

Geometric and algebraic structures related to highly nonlinear functions

S. Yoshiara

Department of Mathematical Sciences
Tokyo Woman's Christian University

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1 Summary

Throughout this talk,

F is a finite field of p^n elements with p a prime number,
 V is the vector space underlying F (so V is of dimension n over $\mathbb{F}_p$).

We consider two classes of functions on V , called **planar**(or **nonlinear (NL)**) and **almost perfectly nonlinear (APN)**, defined only when $p = 2$.

With each of these functions, an algebraic structure and some geometric structures are associated. For a planar function, the associated geometric structure is an affine plane with some transitivity. The associated algebraic structure is commutative **presemifield** iff the function is **Dembowski-Ostrom(DO)**.

For an APN function, the associated geometric structure is a semiplane. The associative algebraic structure is distributive iff the function is **quadratic**. For a quadratic APN function, we may associate another geometric structure, a certain dimensional dual hyperoval over $\mathbb{F}_2$ with second smallest ambient space.

Algebraic structures associated with planar (resp. APN) functions are realized as the epimorphic images of a vector space $W := (V \otimes V)/A(V)$ (resp. $(V \otimes V)/S(V) \cong A(V)$). The corresponding kernel K is a subspace of W of codimension n in W and contains no vectors corresponding to lines (1-dimensional subspaces) of V .

Exhausting DO PN (resp. quadratic APN) functions up to EA-equivalence is essentially equivalent to finding all such subspaces K up to the diagonal action of $GL(V)$.

I discuss explicit descriptions of $(V \otimes V)/S(V) \cong A(V)$ which seems efficient to examine such subspaces. My final aim is to establish the following statement:

Conjecture 1 *The number of such subspaces grows exponentially as n is getting larger.*

2 Highly Nonlinear Functions

2.1 Planar(or PN) and APN functions

For a function f on V and $0 \neq a \in V$, consider the map $\delta(f)_a$ on V defined by $\delta(f)_a(x) := f(x+a) - f(x)$.

If f is linear, then $\delta(f)_a$ takes a single value $f(a)$, namely, $|\delta(f)_a(V)| = 1$ for every $0 \neq a \in V$. So the opposite property to the linearity is that $|\delta(f)_a(V)|$ is large as possible for every $0 \neq a \in V$. Observe that $|\delta(f)_a(V)| \leq |V|$ if p is odd, and $|\delta(f)_a(V)| \leq |V|/2$ if $p = 2$, because $\delta(f)_a(x+a) = \delta(f)_a(x+a)$ ($x \in V$) in this case.

Definition 1 *With the previous notation,*

- *f is called **planar** (or **perfect nonlinear** (PN)) if $|\delta(f)_a(V)| = |V|$.
Equivalently, $\delta(f)_a$ is bijective for every $0 \neq a \in V$.*
- *f is called **almost perfect nonlinear** (APN) if $|\delta(f)_a(V)| = |V|/2$.
Equivalently, $\delta(f)_a$ is a two to one map for every $0 \neq a \in V$.*

It can be shown that if there exists a PN function on V then p is odd.

2.2 Examples of APN functions

The following maps are APN on $F \cong \mathbb{F}_{2^n}$ for every n .

$$\begin{aligned} g(x) &= x^{2^e+1} \text{ with g.c.d.}(e, n) = 1, \\ f(x) &= x^3 + \sum_{i=0}^{n-1} x^{2^i}. \end{aligned}$$

The second one was found around 2007. Including this family, several infinite series of APN functions are constructed recently (see e.g. [1, Table 2]). The following is the first example of a quadratic APN map which is not graph-equivalent to any monomial map.

Example 1 [4] *On $F \cong \mathbb{F}_{2^{10}}$, $f(x) = x^3 + ux^{36}$ ($u \in F$) is APN iff $u \in \omega K^\times \cup \omega^2 K^\times$, where $K = \mathbb{F}_{2^5}$ and $\omega^3 = 1 \neq \omega \in K$.*

2.3 Graph and Extended affine equivalences

Let f and g be functions on V .

Definition 2 *We say that f is **graph-equivalent** (or **CCZ-equivalent**) to g if there are $\mathbb{F}_p$ -linear maps α, β, γ and δ on V and $c, d \in V$ s.t. $(x, y) \mapsto (x^\alpha + y^\gamma, x^\beta + y^\delta) + (c, d)$ is a bijection on $V \oplus V$ sending $\Gamma_f = \{(x, f(x)) \mid x \in V\}$ to Γ_g .*

*If we may take $\gamma = 0$ in the above, f is called **extended affine** (EA)-equivalent to g . Thus f is EA-equivalent to g if $g(x^\alpha + c) = x^\beta + d + f(x)^\delta$ for every $x \in V$.*

2.4 Some properties on equivalence

Proposition 1 (Some properties on equivalence) *If f is PN (resp. APN), then a function g graph-equivalent to f is PN (resp. APN).*

If p is odd, then a function graph-equivalent to f is also EA-equivalent to f . If f is DO, then any function EA-equivalent to f is DO.

Thus in odd characteristic case, the concept of graph-equivalence coincides with that of EA-equivalence. If $p = 2$, there are examples of graph-equivalent APN functions which are EA-inequivalent.

2.5 DO functions and quadratic functions

Definition 3 A function f on a field $F \cong \mathbb{F}_{p^n}$ is called Dembowski-Ostrom (DO), if f is represented by a polynomial in $F[X]$ of shape

$$a + \sum_{i=0}^{n-1} a_i X^{p^i} + \sum_{0 \leq i < j \leq n-1} a_{ij} X^{p^i + p^j}.$$

If $p = 2$, a DO function is referred to as a quadratic function.

3 Structures associated with planar functions

3.1 A geometric interpretation of a planar function

Let f be a function on V . Define an incidence structure $\mathbb{I}(f)$ as follows: the set of points is $V \oplus V$, and the set of lines is $\{L(a, b), L(c) \mid a, b, c \in V\}$, where $L(a, b)$ and $L(c)$ are just symbols indexed by $(a, b) \in V^2$ and $c \in V$. Incidence is given by $(x, y) \in L(a, b)$ iff $y - b = f(x - a)$, and $(x, y) \in L(c)$ iff $x = c$.

The following is easy to verify (e.g. [2]).

Proposition 2 (A geometric interpretation of a PN function) Let f be a function on V . Then f is PN iff $\mathbb{I}(f)$ is an affine plane.

3.2 Algebraic structure associated with a DO planar function

For a function f on V and $0 \neq a \in V$, we consider the following structure on V .

Definition 4 (Algebraic structure $\mathbb{A}(f)$) $\mathbb{A}(f) := (V; +, \circ_f)$, where $\circ_f = \circ$ is an operation on V defined by $x \circ y := f(x + y) + f(x) + f(y) + f(0)$ ($x, y \in V$).

If f is DO and planar (so p is odd), then the algebraic structure $\mathbb{A}(f)$ is a commutative presemifield, whose definition will be given below (notice that this definition involves the even characteristic case).

Definition 5 A presemifield V is a set with operations $+$ and $\circ$, satisfying:

(S1) $(V, +)$ is a group with identity element 0.

(S2) $x \circ (y + z) = x \circ y + x \circ z$ and $(x + y) \circ z = x \circ z + y \circ z$ for all $x, y, z \in V$.

(S3) $x \circ y = 0$ implies $x = 0$ or $y = 0$.

Let f be a DO planar function on V . Then (S2) follows from the assumption that f is DO. (S3) is equivalent to the condition that $\delta_y(f) = f(x + y) + f(y) = f(x) + f(0)$ has a single solution x for each $0 \neq y \in V$, which is the definition of a PN function.

3.3 Coulter-Henderson's result

In fact, Coulter-Henderson showed that the concept of commutative presemifields with p odd is equivalent to the concept of DO planar functions. See [2] for the details.

4 Structures associated with APN functions

4.1 Geometric interpretation of APN functions

Let f be a function on V . Define an incidence structure $\mathbb{I}(f)$ as follows: the set of **points** is $V \oplus V$, and the set of **blocks** is $\{B(a, b) \mid a, b \in V\}$, where $B(a, b)$ is just a symbol indexed by $(a, b) \in V^2$. Incidence is given by $(x, y) \in B(a, b)$ iff $y - b = f(x - a) + f(0)$. (Notice the similarity of the incidence to that of $\mathbb{A}(f)$ for PN functions.)

Proposition 3 [9] *For a function f on V , f is APN iff the incidence structure $\mathbb{I}(f)$ is the incidence graph of a semiplane. Two APN functions f and g are graph-equivalent iff $\mathbb{I}(f)$ is isomorphic to $\mathbb{I}(g)$ as graphs.*

The later part of the proposition was observed by several researchers, including Dillon and Pott [6]. Here we recall a formal definition of a semiplane.

Definition 6 *An incidence structure $(\mathcal{P}, \mathcal{B}; *)$ is called a **semiplane** if for any two distinct elements in $\mathcal{P}$ (resp. $\mathcal{B}$) there are exactly 0 or 2 elements of $\mathcal{B}$ (resp. $\mathcal{P}$) incident with both of them, and its incidence graph is connected, where the incidence graph of $(\mathcal{P}, \mathcal{B}; *)$ is the graph on $\mathcal{P} \cup \mathcal{B}$ in which two vertices are adjacent if the corresponding elements are incident in $(\mathcal{P}, \mathcal{B}; *)$.*

4.2 A geometric interpretation of quadratic APN functions

Theorem 1 [8] *Let f be a function on V with $\dim(V) = n$ over $\mathbb{F}_2$. Then f is quadratic APN iff the associated structure $\mathcal{S}[f]$ is a **DHO** over $\mathbb{F}_2$ (with ambient space of dimension $2n$ if $n \geq 3$). Two quadratic APN functions f and g are extended affine equivalent iff $\mathcal{S}[f]$ is isomorphic to $\mathcal{S}[g]$ as dimensional dual hyperovals.*

We recall a formal definition of a DHO (dimensional dual hyperoval).

Definition 7 *A collection $\mathcal{S}$ of $(d+1)$ -dimensional subspaces of a vector space W over $\mathbb{F}_q$ is called a **d -dimensional dual hyperoval (DHO)** over $\mathbb{F}_q$, if any two distinct members of $\mathcal{S}$ intersect at a 1-dimensional subspace, any three mutually distinct members of $\mathcal{S}$ intersect at the zero subspace, and $|\mathcal{S}| = ((q^{d+1} - 1)/(q - 1)) + 1$.*

*A subspace of W spanned by all members of $\mathcal{S}$ is called the **ambient space** of $\mathcal{S}$.*

5 Universal algebraic observations

In the algebraic structure $\mathbb{A}(f)$ defined for a function f on F (or its underlying space V), the multiplication $\circ$ is given by $x \circ y = f(x + y) + f(x) + f(y) + f(0)$. In particular, $\circ$ is commutative: $x \circ y = y \circ x$.

If f is DO, then $\circ$ satisfies the left and the right distributive laws. If f is PN (so that p is odd), then $x \circ y = 0$ iff $x = 0$ or $y = 0$. Remark in this case, $x \circ x \neq 0$ for $x \neq 0$. If f is APN (so that $p = 2$), then $x \circ y = 0$ iff $x = 0$ or $y = 0$ or $x = y$.

Summarizing, we have

Proposition 4 (Algebraic structures for PN and APN functions) *Assume that f is a function defined on a finite vector space V over $\mathbb{F}_p$ with p odd (resp. $p = 2$). Then f is DO and PN (resp. quadratic APN) iff algebraic system $\mathbb{A}(f)$ satisfies the following (A1)–(A4) (resp. (A1), (A2), (A3') and (A4)):*

(A1) $(V; +)$ is a vector space over $\mathbb{F}_p$.

(A2) $\circ$ is left and right distributive.

(A3) $x \circ y = 0$ if and only if $x = 0$ or $y = 0$.

(A3') $x \circ y = 0$ if and only if $x = 0$ or $y = 0$, or $x = y$.

(A4) $\circ$ is symmetric.

If f is DO PN (so p is odd), the axioms (A1)–(A4) are nothing more than axioms for commutative presemifield.

In the rest of this section, we consider an arbitrary algebraic structure $(V; +, \circ)$ satisfying either axioms (A1)–(A4) (so it is just a commutative semifield) or axioms (A1), (A2), (A3') and (A4). This algebraic consideration allows us to involve commutative presemifields in characteristic $p = 2$. This also makes clear the relation between commutative semifields in characteristic 2 and the algebraic structure corresponding to quadratic APN functions.

By axiom (A3) (resp. (A3')) and (A4), the form $V \times V \ni (x, y) \mapsto x \circ y \in V$ is an **symmetric** (resp. **alternating**) bilinear map on V . From the universality of tensor product, there is an $\mathbb{F}_p$ -linear surjection $\tilde{\rho}$ from $V \otimes V$ onto V such that $\tilde{\rho}(x \otimes y) = x \circ y$ for all $x, y \in V$.

As $\circ$ is symmetric, $\tilde{\rho}$ vanishes on the subspace $A(V)$ of $V \otimes V$ consisting of $x \oplus y + y \oplus x$ for **distinct** $x, y \in V$: $A(V) := \langle x \otimes y + y \otimes x \mid x, y \in V \rangle$. (Notice that $x \otimes x + x \otimes x = 0$ for $x = y$, if $p = 2$.) Thus $\tilde{\rho}$ induces a surjective linear map ρ from $V \otimes V / A(V)$ onto V .

If f is quadratic APN, $\circ$ vanishes on the larger subspace $S(V)$ of $V \otimes V$ spanned by $A(V)$ and $V^{(2)} = \{x \otimes x \mid x \in V\}$: namely, $S(V) = \langle x \otimes y + y \otimes x, x \otimes x \mid x, y \in V \rangle$. Thus $\tilde{\rho}$ induces a surjective linear map ρ from $V \otimes V / S(V)$ onto V .

The kernel $K := \text{Ker}(\rho)$ has codimension n in $(V \otimes V) / A(V)$ or $(V \otimes V) / S(V)$, according as $\circ$ satisfies (A1)–(A4) or (Ai) ($i = 1, 2, 4$) and (A3'). Moreover, K has the following property by axiom (A3), where $x \otimes y \in V \otimes V$ is identified with its image $(x \otimes y) + A(V)$ in $(V \otimes V) / A(V)$:

$$K \cap \{x \otimes y \mid x, y \in V\} = \{0\}.$$

If f is quadratic APN, then the following property follows from (A3'), where $x \otimes y \in V \otimes V$ is identified with its image $(x \otimes y) + S(V)$ in $(V \otimes V) / S(V)$: (notice that as $x \otimes x \in V^{(2)}$, we only need $x \otimes y$ for **distinct** $x, y \in V$.)

$$K \cap \{x \otimes y \mid x \neq y \in V\} = \{0\}.$$

Conversely, if a subspace K of $W := (V \otimes V) / A(V)$ satisfies

$$\text{codim}(K) = \dim(W) - \dim(K) = n \text{ and } K \cap \{x \otimes y \mid x, y \in V\} = \{0\}.$$

then the operation $\circ$ on $(V; +)$ defined by $x \circ y := \alpha((x \otimes y) + K)$ for $x, y \in V$ satisfies the axiom of a commutative presemifield, where α is any isomorphism of W/K with V .

Similar conclusion holds for $\bar{W} := (V \otimes V)/S(V)$. Namely, if a subspace K of $\bar{W} := (V \otimes V)/S(V)$ satisfies the following two properties

$$\text{codim}(K) = \dim(\bar{W}) - \dim(K) = n \text{ and } K \cap \{x \otimes y \mid x \neq y \in V\} = \{0\}.$$

then the operation $\circ$ on $(V; +)$ defined by $x \circ y := \alpha((x \otimes y) + K)$ for $x, y \in V$ satisfies the axioms (A1), (A2), (A3') and (A4), where α is any isomorphism of $\bar{W}/K$ with V .

A canonical form of quadratic APN functions

5.1 Canonical form of a quadratic APN function

Now we return to the case when $\circ = \circ_f$ is determined by a quadratic function f on V : $x \circ_f y = f(x + y) + f(x) + f(y) + f(0)$. Notice that $\circ_f$ coincides with $\circ_g$ iff $f + g$ is an affine function on V . Hence the conclusion of previous section shows the following canonical description of quadratic APN functions, because $A(V)$ can be identified with $(V \otimes V)/S(V)$ via $x \wedge y \mapsto x \otimes y + S(V)$.

This result was first obtained by examining the universal DHO of $S[f]$.

Let Γ be the set of all $\mathbb{F}_2$ -linear surjections γ from $A(V)$ to V with $\text{Ker}(\gamma) \cap \{a \wedge b \mid a, b \in V\} = \{0\}$, and let Af be the set of $\mathbb{F}_2$ -affine maps on V . Fix a basis $\{e_i\}_{i=1}^n$ for V over $\mathbb{F}_2$. For every (γ, α) of $\Gamma \times Af$, the following map $f_{\gamma, \alpha}$ is quadratic APN on V :

$$f_{\gamma, \alpha} : a = \sum_{i=1}^n a_i e_i \mapsto \sum_{1 \leq i < j \leq n} a_i a_j (e_i \wedge e_j)^\gamma + a^\alpha.$$

Theorem 2 [10] *Every quadratic APN map on L is uniquely written as $f_{\gamma, \alpha}$ for (γ, α) . Namely, there is a bijection between the set of quadratic APN maps on L and the set $\Gamma \times Af$.*

5.2 Equivalence

Theorem 3 [10] *For two quadratic APN maps $f_{\gamma, \alpha}$ and $f_{\gamma', \alpha'}$, they are EA-equivalent iff $\text{Ker}(\gamma)$ and $\text{Ker}(\gamma')$ belong to the same orbit under the diagonal action of $GL(V)$: $g(a \wedge b) = g(a) \wedge g(b)$ ($a, b \in V$).*

5.3 Core problem

Thus, finding all the EA-equivalence classes of quadratic APN maps on V is equivalent to finding all $GL(V)$ -orbit on the set of subspaces K of $(V \otimes V)/S(V) =: \bar{W}$ such that:

$$\text{codim}(K) = \dim(\bar{W}) - \dim(K) = n \text{ and } K \cap \{a \wedge b \mid a \neq b \in V\} = \{0\}$$

We call a subspace K of $\bar{W}$ with the above property **line-skew**.

When $p = 2$, $\bar{W} = (V \otimes V)/S(V)$ is a quotient of $W = (V \otimes V)/A(V)$.

(Question) Are there some relations between subspaces K of codimension n in W which yield commutative semifields (namely, $K \cap \{x \otimes y \mid x, y \in V\} = \{0\}$) and subspaces $\bar{K}$ of codimension n in $\bar{W}$ which yield quadratic APN functions (namely, $\bar{K} \cap \{x \otimes y \mid x \neq y \in V\} = \{0\}$).

6 Some explicit description of $A(V)$

6.1 Alternating form scheme $Alt(V)$

We assume that $p = 2$. Then $(V \otimes V)/S(V) \cong A(V)$ by identifying $x \otimes y + S(V)$ with $x \wedge y := x \otimes y + y \otimes x$.

$A(V)$ can also be identified with the space $Alt(V)$ of all alternating bilinear forms on V , by identifying $x \wedge y$ with the alternating form of rank 1 with $f(x, y) = 1$. Here the rank of an alternating form f is $(\dim(V) - \dim Rad(f))/2$.

Recall that $Alt(V)$ is an association scheme with respect to the distance δ given by $\delta(f, g) = \text{the rank of } f - g$. Thus a subspace K of $Alt(V)$ of codimension n is line-skew iff it does not contain form of rank 1 iff any two distinct forms of K are at distance at least 2.

6.2 Line skew subspace as designs in $Alt(V)$

Delsarte and Goethals [3] investigated a subset D of $Alt(V)$ in which two distinct elements are at distance at least d . They obtained the bound $|D| \leq 2^{n(n+1-2d)/2}$ or $|D| \leq 2^{(n-1)(n+2-2d)/2}$ according as n is odd or even. As $\dim(K) = \dim(Alt(V)) - n = n(n-3)/2$, this bound is attained by K if n is odd.

With current terminologies in algebraic combinatorics, we have:

Proposition 5 (Line-skew space as Delsarte design) *Assume that $n = 2m + 1$ is odd. A subspace K of $Alt(V)$ is line-skew iff it is a $(m-1)$ -design in $Alt(V)$ in the sense of Delsarte.*

The previous theorem gives us several strong information about a line-skew subspace, if $\dim(V) = n$ is odd (e.g. [7]). However, so far I could not obtain explicit informations on the numbers of such spaces.

6.3 Another explicit description of $Alt(V)$

We identify V with the field $F \cong \mathbb{F}_{2^n}$, and denote by $F_0 \cong \mathbb{F}_{2^{n/2}}$ the subfield of F of degree 2 if n is even. We set $l = \lfloor n/2 \rfloor$.

Then $Alt(V)$ is isomorphic to $V^l = V^m = \{(b_k)_{k=1}^l \mid b_k \in V\}$ if $n = 2m + 1$ is odd, and to the subspace of V^l with b_l lies in F_0 if $n = 2m + 2$ is even.

The explicit isomorphism can be described. In particular,

Proposition 6 (Subsets corresponding to rank 1 forms) *the set of rank 1 alternating forms corresponds to $\mathcal{L} := \{(x^{2^{k+1}}(y + y^{2^k}))_{k=1}^l \mid x, y \in F \setminus \mathbb{F}_2\}$.*

6.4 Some line-skew subspaces

For every $1 \leq e \leq l$ coprime with n , the e -th entry $x^{2^{e+1}}(y + y^{2^e})$ is nonzero for any vector $(x^{2^{k+1}}(y + y^{2^k}))_{k=1}^l$ of $\mathcal{L}$. Thus the subspace K_e of V^l consisting of all vectors (b_k) with $b_k = 0$ does not contain any vector of $\mathcal{L}$. As K_e has codimension n in $Alt(V)$ (identified with the subspace of V^l described above), K_e is a line-skew subspace. The

canonical projection map $\rho : \text{Alt}(V) \rightarrow \text{Alt}(V)/K_e$ composed with an identification $\text{Alt}(V)/K_e \ni (b_k)_{k=1}^l + K_e \mapsto b_k \in V$ gives $x \wedge y \mapsto x^{2^e}y + xy^{2^e}$. Hence this corresponds to the Gold function $g(x) = x^{2^e+1}$.

We also have line-skew subspace K consisting of $(b_k)_{k=1}^l$ with $b_1 + \sum_{i=0}^{n-1} b_3^{2^i}$. This gives the APN map $f(x) = x^3 + \sum_{i=0}^{n-1} (x^9)^{2^i}$.

When $n = 10$, $K = \{(b_k)_{k=1}^5 \mid b_1 = ub_3^4\}$ is a line-skew subspace yielding APN function $e(x) = x^3 + ux^{36}$.

6.5 Some comments

The last description of $\text{Alt}(V)$ seems explicit enough to find ‘easy’ examples of skew-free subspaces, and so quadratic APN functions.

Recently, Dillon, Edel and Pott [5] introduce the idea of ‘switching’ of APN functions, and produces many new examples of APN functions (including non-quadratic examples). In my setting, switching relation may be interpreted as two line-skew subspaces sharing a hyperplane. I am wondering if this suggests some new direction to generalize the idea of switching.

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