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Semiconcavity results of squared sub-Riemannian distance on Heisenberg group and applications to PDEs

By

Ye ZHANG*

This note is a survey of the results in the papers [9, 18].

§ 1. Euclidean convexity and semiconvexity

Recall that a real function f on an open set $\Omega \subset \mathbb{R}^n$ is called *semiconvex* if there exists a constant $C \geq 0$ such that the following inequality holds:

$$f((1-\lambda)x + \lambda y) \leq (1-\lambda)f(x) + \lambda f(y) + \frac{C}{2}\lambda(1-\lambda)|x-y|^2,$$

for every $x, y \in \Omega$ such that $[x, y] := \{(1-\mu)x + \mu y;\ \mu \in [0,1]\} \subset \Omega$ and $\lambda \in [0,1]$. Here the constant C is called a *semiconvexity constant*. If $f \in C^2(\Omega)$, then it is equivalent to $\operatorname{Hess} f \geq -C \operatorname{Id}_n$. Here we use Hess to denote the usual Hessian matrix on $\mathbb{R}^n$ and Id_n to denote the $n \times n$ identity matrix. If the constant $C = 0$, then it is exactly the usual notion of *convexity* on Euclidean spaces. We call a function f *semiconcave* (resp. *concave*) if $-f$ is semiconvex (resp. convex).

Here two typical examples are given by *distance function from a fixed point* $d_y(x) := |x-y|$, which is convex on $\mathbb{R}$ and semiconcave on $\mathbb{R} \setminus \{y\}$, and squared distance function from a fixed point $d_y^2(x) = |x-y|^2$, which is both convex and semiconcave on $\mathbb{R}$.

The idea is that: the "corner-like" singularity like the one in Figure 1 will break the semiconcavity, and the "corner-like" singularity with the opposite direction (like the graph of $-d_y$) will break the semiconvexity.

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*Analysis on Metric Spaces Unit, Okinawa Institute of Science and Technology, 1919-1 Tancha, Onna-son, Kunigami-gun, Okinawa, 904-0495, Japan.
e-mail: zhangye0217@gmail.com or Ye.Zhang2@oist.jp

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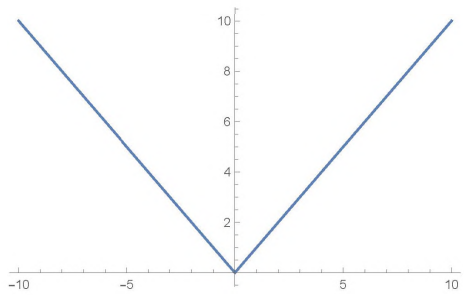


Figure 1. Distance function from a fixed point

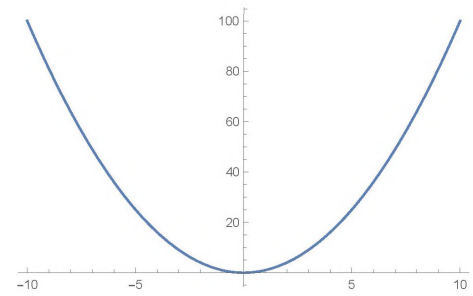


Figure 2. Squared distance function from a fixed point

In the same setting (f is a real function on an open set $\Omega \subset \mathbb{R}^n$), we can also define the notion of *local semiconvexity* (resp. *local semiconcavity*): for every $x \in \Omega$ there exists an open neighbourhood $\Omega_x \subset \Omega$ of x such that f is semiconvex (resp. semiconcave) on Ω_x.

§ 2. 3-dimensional Heisenberg group

In this note we consider a special space which is different from Euclidean spaces and Riemannian manifolds in its natural structure. Recall that the *3-dimensional Heisenberg group* (or *Heisenberg group* in short) $\mathbb{H}$ is $\mathbb{R}^3$ with the group structure given by:

$$(x, y, z) \cdot (x', y', z') = \left(x + x', y + y', z + z' + \frac{1}{2}(xy' - x'y)\right).$$

The Heisenberg group $\mathbb{H}$ is the simplest example of sub-Riemannian manifold, and it plays important roles in several branches of mathematics.

In the following we sometimes use $p = (x, y, z)$ to denote a point in $\mathbb{H}$ for the sake of simplicity. From the multiplication defined above it is easy to see that the identity element is 0 and the inverse element of p is $-p$. The *left-invariant vector fields* (which consist of a basis of the Lie algebra $\mathfrak{g}$) are given by

$$\mathrm{X} := \frac{\partial}{\partial x} - \frac{y}{2}\frac{\partial}{\partial z}, \quad \mathrm{Y} := \frac{\partial}{\partial y} + \frac{x}{2}\frac{\partial}{\partial z}, \quad \mathrm{Z} := [\mathrm{X}, \mathrm{Y}] = \frac{\partial}{\partial z}.$$

In the world of sub-Riemannian manifolds, the motions are only possible along a given set of directions, varying from point to point. To be more precise, for every point we select a subspace of the tangent space of that point, which forms a subbundle $\mathcal{H}$ of the tangent bundle, and we also put a Riemannian metric on $\mathcal{H}$. The pair $(\mathcal{H}, g)$ is called a *sub-Riemannian structure*.

In the setting of Heisenberg group $\mathbb{H}$, there is a canonical choice of the sub-Riemannian structure $(\mathcal{H}, g)$. Here $\mathcal{H} := \cup_{p\in\mathbb{H}}\mathbb{H}_p$ with the *horizontal plane through* $p = (x, y, z)$ given by

$$\mathbb{H}_p := \operatorname{span}\{\mathrm{X}(p), \mathrm{Y}(p)\} \cong \{(x', y', z');\, x'y - xy' + 2z' - 2z = 0\} = p \cdot (\mathbb{R}^2 \times \{0\}),$$

where the "$\cong$" is given by the identification of $T_p\mathbb{H}$ with $p + \mathbb{H}$:

$$a\frac{\partial}{\partial x}\bigg|_p + b\frac{\partial}{\partial y}\bigg|_p + c\frac{\partial}{\partial z}\bigg|_p \mapsto p + (a, b, c),$$

and the metric g is defined by letting $\{\mathrm{X}(p), \mathrm{Y}(p)\}$ as an orthonormal basis at p. From this choice we can see that the sub-Riemannian structure $(\mathcal{H}, g)$ is left invariant under the operator $\cdot$. Note that in the following we will not distinguish $\mathbb{H}_p \subset T_p\mathbb{H}$ from $p \cdot (\mathbb{R}^2 \times \{0\}) \subset \mathbb{H}$. In particular, $\mathbb{H}_0$ is identified with $\mathbb{R}^2 \times \{0\}$.

A *horizontal path* is an absolutely continuous map $\gamma : [0, 1] \to \mathbb{H}$ such that $\dot\gamma(t) \in \mathbb{H}_{\gamma(t)}$ for a.e. t. The *length of such horizontal path* γ can be calculated by the following formula:

$$\ell(\gamma) := \int_0^1 \sqrt{g(\dot\gamma(t), \dot\gamma(t))}dt.$$

The *sub-Riemannian distance* (or *Carnot-Carathéodory distance*) is defined by

$$d(p, q) := \inf\{\ell(\gamma);\, \gamma \text{ horizontal }, \gamma(0) = p, \gamma(1) = q\}.$$

By Chow's theorem (cf. [21, Theorem 1.17]), given two points p and q, there exists an horizontal path joining p and q, so the sub-Riemannian distance is always finite. Furthermore, by [1, Theorem 3.31], the sub-Riemannian distance d is indeed a distance function and its induced topology is the same as the manifold topology. In particular, it is continuous.

Note that since the sub-Riemannian structure $(\mathcal{H}, g)$ is left invariant, the sub-Riemannian distance d is also left invariant:

$$d(p, q) = d(p_* \cdot p, p_* \cdot q), \qquad \text{for all } p_*, p, q \in \mathbb{H}. \tag{2.1}$$

§ 3. Graph of the squared sub-Riemannian distance from a point on $\mathbb{H}$

In the following we use $d_p(q)$ to denote the function $q \mapsto d(q, p)$ on $\mathbb{H}$, namely the *sub-Riemannian distance from the point* $p \in \mathbb{H}$. From the left invariance of d (cf. (2.1)),

we can assume $p = 0$. Actually we have the explicit expression of the sub-Riemannian distance (cf. [11, Lemme 2] or [4, Theorem 1.36]):

(3.1)

$$d_0^2(x,y,z) = \begin{cases} \left(\frac{\theta}{\sin\theta}\right)^2 (x^2+y^2), & \text{if } (x,y) \neq (0,0),\ z \neq 0, \text{ and } \theta = \mu^{-1}\left(\frac{4|z|}{x^2+y^2}\right), \\ x^2+y^2, & \text{if } (x,y) \neq (0,0) \text{ and } z = 0, \\ 4\pi|z|, & \text{if } (x,y) = (0,0). \end{cases}$$

Here

$$\mu(s) := \begin{cases} \frac{2s-\sin(2s)}{2\sin^2 s}, & \text{if } s \neq 0, \\ 0, & \text{if } s = 0 \end{cases}$$

is a strictly increasing diffeomorphism between $(-\pi, \pi)$ and $\mathbb{R}$ (cf. [11, Lemme 3] or [4, Lemma 1.33]). Note that d_0^2 is rotational symmetric in coordinates x and y. So we can draw the graph of d_0^2 on the set $\{(x, 0, z);\ x, z \in \mathbb{R}\}$.

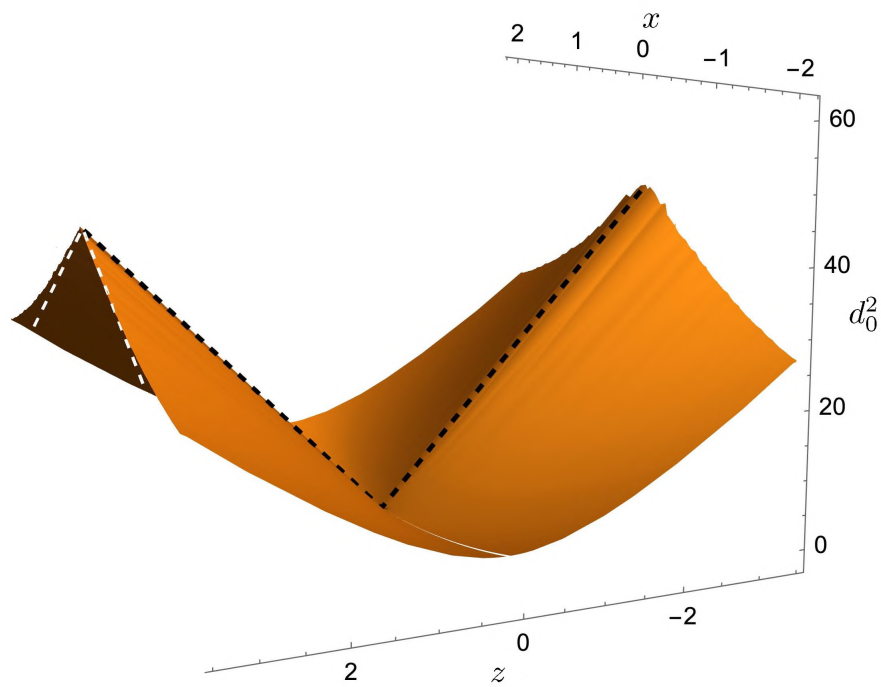


Figure 3. Graph of squared CC distance from the identity 0 on the Heisenberg group $\mathbb{H}$

Define the following sets:

$$\mathrm{SC}^-(p) := \{q;\ d_p^2(q) \text{ fails to be locally semiconcave at } q\},$$
$$\mathrm{SC}^+(p) := \{q;\ d_p^2(q) \text{ fails to be locally semiconvex at } q\}.$$

From the dashed "corner-like" singularities in Figure 3 we can guess

$$\mathrm{SC}^-(0) = \{0\}, \qquad \mathrm{SC}^+(0) = \{(0,0)\} \times \mathbb{R}.$$

Actually, these relations are true on $\mathbb{H}$ and they are closely related to the geometry of $\mathbb{H}$. We will not go into details in this note. For a more detailed explanation, we refer to [3, § 4.2.1].

§ 4. h-convexity and h-semiconvexity

On the Heisenberg group $\mathbb{H}$ (or general Carnot groups) another suitable notion of convexity (resp. concavity) is called h-convexity (resp. h-concavity). Recall that we have the identification $\mathbb{H}_p \cong p \cdot (\mathbb{R}^2 \times \{0\})$. From definition, given an open set $\Omega \subset \mathbb{H}$, a function f is *h-semiconvex* if there exists a constant $C \geq 0$ such that

$$f((1-\lambda)p + \lambda q) \leq (1-\lambda)f(p) + \lambda f(q) + \frac{C}{2}\lambda(1-\lambda)|p-q|^2,$$

holds for every $p \in \Omega$, $q \in \Omega \cap \mathbb{H}_p$ such that $[p,q] \subset \Omega$, and $\lambda \in [0,1]$. As before, the constant C is called an *h-semiconvexity constant*.

Here we add the restriction that the point q should belong to the horizontal plane of p, which is 2-dimensional. As a result, at least locally, h-(semi)convexity is weaker than Euclidean (semi)convexity. If $f \in C^2(\Omega)$, then it is equivalent to the following estimate of symmetrized horizontal Hessian matrix

$$(\mathrm{Hess}_H\, f)^* = \begin{pmatrix} \mathrm{X}^2 f & \frac{1}{2}(\mathrm{XY}f + \mathrm{YX}f) \\ \frac{1}{2}(\mathrm{XY}f + \mathrm{YX}f) & \mathrm{Y}^2 f \end{pmatrix} \geq -C\,\mathrm{Id}_2,$$

where we define the *horizontal Hessian matrix on* $\mathbb{H}$ by

$$\mathrm{Hess}_H\, f := \begin{pmatrix} \mathrm{X}^2 f & \mathrm{XY}f \\ \mathrm{YX}f & \mathrm{Y}^2 f \end{pmatrix}$$

and

$$A^* := \frac{A + A^{\mathrm{T}}}{2}$$

with $\cdot^{\mathrm{T}}$ denoting the *matrix transpose*. For more details of the (symmetrized) horizontal Hessian matrix we refer to [19] or [2]. Note that although $\mathbb{H}$ has dimension 3, this horizontal Hessian matrix is a 2×2 matrix since we only consider the horizontal direction, which has dimension 2 on $\mathbb{H}$. Also we remark that by definition $\mathrm{Hess}_H\, f$ is not necessarily symmetric, so we should consider the symmetrized horizontal Hessian matrix when talking about the lower bound. Similarly, if the constant $C = 0$, then it is exactly the

usual *h-convexity* on Heisenberg group and we call a function f *h-semiconcave* (resp. *h-concave*) if $-f$ is h-semiconvex (resp. h-convex).

From definition, it is easy to show the following proposition.

Proposition 4.1. *Let Ω be an open set of $\mathbb{H}$. Let $\{f_\alpha\}_{\alpha\in\mathcal{A}}$ be a family of h-semiconcave functions on Ω. Assume that for every $\alpha \in \mathcal{A}$, the function f_α is an h-semiconcave function in Ω with h-semiconcavity constant $C \geq 0$ independent of α. Suppose that*

$$f(p) := \inf_{\alpha\in\mathcal{A}} f_\alpha(p) > -\infty$$

for all $p \in \Omega$. Then f is also h-semiconcave in Ω with the same h-semiconcavity constant $C \geq 0$.

§ 5. Main result

Since h-semiconcavity only "sees" the horizontal directions, the black dashed corner in Figure 3, which is in the vertical direction, will not break the h-semiconcavity. Based on this observation, it is reasonable to guess that the squared sub-Riemannian distance from a fixed point is h-semiconcave, which is the main result in [9].

Theorem 5.1 ([9, Theorem 1.1])**.** *On the Heisenberg group $\mathbb{H}$, the squared sub-Riemannian distance from the identity 0, namely d_0^2, is an h-semiconcave function on $\mathbb{H}$.*

In fact, we can generalize this result to the case of step-two Carnot groups and our step-two assumption is essential in the sense that we cannot obtain the h-semiconcavity of the squared sub-Riemannian distance from the identity if we remove this assumption. The counterexample (Engel group), is given in [20, Theorem 1.2].

However, different from the Euclidean case, the h-semiconvexity of the squared sub-Riemannian distance from the identity is not true:

Proposition 5.2 ([9, Proposition 3.4])**.** *On Heisenberg group $\mathbb{H}$, for every $p = (0, 0, \tau) \in \mathbb{H}$ with $\tau \neq 0$, the following holds:*

$$\lim_{h\in\mathbb{H}_0\setminus\{0\},h\to 0} \frac{d_0^2(p\cdot h) + d_0^2(p\cdot h^{-1}) - 2d_0^2(p)}{|h|} = -4d_0(p).$$

In particular,

$$\lim_{h\in\mathbb{H}_0\setminus\{0\},h\to 0} \frac{d_0^2(p\cdot h) + d_0^2(p\cdot h^{-1}) - 2d_0^2(p)}{|h|^2} = -\infty,$$

which implies d_0^2 is not an h-semiconvex function on $\mathbb{H}$.

Here recall that we identified $\mathbb{H}_0$ with $\mathbb{R}^2 \times \{0\} \subset \mathbb{H}$ so we can do the group multiplication and take the inverse of the element $h \in \mathbb{H}_0$. The norm $|\cdot|$ here is the usual Euclidean norm. The proof of Proposition 5.2 is based on the explicit formula of d_0^2. See (3.1).

§ 6. Idea of the proof

The proof of Theorem 5.1 is based on the viscosity theory technique, the local (Euclidean) semiconcavity outside the origin, and the dilation structure of $\mathbb{H}$. Note that similar result holds for general step-two Carnot group case, where another notion of horizontal semiconcavity is used. We refer to the interested readers to [9, Section 4] for more details.

§ 6.1. Viscosity theory technique

We have the following viscosity characterization of h-semiconcavity:

Theorem 6.1 ([2, Proposition 5.1]). *Given an open set $\Omega \subset \mathbb{H}$ and $u \in LSC(\Omega)$ (i.e. the function u is lower semicontinuous in Ω), the following statements are equivalent:*

1. u is h-semiconcave in Ω with h-semiconcavity constant $C \geq 0$.

2. We have

$$-(\mathrm{Hess}_H u)^* \geq -C\,\mathrm{Id}_2 \quad \text{in } \Omega \text{ in the viscosity sense,} \tag{6.1}$$

which means that $-(\mathrm{Hess}_H \varphi(p))^* \geq -C\,\mathrm{Id}_2$ whenever there exist $\varphi \in C^2(\Omega)$ and $p \in \Omega$ such that $u - \varphi$ has a local minimum at p.

By this theorem, it suffices to prove the following **claim:** For every $p \in \mathbb{H}$, there exist $C > 0$ (independent of p) and $c(p) > 0$ such that

$$d_0^2(p \cdot h) + d_0^2(p \cdot h^{-1}) - 2d_0^2(p) \leq C|h|^2, \qquad \text{for all } h \in \mathbb{H}_0, |h| \leq c(p). \tag{6.2}$$

To be more precise, whenever there are $\varphi \in C^2(\mathbb{H})$ and $p \in \mathbb{H}$ such that $d_0^2 - \varphi$ has a local minimum at p, we denote the local minimum by m and in a neighbourhood of p we have

$$d_0^2 \geq \varphi + m$$

and at the point p we have $d_0^2(p) = \varphi(p) + m$. Inserting these into (6.2), for $h \in \mathbb{H}_0$ with $|h|$ small enough, we get

$$\varphi(p \cdot h) + \varphi(p \cdot h^{-1}) - 2\varphi(p) \leq d_0^2(p \cdot h) + d_0^2(p \cdot h^{-1}) - 2d_0^2(p) \leq C|h|^2. \tag{6.3}$$

Now we define the *horizontal gradient* of a function $f \in C^1(\Omega)$ as

$$\nabla_H f := (\mathrm{X}f, \mathrm{Y}f). \tag{6.4}$$

By applying the Taylor expansion in the group (see e.g. [5, § 20]), we can write

$$\begin{aligned} \varphi(p \cdot h) &= \varphi(p) + \langle \nabla_H \varphi(p), h\rangle + \frac{1}{2}\langle (\mathrm{Hess}_H \varphi(p))^* h, h\rangle + o(|h|^2), \\ \varphi(p \cdot h^{-1}) &= \varphi(p) - \langle \nabla_H \varphi(p), h\rangle + \frac{1}{2}\langle (\mathrm{Hess}_H \varphi(p))^* h, h\rangle + o(|h|^2), \end{aligned} \tag{6.5}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product on $\mathbb{R}^2 \cong \mathbb{R}^2 \times \{0\}$. Combining (6.3) and (6.5), we obtain

$$\langle (\mathrm{Hess}_H \varphi(p))^* h, h\rangle + o(|h|^2) \leq C\,|h|^2,$$

for all $h \in \mathbb{H}_0$ with $|h|$ small enough. Dividing this inequality by $|h|^2$ and letting $|h| \to 0^+$, we deduce that $(\mathrm{Hess}_H \varphi(p))^* \leq C\,\mathrm{Id}_m$ and thus we proved Theorem 5.1 by applying Theorem 6.1.

Now we are in a position to prove the claim.

§ 6.2. Local (Euclidean) semiconcavity outside the origin

Recall that $\mathrm{SC}^-(0) = \{0\}$, which means outside the origin, d_0^2 is locally (Euclidean) semiconcave. This actually implies h-semiconcavity on compact set of $\mathbb{H} \setminus \{0\}$. More precisely, we can see this fact from the following computation:

$$\begin{aligned} p \cdot h &= \left(x + h_x, y + h_y, z + \frac{1}{2}(xh_y - yh_x)\right), \\ p \cdot h^{-1} &= \left(x - h_x, y - h_y, z - \frac{1}{2}(xh_y - yh_x)\right) \end{aligned}$$

for $p = (x, y, z) \in \mathbb{H}$ and $h = (h_x, h_y, 0) \in \mathbb{R}^2 \times \{0\} \cong \mathbb{H}_0$. As a result, the claim is true for any compact set $K \subset \mathbb{H} \setminus \{0\}$ and in particular, we choose K to be the unit sphere of the sub-Riemannian distance centred at 0, that is, $K = S := \{p;\, d(p, 0) = d_0(p) = 1\}$. More precisely, there exists $C_* \geq 0$ and $c_* > 0$ such that

$$d_0^2(p \cdot h) + d_0^2(p \cdot h^{-1}) - 2d_0^2(p) \leq C_* |h|^2, \quad \text{for all } p \in S, h \in \mathbb{H}_0, |h| \leq c_*. \tag{6.6}$$

§ 6.3. Dilation structure

Recall that for each $r > 0$ we can define the *dilation* on $\mathbb{H}$ by

$$\delta_r(x, y, z) := (rx, ry, r^2 z), \qquad \text{for all } (x, y, z) \in \mathbb{H},$$

which is a group homomorphism. From the expression of d_0 above, i.e. (3.1), we have

$$d_0(\delta_r(p)) = r\, d_0(p), \qquad \text{for all } r > 0, p \in \mathbb{H}.$$

There are two consequences:

1. The claim is true for all $p \in \mathbb{H} \setminus \{0\}$.

This is because for every $p \neq 0$ we can define $\widetilde{p} = \delta_{1/r}(p)$ with $r = d_0(p)$ (i.e. $p = \delta_r(\widetilde{p})$) so that $\widetilde{p} \in S$. Indeed, we have

$$d_0(\widetilde{p}) = d_0\left(\delta_{1/r}(p)\right) = \frac{1}{r}\, d_0(p) = \frac{1}{r}\, r = 1.$$

Then we can apply (6.6) to $\widetilde{p}$ with $\widetilde{h} = \delta_{1/r}(h)$ for all $h \in \mathbb{H}_0$ such that $|h| \leq c_* r$. It is worth pointing out that, due to the condition $h \in \mathbb{H}_0$, we have

$$\delta_{1/r}(h) = \frac{h}{r}, \quad |\widetilde{h}| = \frac{|h|}{r} \leq \frac{c_*\, r}{r} = c_*.$$

Hence (6.6) gives

$$\begin{aligned} &d_0^2(p \cdot h) + d_0^2(p \cdot h^{-1}) - 2d_0^2(p) = d_0^2(\delta_r(\widetilde{p}) \cdot h) + d_0^2(\delta_r(\widetilde{p}) \cdot h^{-1}) - 2d_0^2(\delta_r(\widetilde{p})) \\ &= r^2[d_0^2(\widetilde{p} \cdot \delta_{1/r}(h)) + d_0^2(\widetilde{p} \cdot \delta_{1/r}(h)^{-1}) - 2d_0^2(\widetilde{p})] = r^2\left[d_0^2(\widetilde{p} \cdot \widetilde{h}) + d_0^2(\widetilde{p} \cdot \widetilde{h}^{-1}) - 2d_0^2(\widetilde{p})\right] \\ &\leq r^2 C_* |\,\widetilde{h}\,|^2 \leq\ r^2 C_* \frac{|h|^2}{r^2} = C_* |h|^2. \end{aligned}$$

2. By the equivalence of the homogeneous norms (cf. [5, Proposition 5.1.4 with Example 5.1.2]) there exists a constant $C_{**} \geq 1$ such that

$$C_{**}^{-1}(x^2 + y^2 + |z|) \leq d_0^2(x, y, z) \leq C_{**}(x^2 + y^2 + |z|), \tag{6.7}$$

which proves the claim for $p = 0$.

We remark that from the estimate (6.7) the constant C in (6.2) cannot be chosen to be 0.

§ 7. Application I: Hamilton-Jacobi equations

We study the regularity problem of the solution to the following *time-dependent Hamilton-Jacobi equations*:

$$\begin{cases} u_t + \Phi\big(|\nabla_H u|\big) = 0, & \text{in } (0, +\infty) \times \mathbb{H}, \\ u(0, \cdot) = \phi, & \text{on } \{0\} \times \mathbb{H}, \end{cases} \tag{7.1}$$

where u_t denotes the time derivative of u, ϕ is the initial datum, ∇_H is the horizontal gradient defined in (6.4), and $\Phi : [0, +\infty) \to [0, +\infty)$ is a continuous, convex, non-decreasing function such that $\Phi(0) = 0$.

By definition, a function $u \in LSC([0, +\infty) \times \mathbb{H})$ is a *lower semicontinuous viscosity solution of the Hamilton-Jacobi equation* (7.1) if, for any test function $\varphi \in C^1((0, +\infty) \times \mathbb{H})$ such that $u - \varphi$ admits a local minimum in (t_0, p_0), it holds

$$\varphi_t(t_0, p_0) + \Phi\big(|\nabla_H \varphi(t_0, p_0)|\big) = 0.$$

The connection of the Hamilton-Jacobi equations and the sub-Riemannian distance is shown by the following theorem.

Theorem 7.1 ([8, Theorem 4])**.** *If $\phi \in LSC(\mathbb{H})$ and there exists a constant $C > 0$ such that $\phi \geq -C(1 + d_0)$, then the lower semicontinuous viscosity solution $u \in LSC([0, +\infty) \times \mathbb{H})$ of the Hamilton-Jacobi equation* (7.1) *can be obtained by the (metric) Hopf-Lax formula*

$$u(t, p) = \inf_{q \in \mathbb{H}} \left[\phi(q) + t\Phi^* \left(\frac{d(p, q)}{t} \right) \right], \quad (t, p) \in [0, +\infty) \times \mathbb{H}, \tag{7.2}$$

where Φ^* is the Legendre-Fenchel function associated to Φ, that is defined by

$$\Phi^*(s) := \sup_{\tau \geq 0} \big\{ s\tau - \Phi(\tau) \}, \quad s \geq 0.$$

For example, if we consider the case $\Phi(\tau) = \frac{\tau^2}{2}$, then $\Phi^*(s) = \frac{s^2}{2}$. Since by the left invariance (2.1) we have $d(p, q) = d(q^{-1} \cdot p, 0) = d_0(q^{-1} \cdot p)$, from the Hopf-Lax formula (7.2)

$$u(t, p) = \inf_{q \in \mathbb{H}} \left\{ \phi(q) + \frac{d_0^2(q^{-1} \cdot p)}{2t} \right\}, \quad (t, p) \in [0, +\infty) \times \mathbb{H}.$$

For fixed $q \in \mathbb{H}$, the function

$$p \mapsto \phi(q) + \frac{d_0^2(q^{-1} \cdot p)}{2t}$$

is h-semiconcave (with h-semiconvexity constant independent of q). As a result, Proposition 4.1 yields the following theorem.

Theorem 7.2 ([9, Theorem 6.2])**.** *Assume $\phi \in LSC(\mathbb{H})$ and there exists a constant $C > 0$ such that $\phi \geq -C(1 + d_0)$. Let $\Phi(\tau) = \tau^2/2$ for $\tau \geq 0$ and u be defined by the Hopf-Lax formula* (7.2)*. Then $u(t, \cdot)$ is h-semiconcave in $\mathbb{H}$, for every $t > 0$.*

In the general case, with similar ideas, similar results hold as well under some assumption on the function Φ.

Theorem 7.3 ([9, Theorem 6.4]). *Assume that $\phi \in LSC(\mathbb{H})$ is bounded. Let $\Phi : [0, +\infty) \to [0, +\infty)$ be a continuous, convex, non-decreasing function with $\Phi(0) = 0$. Assume in addition that Φ is coercive in the sense that*

$$\frac{\Phi(\tau)}{\tau} \to +\infty \quad as\ \tau \to +\infty,$$

and the even extension $\widetilde{\Phi} : \mathbb{R} \to [0, +\infty)$ of Φ is locally strongly convex in $\mathbb{R}$. Let u be defined by the Hopf-Lax formula (7.2). Then $u(t, \cdot)$ is h-semiconcave in $\mathbb{H}$ for every $t > 0$.

Here, similar to the semiconvexity, for a real function f on an open set $\Omega \subset \mathbb{R}^n$, we say f is *strongly convex* if there exists a constant $C > 0$ such that the following inequality holds:

$$f((1-\lambda)x + \lambda y) \le (1-\lambda)f(x) + \lambda f(y) - \frac{C}{2}\lambda(1-\lambda)|x-y|^2,$$

for every $x, y \in \Omega$ such that $[x, y] \subset \Omega$ and $\lambda \in [0, 1]$. It is called *locally strongly semiconvexity* if for every $x \in \Omega$ there exists an open neighbourhood $\Omega_x \subset \Omega$ of x such that f is strongly convex on Ω_x.

On the other hand, for h-semiconvexity, we have the breaking result.

Theorem 7.4 ([18]). *Let $\Phi : [0, +\infty) \to [0, +\infty)$ be a continuous, convex, non-decreasing function with $\Phi(0) = 0$. Assume in addition that*

$$\frac{\Phi(\tau)}{\tau} \to 0^+ \quad as\ \tau \to 0^+, \quad \frac{\Phi(\tau)}{\tau} \to +\infty \quad as\ \tau \to +\infty.$$

Then for every $t > 0$ and $C > 0$, we can find a lower semicontinuous h-convex function ϕ (depending on t and C) such that the function $u(t, \cdot)$ given by the Hopf-Lax formula (7.2) is not h-semiconvex with h-semiconvexity constant C.

In fact, from the assumptions on Φ, Φ^* is continuous and strictly increasing. If we choose $\phi(q) = +\infty$ if $q \neq 0$ and $\phi(q) = 0$ if $q = 0$, then from the Hopf-Lax formula (7.2) we obtain $t\Phi^*(d_0/t)$, which is not h-semiconvex by our assumption and Proposition 5.2. The idea of the proof is to approximate the limit function ϕ by lower semicontinuous h-convex functions.

§ 8. Power concavity

Before the second application, we recall the notion of *power concavity* for nonnegative functions. Let Ω be an open set in $\mathbb{R}^n$ and $\alpha \in \overline{\mathbb{R}} := [-\infty, +\infty] = \mathbb{R} \cup \{\pm\infty\}$. A nonnegative function f on Ω is called *α-concave* if

$$f((1-\lambda)x + \lambda y) \ge \mathcal{M}_\alpha(f(x), f(y); \lambda), \quad \text{for all}\, x, y \in \Omega, [x, y] \subset \Omega, \lambda \in [0, 1],$$

where for $a, b \geq 0$ and $\lambda \in [0,1]$, the function

$$\mathcal{M}_\alpha(a,b;\lambda) := \begin{cases} \max(a,b), & \text{if } \alpha = +\infty, \\ ((1-\lambda)a^\alpha + \lambda b^\alpha)^{\frac{1}{\alpha}}, & \text{if } \alpha \in \mathbb{R} \setminus \{0\}, \\ a^{1-\lambda} b^\lambda, & \text{if } \alpha = 0, \\ \min(a,b), & \text{if } \alpha = -\infty, \end{cases}$$

if $ab > 0$ and $\mathcal{M}_\alpha(a,b;\lambda) := 0$ if $ab = 0$. If $\alpha = 1$, it is the usual notion of *concavity* on the set $\{x \in \Omega;\ f(x) > 0\}$. If $\alpha = 0$ (resp. $\alpha = -\infty$), it is also called *logconcave* (resp. *quasiconcave*). By Jensen's inequality, α-concavity implies β-concavity for $\beta \leq \alpha$. Thus quasiconcavity is the weakest notion of (power) concavity.

Similarly, we have the corresponding horizontal version of power concavity on $\mathbb{H}$. On an open set $\Omega \subset \mathbb{H}$, given $\alpha \in [-\infty, +\infty]$, we call a nonnegative function f *h-α-concave* if

$$f((1-\lambda)p + \lambda q) \geq \mathcal{M}_\alpha(f(p), f(q); \lambda),$$

for all $p \in \Omega, q \in \Omega \cap \mathbb{H}_p, [p,q] \subset \Omega, \lambda \in [0,1]$. As before, for $\alpha = 1, 0, -\infty$, we also call the function f *h-concave*, *h-logconcave*, and *h-quasiconcave* respectively.

§ 9. Application II: heat equation

Now we consider the following *heat equation* on the Euclidean space $\mathbb{R}^n$:

$$\begin{cases} u_t - \Delta u = 0, & \text{in } (0,+\infty) \times \mathbb{R}^n, \\ u(0,\cdot) = \phi, & \text{on } \{0\} \times \mathbb{R}^n. \end{cases} \tag{9.1}$$

In this paper we only consider the initial datum $\phi \in L^\infty(\mathbb{R}^n)$ such that $\phi \geq 0$, that is, nonnegative bounded initial datum. Under these assumptions, the solution can be represented by the following formula

$$u(t,x) = \int_\Omega \phi(x-y) h_t^{\mathbb{R}^n}(y) dy, \qquad (t,x) \in (0,+\infty) \times \mathbb{R}^n$$

with the *heat kernel* given by

$$h_t^{\mathbb{R}^n}(y) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{|y|^2}{4t}}.$$

In this note we focus on the *concavity preserving/breaking problem*, i.e., whether the α-concavity of the initial datum ϕ will be preserved/broken by the heat flow. In [6, Corollary 3.5], it is shown that logconcavity is preserved by the heat flow on Euclidean

spaces. This result comes from an important property of the heat kernel on Euclidean space $\mathbb{R}^n$: logconcavity, that is,

$$\log h_t^{\mathbb{R}^n}(\cdot) = -\frac{|\cdot|^2}{4t} - \frac{n}{2}\log(4\pi t)$$

is a concave function. Note that this fact comes from that the squared Euclidean distance is a convex function.

However, if $n \geq 2$, quasiconcavity and α-concavity (for some $\alpha \in (-\infty, 0)$) is not preserved by the heat flow (see [12, Theorem 1.1] and [13, Theorem 4.1] respectively). In the recent paper [14], the authors showed that the only power concavity preserved by the heat flow is the logconcavity (cf. [14, Corollary 1.7]).

In general we can furthermore consider the heat equation on an open set of $\mathbb{R}^n$ with Dirichlet boundary condition. For the sake of simplicity, we only consider the full space case in this note and refer to [14, Section 3] for more details about the general case.

Now we consider a counterpart of (9.1) on Heisenberg group $\mathbb{H}$, namely

$$\begin{cases} u_t - \Delta_{\text{sub}} u = 0, & \text{in } (0, +\infty) \times \mathbb{H}, \\ u(0, \cdot) = \phi, & \text{on } \{0\} \times \mathbb{H}, \end{cases} \tag{9.2}$$

where the *sub-Laplacian* is given by $\Delta_{\text{sub}} := \mathrm{X}^2 + \mathrm{Y}^2$. In this case the solution can be represented by the following formula:

$$u(t, p) = \int_{\mathbb{H}} \phi(p \cdot q^{-1}) h_t^{\mathbb{H}}(q) dq, \qquad (t, p) \in (0, +\infty) \times \mathbb{H}, \tag{9.3}$$

where the function $h_t^{\mathbb{H}}$ is called the *heat kernel on* $\mathbb{H}$. For more properties of such heat kernel on Heisenberg, we refer to [10, § G of Chapter 1].

On the Heisenberg group $\mathbb{H}$, the following Varadhan's formula holds (cf. [15, Théorème 1] and [16, Théorème II.3]):

$$\lim_{t \to 0^+} 4t \log h_t^{\mathbb{H}}(x, y, z) = -d_0^2(x, y, z).$$

We know that the squared distance function d_0^2 is not h-semiconvex, so the heat kernel on the Heisenberg group $\mathbb{H}$ is not h-logconcave.

Furthermore, combining Proposition 5.2 with the following precise upper and lower bound of the heat kernel established in [17, Theorem 1] (here $f \sim g$ means there exists a universal constant $C \geq 1$ such that $C^{-1} g \leq f \leq C g$):

$$h_t^{\mathbb{H}}(x, y, z) \sim t^{-\frac{3}{2}} \frac{1}{\sqrt{t + \sqrt{x^2 + y^2}\, d_0(x, y, z)}} e^{-\frac{d_0^2(x, y, z)}{4t}},$$

we can further obtain the theorem:

Theorem 9.1 ([18]). *On the Heisenberg group $\mathbb{H}$, the heat kernel $h_t^{\mathbb{H}}(x,y,z)$ is not h-quasiconcave w.r.t. $(x,y,z) \in \mathbb{H}$ for every $t > 0$.*

As a result, we can further obtain the following breaking property of the heat flow on Heisenberg group $\mathbb{H}$:

Theorem 9.2 ([18]). *On the Heisenberg group $\mathbb{H}$, for every $\alpha \in \mathbb{R}$, $t > 0$, there exists an h-α-concave function ϕ (depending on α, t) such that $u(t,\cdot)$ given by (9.3) is not h-quasiconcave.*

The idea of the proof is similar to the one of Theorem 7.4. For initial datum δ_0, we return to the heat kernel case, which is not h-quasiconcave. Then we can prove this theorem by an approximation process.

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