}(\text{tr}(BZ)) \quad (Z \in \mathfrak{H}_n).$$

Then by decomposing each point $Z \in \mathfrak{H}_n$ into the form

$$\begin{pmatrix} \tau' & z \\ t_z & \tau \end{pmatrix} \quad ((\tau, z) \in \mathfrak{H}_{n-1} \times \mathbb{C}^{n-1}, \tau' \in \mathfrak{H}_1),$$

we obtain the Fourier-Jacobi expansion

$$F\left(\begin{pmatrix} \tau' & z \\ t_z & \tau \end{pmatrix}\right) = \sum_{m=1}^{\infty} \phi_m(\tau, z) \mathbf{e}(m\tau'),$$

where

$$\phi_m(\tau, z) := \sum_{\substack{(T,r) \in \text{Sym}_{n-1}^*(\mathbb{Z}) \times \mathbb{Z}^{n-1} \\ 4mT - {}^t r r > 0}} C_F \left(\begin{pmatrix} m & r/2 \\ {}^t r/2 & T \end{pmatrix} \right) \mathbf{e}(\text{tr}(T\tau + {}^t r z)).$$

We note that for each m , the function ϕ_m belongs to the complex vector space $J_{k,m}^{\text{cusp}}(\text{Sp}_{2n-2}(\mathbb{Z})^J)$ consisting of all holomorphic Jacobi cusp forms of weight k and index m with respect to the Jacobi modular group $\text{Sp}_{2n-2}(\mathbb{Z})^J := \text{Sp}_{2n-2}(\mathbb{Z}) \ltimes (\mathbb{Z}^{2n-2} \times \mathbb{Z})$ of genus $2n - 2$ (cf. [28]). Then we define the Dirichlet series $D(s, F)$ attached to F by

$$D(s, F) := \zeta(2s - 2k + 2n) \sum_{m=1}^{\infty} \|\phi_m\|^2 m^{-s},$$

where $\|\phi_m\|^2$ denotes the Petersson norm squared of $\phi_m \in J_{k,m}^{\text{cusp}}(\text{Sp}_{2n-2}(\mathbb{Z})^J)$ introduced to be

$$\begin{aligned} \|\phi_m\|^2 := & \int_{\text{Sp}_{2n-2}(\mathbb{Z})^J \backslash \mathfrak{H}_{n-1} \times \mathbb{C}^{n-1}} |\phi_m(\tau, z)|^2 \det(\text{Im}(\tau))^k \\ & \times \exp(-4\pi m \text{Im}(z) \text{Im}(\tau) {}^t \text{Im}(z)) d\tau^* dz. \end{aligned}$$

We easily see that the Dirichlet series $D(s, F)$ converges absolutely for $\text{Re}(s) > k$. Moreover, Yamazaki showed the following:

Theorem II (cf. [27], see also [22]). *The function*

$$D^*(s, F) := \pi^{k-n-1} (2\pi)^{1-2s} \Gamma(s) D(s, F)$$

has a meromorphic continuation to the whole s -plane, and has simple poles at $s = k, k - n$ with the residue $\|F\|^2$. Furthermore, it satisfies the functional equation

$$D^*(2k - n - s, F) = D^*(s, F).$$

Then, as the first main ingredient of the proof of Theorem 2.1, we have the following:

Theorem 3.1 (cf. [15]). *Let n, k be as in §2. If $f \in S_{2k-n}(\text{SL}_2(\mathbb{Z}))$ is a normalized Hecke eigenform, then*

$$D(s, F_f) = \|\phi_{f,1}\|^2 \zeta(s - k + 1) \zeta(s - k + n) L(s, f),$$

where $\phi_{f,1}$ denotes the first coefficient of the Fourier-Jacobi expansion of F_f .

Moreover, by comparing residues at $s = k$ on both sides, we also obtain

Corollary 3.1. *Under the same situation as above, we have*

$$\frac{\|F_f\|^2}{\|\phi_{f,1}\|^2} = 2^{-k+n-1} \Lambda(k, f) \tilde{\xi}(n). \quad (1)$$

When $n = 2$, the above two results have been obtained by Kohnen and Skoruppa ([21]).

4 The Eichler-Zagier-Ibukiyama isomorphism

Based on the result in the previous section, let us review in this section that there exists a natural correspondence between holomorphic Jacobi forms of integral weight and index 1 and Siegel modular forms of half-integral weight, and explain the coincidence of Petersson norms squared up to scalar.

We put $\Gamma_0^{(2n-2)}(4) := \{ \gamma \in \mathrm{Sp}_{2n-2}(\mathbb{Z}) \mid \gamma \equiv (\mathbf{o}_{n-1}^* \ \ast) \pmod{4} \}$. Then for each $k \in \mathbb{Z}$, we introduce the generalized Kohnen's plus space by

$$S_{k-1/2}^+(\Gamma_0^{(2n-2)}(4)) := \left\{ F(Z) \in S_{k-1/2}(\Gamma_0^{(2n-2)}(4)) \mid \begin{array}{l} C_F(A) = 0 \text{ unless } A \equiv (-1)^{k+1} t_{rr} \\ \pmod{4 \mathrm{Sym}_{n-1}^*(\mathbb{Z})} \text{ for some } r \in \mathbb{Z}^{n-1} \end{array} \right\}.$$

As is mentioned before, for each positive even $k \in \mathbb{Z}$, we have

$$S_{k-1/2}^+(\Gamma_0^{(2)}(4)) \xrightarrow{\cong} S_{2k-2}(\mathrm{SL}_2(\mathbb{Z})).$$

Moreover, Eichler and Zagier ([3]) showed that there exists an isomorphism

$$J_{k,1}^{\mathrm{cusp}}(\mathrm{SL}_2(\mathbb{Z})^J) \xrightarrow{\cong} S_{k-1/2}^+(\Gamma_0^{(1)}(4)),$$

which is compatible with actions of all Hecke operators up to $p = 2$. As a generalization of the isomorphism, Ibukiyama showed the following:

Theorem III (cf. [4]). *If $n \geq 2$, then for each positive even $k \in \mathbb{Z}$, there exists an isomorphism*

$$\sigma : J_{k,1}^{\mathrm{cusp}}(\mathrm{Sp}_{2n-2}(\mathbb{Z})^J) \xrightarrow{\cong} S_{k-1/2}^+(\Gamma_0^{(2n-2)}(4)),$$

which is compatible with actions of Hecke operators up to $p = 2$.

In addition, Eichler and Zagier ([3]) also showed that the isomorphism σ is compatible with Petersson norms squared. As its generalization to higher genus, we obtain the following:

Theorem 4.1. *Under the same assumption as in Theorem III, for each $\phi \in J_{k,1}^{\text{cusp}}(\text{Sp}_{2n-2}(\mathbb{Z})^J)$, we have*

$$\|\phi\|^2 = 2^{2(k-1)(n-1)-1} \|\sigma(\phi)\|^2. \quad (2)$$

Proof. The proof proceeds in a similar way to that of Theorem 5.4 in [3]. $\square$

Thus by combining Corollary 3.1 and Theorem 4.1, we can show Theorem 2.1 in case $n = 2$. Indeed, for a given normalized Hecke eigenform $f \in S_{2k-2}(\text{SL}_2(\mathbb{Z}))$, we denote by $g \in S_{k-1/2}^+(\Gamma_0^{(2)}(4))$ and $\phi_{f,1} \in J_{k,1}^{\text{cusp}}(\text{SL}_2(\mathbb{Z})^J)$ a Hecke eigenform corresponding to f under Shimura's isomorphism and the first coefficient of the Fourier-Jacobi expansion of the Saito-Kurokawa lift $F_f \in S_k(\text{Sp}_{2n}(\mathbb{Z}))$ of f , respectively. Then we have $\sigma(\phi_{f,1}) = g$, and hence by combining the equations (1) and (2), we obtain

$$\frac{\|F_f\|^2}{\|g\|^2} = \frac{\|F_f\|^2}{\|\phi_{f,1}\|^2} \cdot \frac{\|\phi_{f,1}\|^2}{\|g\|^2} = 2^{k-2} \Lambda(k, f) \tilde{\xi}(2),$$

and this proves the assertion. $\square$

5 Rankin-Selberg method for Siegel modular forms of half-integral weight

In this section, we derive an explicit formulae for certain Dirichlet series attached to Siegel modular forms of half-integral weight and apply it to evaluate Petersson norms squared of such forms.

For each positive even $k \in \mathbb{Z}$, we consider

$$F(Z) = \sum_{A \in \text{Sym}_{n-1}^*(\mathbb{Z})_+} C_F(A) \mathbf{e}(\text{tr}(AZ)) \in S_{k-1/2}(\Gamma_0^{(2n-2)}(4)).$$

Then we define the Dirichlet series $R(s, F)$ attached to F by

$$R(s, F) := \sum_{A \in \text{Sym}_{n-1}^*(\mathbb{Z})_+ / \text{SL}_{n-1}(\mathbb{Z})} \frac{|C_F(A)|^2}{e(A) \det A^s},$$

where $e(A) = \#\{X \in \mathrm{SL}_{n-1}(\mathbb{Z}) \mid {}^t X A X = A\}$. This kind of Dirichlet series has been studied by Shimura ([25]) and Kalinin ([11]) in case of integral weight. Then by using a similar method, we easily see the following:

Proposition 5.1. *We put $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2} \Gamma(s/2)$, $\xi(s) = \Gamma_{\mathbb{R}}(s) \zeta(s)$ and*

$$R^*(s, F) := \gamma_{n-1}(s) \xi(2s - 2k + n + 1) \prod_{i=1}^{n/2-1} \xi(4s - 4k - 2i + 2n + 2) R(s, F),$$

where $\gamma_{n-1}(s) = 2^{1-2s(n-1)} \prod_{j=1}^{n-1} \Gamma_{\mathbb{R}}(2s - j + 1)$. Then the function $R^*(s, F)$ has a meromorphic continuation to the whole s -plane and has a simple pole at $s = k - 1/2$ with the residue $\prod_{i=1}^{n/2-1} \xi(2i + 1) \|F\|^2$.

Then we have an explicit formula for the Dirichlet series $R(s, \sigma(\phi_{f,1}))$ as follows:

Theorem 5.2 (cf. [17]). *Under the same situation as in Theorem 3.1, we*

put $\lambda_n = \frac{1}{2} \prod_{i=1}^{n/2-1} \tilde{\xi}(2i)$. Then we have

$$\begin{aligned} R(s, \sigma(\phi_{f,1})) &= \frac{\lambda_n}{2^{(n-1)(s+1/2)}} \zeta(2s + n - 2k + 1)^{-1} \prod_{i=1}^{n/2-1} \zeta(4s + 2n - 4k + 2 - 2i)^{-1} \\ &\quad \times \{ R(s - n/2 + 1, g) \zeta(2s - 2k + 3) \\ &\quad \times \prod_{j=1}^{n/2-1} L(2s - 2k + 2j + 2, f, \mathrm{ad}) \zeta(2s - 2k + 2j + 2) \\ &\quad + (-1)^{n(n-2)/8} R(s, g) \zeta(2s - 2k + n + 1) \\ &\quad \times \prod_{j=1}^{n/2-1} L(2s - 2k + 2j + 1, f, \mathrm{ad}) \zeta(2s - 2k + 2j + 1) \}. \end{aligned}$$

Moreover, by comparing residues at $s = k - 1/2$, we also obtain

Corollary 5.2. *Under the same situation as above, we have*

$$\frac{\|\sigma(\phi_{f,1})\|^2}{\|g\|^2} = 2^{\beta(n,k)} \prod_{i=1}^{n/2-1} \tilde{\xi}(2i) \tilde{\Lambda}(2i + 1, f, \mathrm{ad}), \quad (3)$$

where $\beta(n, k) = -3k(n - 2) + n(n - 3)/2 + 1$.

Therefore, by combining the three equations (1), (2) and (3), we can show Theorem 2.1. Indeed, we have

$$\begin{aligned}
\frac{\|F_f\|^2}{\|g\|^2} &= \frac{\|F_f\|^2}{\|\phi_{f,1}\|^2} \cdot \frac{\|\phi_1\|^2}{\|\sigma(\phi_{f,1})\|^2} \cdot \frac{\|\sigma(\phi_{f,1})\|^2}{\|g\|^2} \\
&= 2^{-k+n-1} \Lambda(k, f) \tilde{\xi}(n) \cdot 2^{2(k-1)(n-1)-1} \cdot 2^{\beta(n,k)} \prod_{i=1}^{n/2-1} \tilde{\xi}(2i) \tilde{\Lambda}(2i+1, f, \text{ad}) \\
&= 2^{-(n-3)(k-n/2)-n+1} \Lambda(k, f) \tilde{\xi}(n) \prod_{i=1}^{n/2-1} \tilde{\xi}(2i) \tilde{\Lambda}(2i+1, f, \text{ad}),
\end{aligned}$$

and this proves the assertion. $\square$

6 Proof of Theorem 5.2

The rest of the paper is devoted to a sketch of a proof of Theorem 5.2. Details will appear in [17]. For each positive $m \in \mathbb{Z}$, we simply write $\mathcal{S}_{m,p} = \text{Sym}_m^*(\mathbb{Z}_p)$ and $\mathcal{S}_{m,p}^\times = \mathcal{S}_{m,p} \cap \text{GL}_m(\mathbb{Q}_p)$. In particular, if m is odd, then we put

$$\mathcal{S}_{m,p}^{(1)} := \{A \in \mathcal{S}_{m,p} \mid A + {}^t r r \in 4\mathcal{S}_{m,p} \text{ for some } r \in \mathbb{Z}_p^m\}.$$

For each $A \in \mathcal{S}_{n-1,p}^{(1)}$, we put

$$\tilde{F}_p^{(1)}(A; X) := \tilde{F}_p\left(\begin{pmatrix} 1 & r/2 \\ {}^t r/2 & (A + {}^t r r)/4 \end{pmatrix}; X\right),$$

where $r = r_A \in \mathbb{Z}_p^{n-1}$ such that $A + {}^t r r \in 4\mathcal{S}_{n-1,p}$. For each $A \in \mathcal{S}_{m,p}^\times$ and $e \geq 0$, we put

$$\mathcal{A}_e(A, A) = \{X \in \text{Mat}_{n-1 \times n-1}(\mathbb{Z}_p)/p^e \text{Mat}_{n-1 \times n-1}(\mathbb{Z}_p) \mid {}^t X A X - A \in p^e \mathcal{S}_{m,p}\}$$

and

$$\alpha_p(A, A) := \frac{1}{2} \lim_{e \rightarrow \infty} p^{e\{-m^2+m(m+1)/2\}} \#\mathcal{A}_e(A, A).$$

For each $\mathfrak{d} \in \mathbb{Z}_p$ and a $\text{GL}_{n-1}(\mathbb{Z}_p)$ -invariant function ω_p on $\mathcal{S}_{n-1,p}^\times$, we put

$$H_p^{(n-1)}(\mathfrak{d}, \omega_p; X, t) := \sum_{l=0}^{\infty} \sum_{A \in \mathcal{A}_p(\mathfrak{d}, l)} \omega_p(A) \frac{|\tilde{F}_p^{(1)}(A; X)|^2}{\alpha_p(A, A)} t^{\text{ord}_p(\det A)},$$

where $\mathcal{A}_p(\mathfrak{d}, l) = \{A \in \mathcal{S}_{n-1,p}^{(1)} \mid \det A = \mathfrak{d} p^{2l+(n-2)\delta_{2,p}}\} / \mathrm{GL}_{n-1}(\mathbb{Z}_p)$. As for $\omega_p : \mathcal{S}_{n-1,p}^\times / \mathrm{GL}_{n-1}(\mathbb{Z}_p) \rightarrow \{\pm 1, 0\}$, we consider either the constant function ι_p on $\mathcal{S}_{n-1,p}^\times$ taking the value 1 or the function ε_p assigning the Hasse invariant of A for $A \in \mathcal{S}_{n-1,p}^\times$ (cf. [19]). Then by using the same method to Ibukiyama and Saito ([8]), similarly to [5, 6], we have

Theorem 6.1. *We have*

$$R(s, \sigma(\phi_{f,1})) = \kappa_{n-1} \sum_{\mathfrak{d}} |c_g(|\mathfrak{d}|)|^2 |\mathfrak{d}|^{-k+n/2+1/2} \times \left\{ \prod_p H_p^{(n-1)}(\mathfrak{d}, \iota_p; \alpha_p, p^{-s+k-1/2}) + \prod_p H_p^{(n-1)}(\mathfrak{d}, \varepsilon_p; \alpha_p, p^{-s+k-1/2}) \right\},$$

where the summation is taken over all fundamental discriminant $\mathfrak{d} \in \mathbb{Z}$ such that $(-1)^{n/2}\mathfrak{d} > 0$ and we put $\kappa_{n-1} = 2^{(n-2)(n-1)/2-\delta_{n,2}} \pi^{-n(n-1)/4} \prod_{i=1}^{n-1} \Gamma(i/2)$.

Moreover, we obtain the following explicit formulae for the power series $H_p^{(n-1)}(\mathfrak{d}, \omega_p; X, t)$:

Theorem 6.2. *Let $\mathfrak{d} \in \mathbb{Z}$ be a fundamental discriminant and $\xi = (\frac{\mathfrak{d}}{p})$, where $(\frac{\mathfrak{d}}{*})$ denotes the Kronecker symbol associated with $\mathfrak{d}$.*

(1) *For $\omega_p = \iota_p$, we have*

$$\begin{aligned} & H_p^{(n-1)}(\mathfrak{d}, \iota_p; X, t) \\ &= \frac{(2^{-(n-1)(n-2)/2} t^{n-2})^{\delta_{2,p}}}{\prod_{i=1}^{n/2-1} (1 - p^{-2i})} (p^{-1}t)^{\mathrm{ord}_p(\mathfrak{d})} (1 - p^{-n}t^2) \prod_{i=1}^{n/2-1} (1 - p^{-2n+2i}t^4) \\ & \quad \times \frac{(1 + p^{-2}t^2)(1 + \xi^2 p^{-3}t^2) - 2\xi p^{-5/2}(X + X^{-1})t^2}{(1 - p^{-2}X^2t^2)(1 - p^{-2}X^{-2}t^2)(1 - p^{-2}t^2)^2} \\ & \quad \times \frac{1}{\prod_{i=1}^{n/2-1} (1 - p^{-2i-1}X^2t^2)(1 - p^{-2i-1}X^{-2}t^2)(1 - p^{-2i-1}t^2)^2}. \end{aligned}$$

(2) *For $\omega_p = \varepsilon_p$, we have*

$$\begin{aligned} & H_p^{(n-1)}(\mathfrak{d}, \varepsilon_p; X, t) = ((-1)^{n(n-2)/8} 2^{-(n-1)(n-2)/2} t^{n-2})^{\delta_{2,p}} \\ & \quad \times \frac{((-1)^{n/2}, (-1)^{n/2}\mathfrak{d})_p}{\prod_{i=1}^{n/2-1} (1 - p^{-2i})} (p^{-n/2}t)^{\mathrm{ord}_p(\mathfrak{d})} (1 - p^{-n}t^2) \prod_{i=1}^{n/2-1} (1 - p^{-2n+2i}t^4) \\ & \quad \times \frac{(1 + p^{-n}t^2)(1 + \xi^2 p^{-n-1}t^2) - 2\xi p^{-1/2-n}(X + X^{-1})t^2}{(1 - p^{-n}X^2t^2)(1 - p^{-n}X^{-2}t^2)(1 - p^{-n}t^2)^2} \\ & \quad \times \frac{1}{\prod_{i=1}^{n/2-1} (1 - p^{-2i}X^2t^2)(1 - p^{-2i}X^{-2}t^2)(1 - p^{-2i}t^2)^2}, \end{aligned}$$

where $(*, *)_p$ denotes the Hilbert symbol over $\mathbb{Q}_p$.

On the other hand, by using the same argument as in Theorem 6.1, we obtain the following:

Proposition 6.3. *Let f and g be a couple of Hecke eigenforms as in § 2. Then we have*

$$R(s, g) = L(2s - 2k + n + 1, f, \text{ad}) \sum_{\mathfrak{d}} |c_g(|\mathfrak{d}|)|^2 |\mathfrak{d}|^{-s} \\ \times \prod_p \left\{ (1 + p^{-2s+2k-n-1}) \left(1 + \left(\frac{\mathfrak{d}}{p} \right)^2 p^{-2s+2k-n-2} - 2 \left(\frac{\mathfrak{d}}{p} \right) a_f(p) p^{-2s+2k-n-3/2} \right) \right\},$$

where the summation is taken over all fundamental discriminant $\mathfrak{d} \in \mathbb{Z}$ such that $(-1)^{n/2} \mathfrak{d} > 0$.

By combining Theorems 6.1, 6.2 and Proposition 6.3, we can prove Theorem 5.2. $\square$

References

- [1] S. Böcherer, *Über die Fourier-Jacobi-Entwicklung Siegelscher Eisensteinreihen II*, Math. Z. **189** (1985), 81–110.
- [2] S. Breulmann and M. Kuss, *On a conjecture of Duke-Imamoğlu*, Proc. Amer. Math. Soc. **128** (2000), no. 6, 1595–1604.
- [3] M. Eichler and D. Zagier, *The theory of Jacobi forms*, Progress in Math., vol. 55, Birkhäuser Boston Inc., Boston, Mass., 1985.
- [4] T. Ibukiyama, *On Jacobi forms and Siegel modular forms of half integral weights*, Comment. Math. Univ. St. Paul. **41** (1992), no. 2, 109–124.
- [5] T. Ibukiyama and H. Katsurada, *An explicit formula for Koecher-Maaß Dirichlet series for Eisenstein series of Klingen type*, J. Number Theory **102** (2003), 223–256.
- [6] _____, *An explicit formula for the Koecher-Maaß Dirichlet series for the Ikeda lifting*, Abh. Math. Sem. Univ. Hamburg **74** (2004), 101–121.

- [7] _____, *Koecher-Maaß series for real analytic Siegel Eisenstein series*, Automorphic forms and zeta functions, 170–197, World Sci. Publ., Hackensack, NJ, 2006.
- [8] T. Ibukiyama and H. Saito, *On zeta functions associated to symmetric matrices. I. An explicit form of zeta functions*, Amer. J. Math. **117** (1995), 1097–1155.
- [9] T. Ikeda, *On the lifting of elliptic modular forms to Siegel cusp forms of degree $2n$* , Ann. of Math. **154** (2001), no. 3, 641–681.
- [10] _____, *Pullback of the lifting of elliptic cusp forms and Miyawaki's conjecture*, Duke Math. J. **131** (2006), no. 3, 469–497.
- [11] V. L. Kalinin, *Analytic properties of the convolution of Siegel modular forms of genus n* , Math. USSR Sbornik, vol. **48** (1984), no. 1, 193–200.
- [12] H. Katsurada, *An explicit formula for Siegel series*, Amer. J. Math. **121** (1999), 415–452.
- [13] _____, *Congruence of Siegel modular forms and special values of their standard zeta functions*, Math. Z. **259** (2008), 97–111.
- [14] _____, *Congruence between Ikeda lifts and non-Ikeda lifts*, Preprint.
- [15] H. Katsurada and H. Kawamura, *A certain Dirichlet series of Rankin-Selberg type associated with the Ikeda lifting*, J. Number Theory **128** (2008), 2025–2052.
- [16] _____, *On Ikeda's conjecture on the period of the Ikeda lift and its application*, to appear in RIMS Koukyuroku Bessatsu.
- [17] _____, *Ikeda's conjecture on the period of the Ikeda lift*, Preprint.
- [18] Y. Kitaoka, *Dirichlet series in the theory of Siegel modular forms*, Nagoya Math. J., **95** (1984), 73–84.
- [19] _____, *Arithmetic of quadratic forms*, Cambridge Tracts Math. 106, Cambridge Univ. Press, Cambridge, 1993.

- [20] W. Kohnen, *Modular forms of half-integral weight on $\Gamma_0(4)$* , Math. Ann. **248** (1980), 249–266.
- [21] W. Kohnen and N.-P. Skoruppa, *A certain Dirichlet series attached to Siegel modular forms of degree two*, Invent. Math. **95** (1989), 541–558.
- [22] A. Krieg, *A Dirichlet series for modular forms of degree n* , Acta Arith. **59** (1991), 243–259.
- [23] T. Oda, *On the poles of Andrianov L -functions*, Math. Ann. **256** (1981), 323–340.
- [24] S. Rallis, *L -functions and Oscillator representation*, Lect. Notes. Math. vol. 1245, Springer-Verlag, 1988.
- [25] G. Shimura, *The special values of the zeta functions associated with cusp forms*, Comm. Pure Appl. Math. **29** (1976), 783–804.
- [26] J. Sturm, *Special values of zeta functions and Eisenstein series of half integral weight*, Amer. J. Math. **102** (1980), 219–240.
- [27] T. Yamazaki, *Rankin-Selberg method for Siegel cusp forms*, Nagoya Math. J. **120** (1990), 35–49.
- [28] C. Ziegler, *Jacobi forms of higher degree*, Abh. Math. Sem. Univ. Hamburg **59** (1989), 191–224.

Hidenori Katsurada

Muroran Institute of Technology
27-1, Mizumoto, Muroran,
050-8585, Japan.

hidenori@mmm.muroran-it.ac.jp

Hisa-aki Kawamura

Department of Mathematics,
Hokkaido University

Kita 10 Nishi 8, Kita-Ku, Sapporo,
060-0810, Japan.

kawamura@math.sci.hokudai.ac.jp