

A Generalization of a Prime Number Theorem of Rodosskii-

Tatsuzawa for an algebraic Field

Teluhiko Hilano (University of Tokyo)

1. Introduction

The Prime Number Theorem in an Arithmetic Progressions is stated as follows ;

Theorem A . If $(q, l) = 1$ and $q \leq \exp(c \sqrt{\log x})$ then

$$\pi(x, q, l) = \frac{1}{\varphi(q)} \int_2^x \frac{dt}{\log t} + O\left(\frac{x^{\beta_1}}{\varphi(q) \log x}\right) + O(xe^{-c\sqrt{\log x}})$$

where β_1 is a so-called Siegel zero.

For q , which does not satisfy the above restriction, we have

the following theorem ;

Theorem B (Rodosskii - Tatsuzawa) (See , Prachar (8) Chap . 9 Satz 2 . 2)

There exist constant c_1 and c_2 , such that, if

$$c_1 \log q \log \log q \leq \log x \leq c_2 (\log q)^2$$

$$\pi(x, q, l) = \frac{1}{\varphi(q)} \int_2^x \frac{dt}{\log t} + O\left(\frac{x^{\beta_1}}{\varphi(q) \log x}\right) + O\left(\frac{x}{\varphi(q) \log x} e^{-c \frac{\log x}{\log q}}\right)$$

The prime ideal theorem in an ideal classes of an algebraic field,

which is a generalization of theorem A, has been already obtained under

the similar restriction. (see, Mitsui (5))

The purpose of this paper is to offer a generalization of theorem B for an algebraic field under the similar condition. We now write it down ;

Theorem K is an algebraic field of degree n and $\mathfrak{f}$ is an integral ideal in K . We denote a product of $\mathfrak{f}$ and some infinite prime ideals by $\tilde{\mathfrak{f}}$ $h(\tilde{\mathfrak{f}})$ is the number of ideal classes mod $\tilde{\mathfrak{f}}$ and $\mathcal{L}$ is one of the ideal classes . Then we have

for all $c > 0$, there exist c_1 and c_2 such that, under the condition

$$c_1 \log N(\mathfrak{f}) \log \log N(\mathfrak{f}) \leq \log x \leq c (\log N(\mathfrak{f}))^2$$

$$\pi(x, \mathcal{L}) = \sum_{\substack{N(\mathfrak{f}) < x \\ \mathfrak{f} \in \mathcal{L}}} 1 = \frac{1}{h(\tilde{\mathfrak{f}})} \int_2^x \frac{dt}{\log t} + O\left(\frac{x^{c_1}}{h(\tilde{\mathfrak{f}}) \log x}\right) + O\left(\frac{x}{h(\tilde{\mathfrak{f}}) \log x} e^{-c_2 \frac{\log x}{\log N(\mathfrak{f})}}\right)$$

(The constant in O depend only on K and c .)

The author has already given another proof of theorem B in (2) , using Gallagher's mean value theorem (see, Gallagher (1) Theorem 2) In this paper, we use its generalization for an algebraic field , again .

The functions $\mu(\alpha)$, $d(\alpha)$, $\lambda(\alpha)$ and $\varphi(\alpha)$ are that of generalization for an algebraic field of Möbius , divisor , von Mangoldt and Euler function , respectively . All constants of all estimations depend only on K and c .

The author wishes to express his thanks to Prof. T . Tatszawa , who

taught him Theorem 2 and 3 in this paper, and encouraged him during preparing this paper.

2. The number of ideals of an ideal class mod $\tilde{f}$ whose norm lie in a given interval

Let K be an algebraic field of degree n with r_1 real conjugates and $2r_2$ complex conjugates and $\mathfrak{f}$ an integral ideal of K . And we put $\tilde{f} = \mathfrak{f} \mathfrak{f}_1^{(1)} \cdots \mathfrak{f}_r^{(r)}$ and $h(\tilde{f})$ is the number of ideal classes mod $\tilde{f}$. Then,

Theorem 1

$$R(\tilde{f}) = \frac{h(\tilde{f})}{e(\tilde{f})} \quad (1)$$

where h is the ideal class number of K , $e(\tilde{f})$ the number of residue classes mod $\tilde{f}$ of units of K , and $\varphi(\tilde{f}) = 2^{\delta} \varphi(\mathfrak{f})$

The proof of this theorem, for example, appears in Suetsuna (9) Chap.

2 Th. 4 (p. 55)

Cor.

$$R(\tilde{f}) \ll N(\mathfrak{f}) \quad (2)$$

Theorem 2

$$\sum_{\substack{N(\alpha) < x \\ \alpha \in \mathcal{L}}} 1 = \frac{1}{h(\tilde{f})} \frac{\varphi(\mathfrak{f})}{N(\mathfrak{f})} c_K x + O\left(\frac{1}{h(\tilde{f})} \frac{\varphi(\mathfrak{f})}{N(\mathfrak{f})} N(\mathfrak{f})^{\frac{1}{n}} x^{1-\frac{1}{n}}\right) \quad (3)$$

where c_K is the constant only determined by K .

proof) see, Tatsuzawa (10)

Theorem 3 If $y \ll x$, then

$$\sum_{\substack{\chi \in N(\mathfrak{f}) \\ \sigma \in \mathcal{L}}} 1 \ll \frac{y}{h(\tilde{\mathfrak{f}})} + \frac{N(\mathfrak{f})^{\frac{1}{n}}}{h(\tilde{\mathfrak{f}})} x^{1-\frac{1}{n}} + 1 \quad (4)$$

We now choose a ideal class $\mathcal{L} \pmod{\mathfrak{f}}$, and fix it.

3. A Generalization of Gallagher's Mean Value Theorem

Lemma If $z \ll y$, and $a(\sigma)$ are complex numbers, then

$$\sum_{\chi \pmod{\tilde{\mathfrak{f}}}} \left| \sum_{\substack{y \leq N(\sigma) \leq y+z}} a(\sigma) \chi(\sigma) \right|^2 \ll (z + N(\mathfrak{f})^{\frac{1}{n}} y^{1-\frac{1}{n}} + h(\tilde{\mathfrak{f}})) \sum_{\substack{y \leq N(\sigma) \leq y+z}} |a(\sigma)|^2 \quad (5)$$

proof) By the orthogonal relation, Schwarz inequality and theorem 3,

we have

$$\begin{aligned} \sum_{\chi \pmod{\tilde{\mathfrak{f}}}} \left| \sum_{\substack{y \leq N(\sigma) \leq y+z}} a(\sigma) \chi(\sigma) \right|^2 &\ll \sum_{\mathcal{L} \pmod{\tilde{\mathfrak{f}}}} h(\tilde{\mathfrak{f}}) \left| \sum_{\substack{y \leq N(\sigma) \leq y+z \\ \sigma \in \mathcal{L}}} a(\sigma) \right|^2 \\ &\ll \sum_{\mathcal{L} \pmod{\tilde{\mathfrak{f}}}} h(\tilde{\mathfrak{f}}) \left(\sum_{\substack{y \leq N(\sigma) \leq y+z \\ \sigma \in \mathcal{L}}} 1 \right) \sum_{\substack{y \leq N(\sigma) \leq y+z \\ \sigma \in \mathcal{L}}} |a(\sigma)|^2 \\ &\ll (z + N(\mathfrak{f})^{\frac{1}{n}} x^{1-\frac{1}{n}} + h(\tilde{\mathfrak{f}})) \sum_{y \leq N(\sigma) \leq y+z} |a(\sigma)|^2 \end{aligned}$$

Theorem 4 . χ is a character $\pmod{\tilde{\mathfrak{f}}}$, and

$$S(\chi, t) = \sum a(\sigma) \chi(\sigma) N(\sigma)^{-it}$$

is an absolute convergent Dirichlet series. Then, if $T \geq 1$, we have

$$\sum_{\chi \pmod{\tilde{\mathfrak{f}}}} \int_{-T}^T |S(\chi, t)|^2 dt \ll \sum (h(\tilde{\mathfrak{f}})T + N(\mathfrak{f})^{\frac{1}{n}} T N(\sigma)^{1-\frac{1}{n}} + N(\sigma)) |a(\sigma)|^2 \quad (6)$$

proof) We first refer the following inequality.

$S(t) = \sum a_n n^{it}$ is an absolute convergent Dirichlet series.

and $\tau = e^{\frac{1}{y}}$. Then

$$\int_{-T}^T |S(t)|^2 dt \ll T^2 \int_0^{\frac{yT}{y}} \left| \sum_{\substack{y \leq n \leq y\tau}} a_n \right|^2 \frac{dy}{y}$$

(See Gallagher (1) Theorem 1) Then we have

$$\sum_{\chi \bmod \tilde{f}} \int_{-T}^T |S(\chi, t)|^2 dt \ll T^2 \int_0^k \sum_{\chi \bmod \tilde{f}} \left| \sum_{y \leq N(\chi) \leq y\tau} a_\chi \chi(n) \right|^2 \frac{dy}{y}$$

By the assumption $T \geq 1$, we can apply the above lemma to the inner sum.

$$\ll T^2 \int_0^k (y(\tau-1) + N(\tilde{f})^{\frac{1}{n}} y^{1-\frac{1}{n}} + R(\tilde{f})) \sum_{y \leq N(\chi) \leq y\tau} |a_\chi|^2 \frac{dy}{y}$$

The coefficient of $|a_n|^2$ is ($n \geq 1$)

$$\begin{aligned} & T^2 \int_{N(\chi)/\tau}^{N(\chi)} (y(\tau-1) + N(\tilde{f})^{\frac{1}{n}} y^{1-\frac{1}{n}} + R(\tilde{f})) \frac{dy}{y} \\ &= R(\tilde{f})T + T^2 N(\chi) (\tau-1) (1-\tau^{-1}) \\ & \quad + \frac{1}{1-\frac{1}{n}} N(\tilde{f})^{\frac{1}{n}} N(\chi)^{1-\frac{1}{n}} (1-\tau^{-1+\frac{1}{n}}) T^2 \\ &\ll R(\tilde{f})T + N(\chi) + N(\tilde{f})^{\frac{1}{n}} T N(\chi)^{1-\frac{1}{n}} \end{aligned}$$

In the case $n = 1$, the proof is similarly done. Then we get the theorem.

4. Some properties on $L(s, \chi)$

In this paragraph we pick up some properties of $L(s, \chi)$. These

proofs can be found in Mitsui (5) or be similarly done as the case

of Dirichlet L -functions, using theorem 2 and 3.

Theorem 5. The following estimations are true uniformly on $\sigma \geq 1 - \frac{1}{2n}$

$$L(s, \chi) = \sum_{N(f) < M} \frac{\chi(\alpha)}{N(\alpha)^s} + E(\chi) \frac{N(f)}{h(\tilde{f}) \varphi(f)} C_K \frac{M^{1-s}}{s-1} + O\left(|s| \frac{N(f)^{\frac{1}{n}}}{M^{\sigma + \frac{1}{n} - 1}}\right) \quad (7)$$

where $E(\chi)$ is 1 if χ is principal, 0 if not.

Theorem 6

$$L(s, \chi) (s-1)^{E(\chi)} \ll N(f)^2 (1+t+2)^{\frac{2}{3}n(1-\sigma)+E(\chi)} \log((1+t+2)N(f)) \quad (8)$$

(uniformly $-\frac{1}{2} \leq \sigma \leq 4$)

Theorem 7

$$\frac{L'}{L}(s, \chi) = \sum_{|\gamma_0 - t| \leq 1} \frac{1}{s - \gamma_0} - \frac{E(\chi)}{s-1} + O(\log(N(f)(1+t+2))) \quad (9)$$

where $\gamma_0 = \beta_0 + i\gamma_0$ is a zero of $L(s, \chi)$

Theorem 8 There exists a constant $A > 0$, such that, in the region

$$\sigma \geq 1 - \frac{A}{\log(N(f)(1+t+2))}, \quad |t| > 0 \quad (10)$$

all $L(s, \chi)$ whose character are defined mod $\tilde{f}$, are zero-free.

For $t = 0$, all $L(s, \chi)$ except one $L(s, \chi_1)$ where χ_1 is a real

character mod $\tilde{f}$, are zero-free in the interval $1 \geq \sigma \geq 1 - \frac{A}{\log(N(f))}$

The number of the exceptional zeros is at most one with multiplicity.

5. Notations and Remarks

Now we put the restriction

$$C_1 \log N(f) \log \log N(f) \leq \log x \leq C_2 (\log N(f))^2 \quad (11)$$

and $N(f)$ is sufficiently large. We define the following functions.

$$\psi(x, x) = \sum_{N(n) < x} \chi(n) \Lambda(n) \quad (12)$$

$$\psi_1(x, x) = \sum_{N(n) < x} \chi(n) \Lambda(n) \log \frac{x}{N(n)} \quad (13)$$

$$Q(s, x) = \sum_{N(n) < N(f) T^n} \frac{\mu(n) \chi(n)}{N(n)^s} \quad (14)$$

Furthermore, we assume that

$$\log T \leq \log N(f) \quad (15)$$

$$\alpha = 1 + \frac{1}{\log x}, \quad \beta = 1 - \frac{1}{2n} + \frac{1}{\log x} \quad (16)$$

$$\psi(x, z) = \sum_{\substack{N(n) < x \\ n \in z}} \Lambda(n) \quad (17)$$

and

$$\psi_1(x, z) = \sum_{\substack{N(n) < x \\ n \in z}} \Lambda(n) \log \frac{x}{N(n)} \quad (18)$$

Then,

$$\psi_1(x, z) = \frac{1}{h(f)} \sum_{x \bmod f} \overline{\chi(z)} \psi_1(x, x) \quad (19)$$

We begin with the estimation of $\psi_1(x, z)$

6. The estimation of $\psi_1(x, z)$

6.1 Transformation of $\psi_1(x, z)$

Lemma $a > 0$, then

$$\frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \frac{x^s}{s^2} ds = \begin{cases} 0 & 0 < x \leq 1 \\ \log x & x \geq 1 \end{cases} \quad (20)$$

The proof of this lemma is immediately done .

By the above lemma and the definition of $L(s, \chi)$, we have

$$\begin{aligned} \psi_1(\chi, \chi) &= -\frac{1}{2\pi i} \int_{\alpha-i\infty}^{\alpha+i\infty} \frac{L'(s, \chi)}{L(s, \chi)} \frac{x^s}{s^2} ds \\ &= -\frac{1}{2\pi i} \left(\int_{\alpha-iT}^{\alpha+iT} + \int_{\alpha-i\infty}^{\alpha-iT} + \int_{\alpha+iT}^{\alpha+i\infty} \right) \frac{L'(s, \chi)}{L(s, \chi)} \frac{x^s}{s^2} ds \\ &= -\frac{1}{2\pi i} \{ I_1(\chi) + I_2(\chi) + I_3(\chi) \} \quad (21) \end{aligned}$$

(Say !)

By theorem 8 and the condition (15), we may assume the following conditions .

1) All $L(s, \chi)$ except one real zero of $L(s, \chi_1)$ are zero-free in the region

$$\sigma \geq 1 - \frac{A}{\log N(t)}, \quad |t| \leq T \quad (A > 0) \quad (22)$$

2) All zeros and poles of all $L(s, \chi) \bmod \mathcal{F}$ are far from the line

$$\sigma = 1 - \frac{A}{\log N(t)}, \quad |t| \leq T \quad (23)$$

by $\gg \frac{1}{\log N(t)}$.

We put $\gamma = 1 - \frac{A}{\log N(t)}$ (24)

We change the way of integration, and have

$$\begin{aligned} I_1(\chi) &= -2\pi i E(\chi) x + 2\pi i E_1(\chi) \frac{x^{\beta_1}}{\beta_1^2} + \int_{\gamma-iT}^{\gamma+iT} \frac{L'}{L} \frac{x^s}{s^2} ds \\ &\quad + \left(\int_{\gamma+iT}^{\alpha+iT} - \int_{\gamma-iT}^{\alpha-iT} \right) \frac{L'}{L}(s, \chi) \frac{x^s}{s^2} ds \\ &= -2\pi i E(\chi) x + 2\pi i E_1(\chi) \frac{x^{\beta_1}}{\beta_1^2} + I_4(\chi) + I_5(\chi) \quad (25) \end{aligned}$$

(Say !)

$$E_1(\chi) = \begin{cases} 1 & \text{if there is a zero in the region} \\ 0 & \text{otherwise} \end{cases} \quad (26)$$

We put $E_1^* = \sum_{\chi \bmod f} E_1(\chi) \overline{\chi(z)}$ (27)

Using (14), we transform $I_4(\chi)$ as follows ;

$$\begin{aligned} I_4(\chi) &= \int_{\delta-iT}^{\delta+iT} \frac{L'(s, \chi) (1 - L(s, \chi) Q(s, \chi))^2}{L(s, \chi)} \frac{\chi^s}{s^2} ds \\ &\quad + \left(\int_{\beta-iT}^{\beta+iT} + \int_{\beta+iT}^{\delta+iT} - \int_{\beta-iT}^{\delta-iT} \right) (2L'Q - L'LQ^2) \frac{\chi^s}{s^2} ds \\ &= I_7(\chi) + I_8(\chi) + I_9(\chi) + I_{10}(\chi) \quad (\text{say}) \quad (28) \end{aligned}$$

6.2 Estimation of $I_2(\chi)$ and $I_3(\chi)$

By theorem 7, we have the following estimations on the line $\sigma = \alpha$

$$\frac{L'}{L}(s, \chi) \ll \log(Nf)(|t|+2) \log x \quad (29)$$

Hence ,

$$I_3(\chi) + I_4(\chi) \ll x \log x \int_T^{\infty} \frac{\log Nf(1H+2)}{\alpha^2 + T^2} dt \ll \frac{x \log^2 x}{T} \quad (30)$$

6.3 Estimation of $I_5(\chi)$ and $I_6(\chi)$

As the above paragraph, we have

$$I_5(\chi) + I_6(\chi) \ll \frac{\log^2 x}{T^2} \int_T^{\infty} x^{\sigma} d\sigma \ll \frac{x \log^2 x}{T^2} \quad (31)$$

6.4 Estimation of $I_9(\chi)$ and $I_{10}(\chi)$

By using Cauchy's integration formula, we have

$$L'(s, \chi) \ll N(f)^2 (1+t+2)^{\frac{2}{3}n(1-\sigma)} \log^3 N(f) (1+t+2) \quad (32)$$

On the other hand, the following estimations are obtained by using convexity argument.

$$Q(s, \chi) \ll N(f) T^{n(1-\sigma)} \log \chi$$

for $\beta \leq \sigma \leq \alpha$. (33)

Therefore we have

$$\begin{aligned} I_9(\chi) + I_{10}(\chi) &\ll N(f)^6 \frac{\chi}{T^2} \log^5 \chi \int_{\beta}^{\gamma} \left(\frac{T^{\frac{10}{3}n}}{\chi} \right)^{1-\sigma} d\sigma \\ &\ll N(f)^6 \frac{\chi^{\gamma}}{T^2} \log^5 \chi \\ &\quad \left(\text{if } T^{\frac{10}{3}n} \leq \chi \right) \end{aligned} \quad (34)$$

6.5 Estimation of $I_8(\chi)$

By theorem 6, (32), and (33), we have

$$\begin{aligned} I_8(\chi) &\ll N(f)^6 T^{\frac{10}{3}n(1-\beta)} \chi^{\beta} \log^6 \chi \\ &= N(f)^6 T^{\frac{5}{3}n} \chi^{\beta} \log^6 \chi \end{aligned} \quad (35)$$

6.6 Estimations of $I_7(\chi)$

Putting $M = N(f) T^n$ in theorem 5, we have on the line $\sigma = \alpha$

$$L(s, \chi) Q(s, \chi) - 1 = \sum_{N(f)T^n \leq N(\sigma) \leq (N(f)T^n)^2} b(\sigma) \chi(\sigma) N(\sigma)^{-s} + O\left(|Q(s, \chi)| \left(\frac{1}{T} + E(\sigma) \log \chi\right)\right) \quad (36)$$

where

$$b(\sigma) = \sum_{\substack{\beta | \sigma, \\ N(\beta) < N(f) T^n}} \mu(\beta) \quad (37)$$

Therefore ,

$$\sum_{x \bmod f} |I_T(x)| \ll x^\delta \log^2 x \left\{ \int_{-T}^T x \sum_{\substack{N(f)T^n \leq N(n) \leq (N(f)T^n)^2}} \frac{|b(n) x(n) N(n)^{-\gamma-it}|^2}{|x+it|^2} dt \right. \\ \left. + \frac{1}{T^2} \int_{-T}^T \sum_{x \bmod f} |Q(x+it, x)|^2 dt + \log^2 x \int_{-T}^T \frac{|Q(x+it, x)|^2}{|x+it|^2} dt \right\} \quad (38)$$

We now apply Gallagher's mean value theorem to the above integrals .

Then we have

$$\sum_{x \bmod f} \int_{-T}^T |Q(x+it, x)|^2 dt \ll \sum_{N(n) < N(f)T^n} (R(f)T + N(f)^{\frac{1}{n}} T N(n)^{-\frac{1}{n} + N(n)}) \frac{1}{N(n)^{\delta}} \\ \ll N(f)T \quad (39)$$

and

$$\sum_{x \bmod f} \int_{-T}^T \frac{\left| \sum_{\substack{N(f)T^n \leq N(n) \leq (N(f)T^n)^2}} b(n) x(n) N(n)^{-\gamma-it} \right|^2}{|x+it|^2} dt \\ \ll \sum_{\substack{N(f)T^n \leq N(n) \leq (N(f)T^n)^2}} (R(f)T + N(f)^{\frac{1}{n}} T N(n)^{-\frac{1}{n} + N(n)}) \frac{|b(n)|^2}{N(n)^{2\delta}} \\ \ll \log^4 x \quad (40)$$

(We use $\sum_{N(n) < x} |b(n)|^2 \ll \sum_{N(n) \leq x} |d(n)|^2 \ll x \log^3 x$)

The last integral is trivially estimated and we obtain

$$\sum_{x \bmod f} I_T(x) \ll x^\delta \left(\log^4 x + \frac{N(f)}{T} \right) \quad (41)$$

6.7 Estimation of $\psi(x, 2)$

Now we collect the results of preceding paragraphs , then we have

$$\psi_1(x, 2) = \frac{x}{R(f)} - E_1^* \frac{x^{\beta_1}}{R(f)^{\beta_1}} + o\left(\frac{x}{T} \log^2 x\right)$$

$$+ O\left(x^{1-\frac{1}{2n}} N(f)^6 T^{\frac{5}{3}} \log^6 x\right) + O\left(\frac{x^6}{R(f)} \log^4 x\right) \quad (42)$$

Taking $T = N(f)^7$, we have

$$\frac{x}{T} \log^2 x \ll \frac{x}{R(f)} e^{-c_3 \frac{\log x}{\log N(f)}} \quad (43)$$

$$\begin{aligned} x^{1-\frac{1}{2n}} N(f)^6 T^{\frac{5}{3}} \log^5 x &\ll \frac{x}{R(f)} N(f)^{-c_2 \log \log N(f)} \\ &\ll \frac{x}{R(f)} e^{-c_5 \frac{\log x}{\log N(f)}} \end{aligned} \quad (44)$$

and

$$\frac{x^7}{R(f)} \log^4 x \ll \frac{x}{R(f)} e^{-\frac{A}{2} \frac{\log x}{\log N(f)}} \quad (45)$$

Therefore we get

$$\psi_1(x, z) = \frac{x}{R(f)} - E_1^* \frac{x^{\beta_1}}{R(f)^{\beta_1}} + O\left(\frac{x}{R(f)} e^{-c_6 \frac{\log x}{\log N(f)}}\right) \quad (46)$$

7. Estimation of $\psi(x, z)$

Proposition 1. For all $c^* > 0$, there exists a constant $c,^* 0$,

such that, under the condition

$$c_1^* \log N(f) \log \log N(f) \leq \log x \leq c^* (\log N(f))^2$$

$$\psi(x, z) = \frac{x}{R(f)} - E_1^* \frac{x^{\beta_1}}{R(f)^{\beta_1}} + O\left(\frac{x}{R(f)} e^{-c_2^* \frac{\log x}{\log N(f)}}\right) \quad (47)$$

proof) Let $c_2 = 2 c^*$ and take θ , such that $0 \leq \theta \leq 1$,

we have by (46),

$$\begin{aligned} \psi_1(x + \theta x, z) - \psi_1(x, z) &= \frac{\theta x}{R(f)} - E_1^* \frac{x^{\beta_1}}{R(f)^{\beta_1}} (\theta + o(\theta^2)) \\ &\quad + O\left(\frac{x}{R(f)} e^{-c_6 \frac{\log x}{\log N(f)}}\right) \end{aligned} \quad (48)$$

And by the definition of $\psi_1(x, z)$, we get

$$\psi_1(x+\theta x, \ell) - \psi_1(x, \ell) = \sum_{N(\alpha) \leq x, \alpha \in \ell} \Lambda(\alpha) \log(1+\theta) + \sum_{\substack{x \leq N(\alpha) < x+\theta x \\ \alpha \in \ell}} \Lambda(\alpha) \log \frac{x}{N(\alpha)} \quad (49)$$

Using theorem 3, we obtain

$$\sum_{\substack{x \leq N(\alpha) < x+\theta x \\ \alpha \in \ell}} \Lambda(\alpha) \log \frac{x}{N(\alpha)} = O\left(\theta \log x \left(\frac{\theta x}{h(\ell)} + \frac{N(\ell)^{\frac{1}{n}}}{h(\ell)} x^{1-\frac{1}{n}} + 1\right)\right) \quad (50)$$

and

$$\sum_{N(\alpha) \leq x, \alpha \in \ell} \Lambda(\alpha) \log(1+\theta) = \theta \psi(x, \ell) + O\left(\frac{\theta^2 x \log x}{h(\ell)}\right) \quad (51)$$

The proposition is immediately deduced, taking

$$\theta = e^{-\frac{C_6}{2} \frac{\log x}{\log N(\ell)}} \quad (52)$$

The theorem is easily proved from this proposition.

References

1. Gallagher, P. X., A large density estimate near $\sigma = 1$ Invent.

Math. 11 (1970) p. 329 - 339

2. Hilano, T., A remark on the prime number theorem of Rodosskii

(In Japanese) 解析数論七十一報告集 (1972)

3. Landau, E. Einführung in der elementare und analytische Theorie

der algebraischen Zahlen und der Ideale, Leipzig. 1918

4. Landau, E., Über Ideale und Primideale in Idealklassen, Math.

Zeit., 2 (1918) p. 52 - p. 154

5. Mitsui, T., Generalized prime number theorem Jap. J. Math. 26 (1956)

$p . 1 - p . 42$

6 . Montgomery , H . L . , *Topics in multiplicative number theory* ,

Springer Lecture Note 227 (1970) , Springer Verlag

7 . Page , A . , *On the number of primes in an arithmetic progression* ,

Proc . London Math . Soc . 39 (1935) , p . 116 - p . 141

8 . Prachar , K . , *Primzahlverteilung* , Springer (1957)

9 . Suetsuna , Z . , *Analytic number theory* , (In Japanese) Iwanami

(1950)

10 . Tatsuzawa , T . , *On the number of integral ideals in algebraic*

number fields , whose norms not exceeding x

to appear