

On the $K(\pi, 1)$ -property for rings of integers in the mixed case

By

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Abstract

We investigate the Galois group $G_S(p)$ of the maximal p -extension unramified outside a finite set S of primes of a number field in the (mixed) case, when there are primes dividing p inside and outside S . We show that the cohomology of $G_S(p)$ is ‘often’ isomorphic to the étale cohomology of the scheme $\mathrm{Spec}(\mathcal{O}_k \setminus S)$, in particular, $G_S(p)$ is of cohomological dimension 2 then. We deduce this from the results in our previous paper [Sch2], which mainly dealt with the tame case.

§ 1. Introduction

Let Y be a connected locally noetherian scheme and let p be a prime number. We denote the étale fundamental group of Y by $\pi_1(Y)$ and its maximal pro- p factor group by $\pi_1(Y)(p)$. The Hochschild-Serre spectral sequence induces natural homomorphisms

$$\phi_i : H^i(\pi_1^{et}(Y)(p), \mathbb{Z}/p\mathbb{Z}) \longrightarrow H_{et}^i(Y, \mathbb{Z}/p\mathbb{Z}), \quad i \geq 0,$$

and we call Y a ‘ $K(\pi, 1)$ for p ’ if all ϕ_i are isomorphisms; see [Sch2] Proposition 2.1 for equivalent conditions. See [Wi2] for a purely Galois cohomological approach to the $K(\pi, 1)$ -property. Our main result is the following

Theorem 1.1. *Let k be a number field and let p be a prime number. Assume that k does not contain a primitive p -th root of unity and that the class number of k is prime to p . Then the following holds:*

Let S be a finite set of primes of k and let T be a set of primes of k of Dirichlet density $\delta(T) = 1$. Then there exists a finite subset $T_1 \subset T$ such that $\mathrm{Spec}(\mathcal{O}_k) \setminus (S \cup T_1)$ is a $K(\pi, 1)$ for p .

Received January 15, 2008. Revised June 2, 2008.

2000 Mathematics Subject Classification(s): 11R34, 12G10

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Remarks. 1. If S contains the set S_p of primes dividing p , then Theorem 1.1 holds with $T_1 = \emptyset$ and even without the condition $\zeta_p \notin k$ and $Cl(k)(p) = 0$, see [Sch2], Proposition 2.3. In the tame case $S \cap S_p = \emptyset$, the statement of Theorem 1.1 is the main result of [Sch2]. Here we provide the extension to the ‘mixed’ case $\emptyset \subsetneq S \cap S_p \subsetneq S_p$.

2. For a given number field k , all but finitely many prime numbers p satisfy the condition of Theorem 1.1. We conjecture that Theorem 1.1 holds without the restricting assumption on p .

Let S be a finite set of places of a number field k . Let $k_S(p)$ be the maximal p -extension of k unramified outside S and put $G_S(p) = \text{Gal}(k_S(p)|k)$. If $S_{\mathbb{R}}$ denotes the set of real places of k , then $G_{S \cup S_{\mathbb{R}}}(p) \cong \pi_1(\text{Spec}(\mathcal{O}_k) \setminus S)(p)$ (we have $G_S(p) = G_{S \cup S_{\mathbb{R}}}(p)$ if p is odd or k is totally imaginary). The following Theorem 1.2 sharpens Theorem 1.1.

Theorem 1.2. *The set $T_1 \subset T$ in Theorem 1.1 may be chosen such that*

- (i) T_1 consists of primes $\mathfrak{p}$ of degree 1 with $N(\mathfrak{p}) \equiv 1 \pmod{p}$,
- (ii) $(k_{S \cup T_1}(p))_{\mathfrak{p}} = k_{\mathfrak{p}}(p)$ for all primes $\mathfrak{p} \in S \cup T_1$.

Note that Theorem 1.2 provides nontrivial information even in the case $S \supset S_p$, where assertion (ii) was only known when k contains a primitive p -th root of unity (Kuz’min’s theorem, see [Kuz] or [NSW], 10.6.4 or [NSW²], 10.8.4, respectively) and for certain CM fields (by a result of Mukhamedov, see [Muk] or [NSW], X §6 exercise or [NSW²], X §8 exercise, respectively).

By Theorem 3.3 below, Theorem 1.2 provides many examples of $G_S(p)$ being a duality group. If $\zeta_p \notin k$, this is interesting even in the case that $S \supset S_p$, where examples of $G_S(p)$ being a duality group were previously known only for real abelian fields and for certain CM-fields (see [NSW], 10.7.15 and [NSW²], 10.9.15, respectively, and the remark following there).

Previous results in the mixed case had been achieved by K. Wingberg [W11], Ch. Maire [Mai] and D. Vogel [Vog]. Though not explicitly visible in this paper, the present progress in the subject was only possible due to the results on mild pro- p groups obtained by J. Labute in [Lab].

I would like to thank K. Wingberg for pointing out that the proof of Proposition 8.1 in my paper [Sch2] did not use the assumption that the sets S and S' are disjoint from S_p . This was the key observation for the present paper. The main part of this text was written while I was a guest at the Department of Mathematical Sciences of Tokyo University and of the Research Institute for Mathematical Sciences in Kyoto. I want to thank these institutions for their kind hospitality.

§ 2. Proof of Theorems 1.1 and 1.2

We start with the observation that the proofs of Proposition 8.1 and Corollary 8.2 in [Sch2] did not use the assumption that the sets S and S' are disjoint from S_p . Therefore, with the same proof (which we repeat for the convenience of the reader) as in loc. cit., we obtain

Proposition 2.1. *Let k be a number field and let p be a prime number. Assume k to be totally imaginary if $p = 2$. Put $X = \text{Spec}(\mathcal{O}_k)$ and let $S \subset S'$ be finite sets of primes of k . Assume that $X \setminus S$ is a $K(\pi, 1)$ for p and that $G_S(p) \neq 1$. Further assume that each $\mathfrak{p} \in S' \setminus S$ does not split completely in $k_S(p)$. Then the following hold.*

- (i) $X \setminus S'$ is a $K(\pi, 1)$ for p .
- (ii) $k_{S'}(p)_{\mathfrak{p}} = k_{\mathfrak{p}}(p)$ for all $\mathfrak{p} \in S' \setminus S$.

Furthermore, the arithmetic form of Riemann's existence theorem holds, i.e., setting $K = k_S(p)$, the natural homomorphism

$$\bigstar_{\mathfrak{p} \in S' \setminus S(K)} T(K_{\mathfrak{p}}(p)|K_{\mathfrak{p}}) \longrightarrow \text{Gal}(k_{S'}(p)|K)$$

is an isomorphism. Here $T(K_{\mathfrak{p}}(p)|K_{\mathfrak{p}})$ is the inertia group and $\bigstar$ denotes the free pro- p -product of a bundle of pro- p -groups, cf. [NSW], Ch. IV, §3. In particular, the group $\text{Gal}(k_{S'}(p)|k_S(p))$ is a free pro- p -group.

Proof. The $K(\pi, 1)$ -property implies

$$H^i(G_S(p), \mathbb{Z}/p\mathbb{Z}) \cong H_{et}^i(X \setminus S, \mathbb{Z}/p\mathbb{Z}) = 0 \text{ for } i \geq 4,$$

hence $cd \ G_S(p) \leq 3$. Let $\mathfrak{p} \in S' \setminus S$. Since $\mathfrak{p}$ does not split completely in $k_S(p)$ and since $cd \ G_S(p) < \infty$, the decomposition group of $\mathfrak{p}$ in $k_S(p)|k$ is a non-trivial and torsion-free quotient of $\mathbb{Z}_p \cong \text{Gal}(k_{\mathfrak{p}}^{nr}(p)|k_{\mathfrak{p}})$. Therefore $k_S(p)_{\mathfrak{p}}$ is the maximal unramified p -extension of $k_{\mathfrak{p}}$. We denote the normalization of an integral normal scheme Y in an algebraic extension L of its function field by Y_L . Then $(X \setminus S)_{k_S(p)}$ is the universal pro- p covering of $X \setminus S$. We consider the étale excision sequence for the pair $((X \setminus S)_{k_S(p)}, (X \setminus S')_{k_S(p)})$. By assumption, $X \setminus S$ is a $K(\pi, 1)$ for p , hence $H_{et}^i((X \setminus S)_{k_S(p)}, \mathbb{Z}/p\mathbb{Z}) = 0$ for $i \geq 1$ by [Sch2], Proposition 2.1. Omitting the coefficients $\mathbb{Z}/p\mathbb{Z}$ from the notation, this implies isomorphisms

$$H_{et}^i((X \setminus S')_{k_S(p)}) \xrightarrow{\sim} \bigoplus'_{\mathfrak{p} \in S' \setminus S(k_S(p))} H_{\mathfrak{p}}^{i+1}(((X \setminus S)_{k_S(p)})_{\mathfrak{p}})$$

for $i \geq 1$. Here (and in variants also below) we use the notational convention

$$\bigoplus'_{\mathfrak{p} \in S' \setminus S(k_S(p))} H_{\mathfrak{p}}^{i+1}(((X \setminus S)_{k_S(p)})_{\mathfrak{p}}) := \varinjlim_{K \subset k_S(p)} \bigoplus_{\mathfrak{p} \in S' \setminus S(K)} H_{\mathfrak{p}}^{i+1}(((X \setminus S)_K)_{\mathfrak{p}}),$$

where K runs through the finite extensions of k inside $k_S(p)$. As $k_S(p)$ realizes the maximal unramified p -extension of $k_{\mathfrak{p}}$ for all $\mathfrak{p} \in S' \setminus S$, the schemes $((X \setminus S)_{k_S(p)})_{\mathfrak{p}}$, $\mathfrak{p} \in S' \setminus S(k_S(p))$, have trivial cohomology with values in $\mathbb{Z}/p\mathbb{Z}$ and we obtain isomorphisms

$$H^i((k_S(p))_{\mathfrak{p}}) \xrightarrow{\sim} H_{\mathfrak{p}}^{i+1}(((X \setminus S)_{k_S(p)})_{\mathfrak{p}})$$

for $i \geq 1$. These groups vanish for $i \geq 2$. This implies

$$H_{et}^i((X \setminus S')_{k_S(p)}) = 0$$

for $i \geq 2$. Since the scheme $(X \setminus S')_{k_{S'}(p)}$ is the universal pro- p covering of $(X \setminus S')_{k_S(p)}$, the Hochschild-Serre spectral sequence

$$E_2^{ij} = H^i(Gal(k_{S'}(p)|k_S(p)), H_{et}^j((X \setminus S')_{k_{S'}(p)})) \Rightarrow H_{et}^{i+j}((X \setminus S')_{k_S(p)})$$

yields an inclusion

$$H^2(Gal(k_{S'}(p)|k_S(p))) \hookrightarrow H_{et}^2((X \setminus S')_{k_S(p)}) = 0.$$

Hence $Gal(k_{S'}(p)|k_S(p))$ is a free pro- p -group and

$$H^1(Gal(k_{S'}(p)|k_S(p))) \xrightarrow{\sim} H_{et}^1((X \setminus S')_{k_S(p)}) \cong \bigoplus'_{\mathfrak{p} \in S' \setminus S(k_S(p))} H^1(k_S(p)_{\mathfrak{p}}).$$

We set $K = k_S(p)$ and consider the natural homomorphism

$$\phi : \bigstar_{\mathfrak{p} \in S' \setminus S(K)} T(K_{\mathfrak{p}}(p)|K_{\mathfrak{p}}) \longrightarrow Gal(k_{S'}(p)|K).$$

By the calculation of the cohomology of a free product ([NSW], 4.3.10 and 4.1.4), ϕ is a homomorphism between free pro- p -groups which induces an isomorphism on mod p cohomology. Therefore ϕ is an isomorphism. In particular, $k_{S'}(p)_{\mathfrak{p}} = k_{\mathfrak{p}}(p)$ for all $\mathfrak{p} \in S' \setminus S$. Using that $Gal(k_{S'}(p)|k_S(p))$ is free, the Hochschild-Serre spectral sequence induces an isomorphism

$$0 = H_{et}^2((X \setminus S')_{k_S(p)}) \xrightarrow{\sim} H_{et}^2((X \setminus S')_{k_{S'}(p)})^{Gal(k_{S'}(p)|k_S(p))}.$$

Hence $H_{et}^2((X \setminus S')_{k_{S'}(p)}) = 0$, since $Gal(k_{S'}(p)|k_S(p))$ is a pro- p -group. Now [Sch2], Proposition 2.1 implies that $X \setminus S'$ is a $K(\pi, 1)$ for p . $\square$

In order to prove Theorem 1.1, we first provide the following lemma. For an extension field $K|k$ and a set of primes T of k , we write $T(K)$ for the set of prolongations of primes in T to K and $\delta_K(T)$ for the Dirichlet density of the set of primes $T(K)$ of K .

Lemma 2.2. *Let k be a number field, p a prime number and S a finite set of nonarchimedean primes of k . Let T be a set of primes of k with $\delta_{k(\mu_p)}(T) = 1$. Then there exists a finite subset $T_0 \subset T$ such that all primes $\mathfrak{p} \in S$ do not split completely in the extension $k_{T_0}(p)|k$.*

Proof. By [NSW], 9.2.2 (ii) or [NSW²], 9.2.3 (ii), respectively, the restriction map

$$H^1(G_{T \cup S \cup S_p \cup S_{\mathbb{R}}}(p), \mathbb{Z}/p\mathbb{Z}) \longrightarrow \prod_{\mathfrak{p} \in S \cup S_p \cup S_{\mathbb{R}}} H^1(k_{\mathfrak{p}}, \mathbb{Z}/p\mathbb{Z})$$

is surjective. A class $\alpha \in H^1(G_{T \cup S \cup S_p \cup S_{\mathbb{R}}}(p), \mathbb{Z}/p\mathbb{Z})$ which restricts to an unramified class $\alpha_{\mathfrak{p}} \in H_{nr}^1(k_{\mathfrak{p}}, \mathbb{Z}/p\mathbb{Z})$ for all $\mathfrak{p} \in S \cup S_p \cup S_{\mathbb{R}}$ is contained in $H^1(G_T(p), \mathbb{Z}/p\mathbb{Z})$. Therefore the image of the composite map

$$H^1(G_T(p), \mathbb{Z}/p\mathbb{Z}) \hookrightarrow H^1(G_{T \cup S \cup S_p \cup S_{\mathbb{R}}}(p), \mathbb{Z}/p\mathbb{Z}) \rightarrow \prod_{\mathfrak{p} \in S} H^1(k_{\mathfrak{p}}, \mathbb{Z}/p\mathbb{Z})$$

contains the subgroup $\prod_{\mathfrak{p} \in S} H_{nr}^1(k_{\mathfrak{p}}, \mathbb{Z}/p\mathbb{Z})$. As this group is finite, it is already contained in the image of $H^1(G_{T_0}(p), \mathbb{Z}/p\mathbb{Z})$ for some finite subset $T_0 \subset T$. We conclude that no prime in S splits completely in the maximal elementary abelian p -extension of k unramified outside T_0 . $\square$

Proof of Theorems 1.1 and 1.2. As $p \neq 2$, we may ignore archimedean primes. Furthermore, we may remove the primes in $S \cup S_p$ and all primes of degree greater than 1 from T . In addition, we remove all primes $\mathfrak{p}$ with $N(\mathfrak{p}) \not\equiv 1 \pmod{p}$ from T . After these changes, we still have $\delta_{k(\mu_p)}(T) = 1$.

By Lemma 2.2, we find a finite subset $T_0 \subset T$ such that no prime in S splits completely in $k_{T_0}(p)|k$. Put $X = \text{Spec}(\mathcal{O}_k)$. By [Sch2], Theorem 6.2, applied to T_0 and $T \setminus T_0$, we find a finite subset $T_2 \subset T \setminus T_0$ such that $X \setminus (T_0 \cup T_2)$ is a $K(\pi, 1)$ for p . Then Proposition 2.1 applied to $T_0 \cup T_2 \subset S \cup T_0 \cup T_2$, shows that also $X \setminus (S \cup T_0 \cup T_2)$ is a $K(\pi, 1)$ for p . Now put $T_1 = T_0 \cup T_2 \subset T$.

It remains to show Theorem 1.2. Assertion (i) holds by construction of T_1 . Again by construction, $X \setminus T_1$ is a $K(\pi, 1)$ for p . By [Sch2], Theorem 3, the field $k_{T_1}(p)$ realizes $k_{\mathfrak{p}}(p)$ for $\mathfrak{p} \in T_1$, showing (ii) for these primes. Finally, assertion (ii) for $\mathfrak{p} \in S$ follows from Proposition 2.1. $\square$

§ 3. Duality

We start by investigating the relation between the $K(\pi, 1)$ -property and the universal norms of global units.

Let us first remove redundant primes from S : If $\mathfrak{p} \nmid p$ is a prime with $\zeta_p \notin k_{\mathfrak{p}}$, then every p -extension of the local field $k_{\mathfrak{p}}$ is unramified (see [NSW], 7.5.1 or [NSW²], 7.5.9, respectively). Therefore primes $\mathfrak{p} \notin S_p$ with $N(\mathfrak{p}) \not\equiv 1 \pmod{p}$ cannot ramify in a p -extension. Removing all these redundant primes from S , we obtain a subset $S_{\min} \subset S$, which has the property that $G_S(p) = G_{S_{\min}}(p)$. Furthermore, by [Sch2], Lemma 4.1, $X \setminus S$ is a $K(\pi, 1)$ for p if and only if $X \setminus S_{\min}$ is a $K(\pi, 1)$ for p .

Theorem 3.1. *Let k be a number field and let p be a prime number. Assume that k is totally imaginary if $p = 2$. Let S be a finite set of nonarchimedean primes of k . Then any two of the following conditions (a) – (c) imply the third.*

- (a) $\text{Spec}(\mathcal{O}_k) \setminus S$ is a $K(\pi, 1)$ for p .
- (b) $\varprojlim_{K \subset k_S(p)} \mathcal{O}_K^{\times} \otimes \mathbb{Z}_p = 0$.
- (c) $(k_S(p))_{\mathfrak{p}} = k_{\mathfrak{p}}(p)$ for all primes $\mathfrak{p} \in S_{\min}$.

The limit in (b) runs through all finite extensions K of k inside $k_S(p)$. If (a)–(c) hold, then also

$$\varprojlim_{K \subset k_S(p)} \mathcal{O}_{K, S_{\min}}^{\times} \otimes \mathbb{Z}_p = 0.$$

Remarks: 1. Assume that $\zeta_p \in k$ and $S \supset S_p$. Then (a) holds and condition (c) holds for $p > 2$ if $\#S > r_2 + 2$ (see [NSW²], Remark 2 after 10.9.3). In the case $k = \mathbb{Q}(\zeta_p)$, $S = S_p$, condition (c) holds if and only if p is an irregular prime number.

2. Assume that $S \cap S_p = \emptyset$ and $S_{\min} \neq \emptyset$. If condition (a) holds, then either $G_S(p) = 1$ (which only happens in very special situations, see [Sch2], Proposition 7.4) or (c) holds by [Sch2], Theorem 3 (or by Proposition 3.2 below).

Proof of Theorem 3.1. We may assume $S = S_{\min}$ in the proof. Let K run through the finite extensions of k in $k_S(p)$ and put $X_K = \text{Spec}(\mathcal{O}_K)$. Applying the topological Nakayama-Lemma ([NSW], 5.2.18) to the compact $\mathbb{Z}_p$ -module $\varprojlim \mathcal{O}_K^{\times} \otimes \mathbb{Z}_p$, we see that condition (b) is equivalent to

$$(b)' \quad \varprojlim_{K \subset k_S(p)} \mathcal{O}_K^{\times} / p = 0.$$

Furthermore, by [Sch2], Proposition 2.1, condition (a) is equivalent to

$$(a)' \quad \varinjlim_{K \subset k_S(p)} H_{\text{et}}^i((X \setminus S)_K, \mathbb{Z}/p\mathbb{Z}) = 0 \text{ for } i \geq 1.$$

Condition (a)' always holds for $i = 1$, $i \geq 4$, and it holds for $i = 3$ provided that $G_S(p)$ is infinite or S is nonempty or $\zeta_p \notin k$ (see [Sch2], Lemma 3.7). The flat Kummer sequence $0 \rightarrow \mu_p \rightarrow \mathbb{G}_m \xrightarrow{p} \mathbb{G}_m \rightarrow 0$ induces exact sequences

$$0 \longrightarrow \mathcal{O}_K^{\times} / p \longrightarrow H_{\text{f}}^1(X_K, \mu_p) \longrightarrow {}_p\text{Pic}(X_K) \rightarrow 0$$

for all K . As the field $k_S(p)$ does not have nontrivial unramified p -extensions, class field theory implies

$$\varprojlim_{K \subset k_S(p)} {}_p\text{Pic}(X_K) \subset \varprojlim_{K \subset k_S(p)} \text{Pic}(X_K) \otimes \mathbb{Z}_p = 0.$$

As we assumed k to be totally imaginary if $p = 2$, the flat duality theorem of Artin-Mazur ([Mil], III Corollary 3.2) induces natural isomorphisms

$$H_{et}^2(X_K, \mathbb{Z}/p\mathbb{Z}) = H_{\text{fl}}^2(X_K, \mathbb{Z}/p\mathbb{Z}) \cong H_{\text{fl}}^1(X_K, \mu_p)^\vee.$$

We conclude that

$$(*) \quad \varinjlim_{K \subset k_S(p)} H_{et}^2(X_K, \mathbb{Z}/p\mathbb{Z}) \cong \left(\varprojlim_{K \subset k_S(p)} \mathcal{O}_K^\times / p \right)^\vee.$$

We first show the equivalence of (a) and (b) in the case $S = \emptyset$. If (a)' holds, then (*) shows (b)'. If (b) holds, then $\zeta_{\mathfrak{p}} \notin k$ or $G_S(p)$ is infinite. Hence we obtain (a)' for $i = 3$. Furthermore, (b)' implies (a)' for $i = 2$ by (*). This finishes the proof of the case $S = \emptyset$.

Now we assume that $S \neq \emptyset$. For $\mathfrak{p} \in S(K)$, a standard calculation of local cohomology shows that

$$H_{\mathfrak{p}}^i(X_K, \mathbb{Z}/p\mathbb{Z}) \cong \begin{cases} 0 & \text{for } i \leq 1, \\ H^1(K_{\mathfrak{p}}, \mathbb{Z}/p\mathbb{Z}) / H_{nr}^1(K_{\mathfrak{p}}, \mathbb{Z}/p\mathbb{Z}) & \text{for } i = 2, \\ H^2(K_{\mathfrak{p}}, \mathbb{Z}/p\mathbb{Z}) & \text{for } i = 3. \\ 0 & \text{for } i \geq 4. \end{cases}$$

For $\mathfrak{p} \in S = S_{\min}$, every proper Galois subextension of $k_{\mathfrak{p}}(p)|k_{\mathfrak{p}}$ admits ramified p -extensions. Hence condition (c) is equivalent to

$$(c)' \quad \varinjlim_{K \subset k_S(p)} \bigoplus_{\mathfrak{p} \in S(K)} H_{\mathfrak{p}}^i(X_K, \mathbb{Z}/p\mathbb{Z}) = 0 \text{ for all } i,$$

and to

$$(c)'' \quad \varinjlim_{K \subset k_S(p)} \bigoplus_{\mathfrak{p} \in S(K)} H_{\mathfrak{p}}^2(X_K, \mathbb{Z}/p\mathbb{Z}) = 0.$$

Consider the direct limit over all K of the excision sequences

$$\cdots \rightarrow \bigoplus_{\mathfrak{p} \in S(K)} H_{\mathfrak{p}}^i(X_K, \mathbb{Z}/p\mathbb{Z}) \rightarrow H_{et}^i(X_K, \mathbb{Z}/p\mathbb{Z}) \rightarrow H_{et}^i((X \setminus S)_K, \mathbb{Z}/p\mathbb{Z}) \rightarrow \cdots.$$

Assume that (a)' holds, i.e. the right hand terms vanish in the limit for $i \geq 1$. Then (*) shows that (b)' is equivalent to (c)'.

Now assume that (b) and (c) hold. As above, (b) implies the vanishing of the middle term for $i = 2$ in the limit. Condition (c)' then shows (a)'.

We have proven that any two of the conditions (a)–(c) imply the third.

Finally, assume that (a)–(c) hold. Tensoring the exact sequences (cf. [NSW], 10.3.11 or [NSW²], 10.3.12, respectively)

$$0 \rightarrow \mathcal{O}_K^\times \rightarrow \mathcal{O}_{K,S}^\times \rightarrow \bigoplus_{\mathfrak{p} \in S(K)} (K_{\mathfrak{p}}^\times / U_{\mathfrak{p}}) \rightarrow \text{Pic}(X_K) \rightarrow \text{Pic}((X \setminus S)_K) \rightarrow 0$$

by (the flat $\mathbb{Z}$ -algebra) $\mathbb{Z}_p$, we obtain exact sequences of finitely generated, hence compact, $\mathbb{Z}_p$ -modules. Passing to the projective limit over the finite extensions K of k inside $k_S(p)$ and using $\varprojlim \text{Pic}(X_K) \otimes \mathbb{Z}_p = 0$, we obtain the exact sequence

$$0 \rightarrow \varprojlim_{K \subset k_S(p)} \mathcal{O}_K^\times \otimes \mathbb{Z}_p \rightarrow \varprojlim_{K \subset k_S(p)} \mathcal{O}_{K,S}^\times \otimes \mathbb{Z}_p \rightarrow \varprojlim_{K \subset k_S(p)} \bigoplus_{\mathfrak{p} \in S(K)} (K_{\mathfrak{p}}^\times / U_{\mathfrak{p}}) \otimes \mathbb{Z}_p \rightarrow 0.$$

Condition (c) and local class field theory imply the vanishing of the right hand limit. Therefore (b) implies the vanishing of the projective limit in the middle. $\square$

If $G_S(p) \neq 1$ and condition (a) of Theorem 1.1 holds, then the failure in condition (c) can only come from primes dividing p . This follows from the next

Proposition 3.2. *Let k be a number field and let p be a prime number. Assume that k is totally imaginary if $p = 2$. Let S be a finite set of nonarchimedean primes of k . If $\text{Spec}(\mathcal{O}_k) \setminus S$ is a $K(\pi, 1)$ for p and $G_S(p) \neq 1$, then every prime $\mathfrak{p} \in S$ with $\zeta_p \in k_{\mathfrak{p}}$ has an infinite inertia group in $G_S(p)$. Moreover, we have*

$$k_S(p)_{\mathfrak{p}} = k_{\mathfrak{p}}(p)$$

for all $\mathfrak{p} \in S_{\min} \setminus S_p$.

Proof. We may assume $S = S_{\min}$. Suppose $\mathfrak{p} \in S$ with $\zeta_p \in k_{\mathfrak{p}}$ does not ramify in $k_S(p)|k$. Setting $S' = S \setminus \{\mathfrak{p}\}$, we have $k_{S'}(p) = k_S(p)$, in particular,

$$H_{et}^1(X \setminus S', \mathbb{Z}/p\mathbb{Z}) \xrightarrow{\sim} H_{et}^1(X \setminus S, \mathbb{Z}/p\mathbb{Z}).$$

In the following, we omit the coefficients $\mathbb{Z}/p\mathbb{Z}$ from the notation. Using the vanishing of $H_{et}^3(X \setminus S)$, the étale excision sequence yields a commutative exact diagram

$$\begin{array}{ccccccc} & & H^2(G_{S'}(p)) & \xrightarrow{\sim} & H^2(G_S(p)) & & \\ & & \downarrow & & \downarrow \wr & & \\ H_{\mathfrak{p}}^2(X) & \hookrightarrow & H_{et}^2(X \setminus S') & \xrightarrow{\alpha} & H_{et}^2(X \setminus S) & \longrightarrow & H_{\mathfrak{p}}^3(X) \twoheadrightarrow H_{et}^3(X \setminus S'). \end{array}$$

Hence α is split-surjective and $\mathbb{Z}/p\mathbb{Z} \cong H_{\mathfrak{p}}^3(X) \xrightarrow{\sim} H_{et}^3(X \setminus S')$. This implies $S' = \emptyset$, hence $S = \{\mathfrak{p}\}$, and $\zeta_p \in k$. The same applies to every finite extension of k in $k_S(p)$, hence $\mathfrak{p}$ is inert in $k_S(p) = k_{\emptyset}(p)$. This implies that the natural homomorphism

$$\text{Gal}(k_{\mathfrak{p}}^{nr}(p)|k_{\mathfrak{p}}) \longrightarrow G_{\emptyset}(k)(p)$$

is surjective. Therefore $G_S(p) = G_\emptyset(p)$ is abelian, hence finite by class field theory. Since this group has finite cohomological dimension by the $K(\pi, 1)$ -property, it is trivial, in contradiction to our assumptions.

This shows that all $\mathfrak{p} \in S$ with $\zeta_p \in k_{\mathfrak{p}}$ ramify in $k_S(p)$. As this applies to every finite extension of k inside $k_S(p)$, the inertia groups must be infinite. For $\mathfrak{p} \in S_{\min} \setminus S_p$ this implies $k_S(p)_{\mathfrak{p}} = k_{\mathfrak{p}}(p)$. $\square$

Theorem 3.3. *Let k be a number field and let p be a prime number. Assume that k is totally imaginary if $p = 2$. Let S be a finite nonempty set of nonarchimedean primes of k . Assume that conditions (a)–(c) of Theorem 3.1 hold and that $\zeta_p \in k_{\mathfrak{p}}$ for all $\mathfrak{p} \in S$. Then $G_S(p)$ is a pro- p duality group of dimension 2.*

Proof. Condition (a) implies $H^3(G_S(p), \mathbb{Z}/p\mathbb{Z}) \xrightarrow{\sim} H_{et}^3(X \setminus S, \mathbb{Z}/p\mathbb{Z}) = 0$. Hence $\text{cd } G_S(p) \leq 2$. On the other hand, by (c), the group $G_S(p)$ contains $\text{Gal}(k_{\mathfrak{p}}(p)|k_{\mathfrak{p}})$ as a subgroup for all $\mathfrak{p} \in S$. As $\zeta_p \in k_{\mathfrak{p}}$ for $\mathfrak{p} \in S$, these local groups have cohomological dimension 2, hence so does $G_S(p)$.

In order to show that $G_S(p)$ is a duality group, we have to show that

$$D_i(G_S(p), \mathbb{Z}/p\mathbb{Z}) := \varinjlim_{\substack{U \subset G_S(p) \\ \text{cor}^\vee}} H^i(U, \mathbb{Z}/p\mathbb{Z})^\vee$$

vanish for $i = 0, 1$, where U runs through the open subgroups of $G_S(p)$ and the transition maps are the duals of the corestriction homomorphisms; see [NSW], 3.4.6. The vanishing of D_0 is obvious, as $G_S(p)$ is infinite. We therefore have to show that

$$\varinjlim_{K \subset k_S(p)} H^1((X \setminus S)_K, \mathbb{Z}/p\mathbb{Z})^\vee = 0.$$

We put $X = \text{Spec}(\mathcal{O}_k)$ and denote the embedding by $j : (X \setminus S)_K \rightarrow X_K$. By the flat duality theorem of Artin-Mazur, we have natural isomorphisms

$$H^1((X \setminus S)_K, \mathbb{Z}/p\mathbb{Z})^\vee \cong H_{\mathcal{A}, c}^2((X \setminus S)_K, \mu_p) = H_{\mathcal{A}}^2(X_K, j_! \mu_p).$$

The excision sequence together with a straightforward calculation of local cohomology groups shows an exact sequence

$$(*) \quad \bigoplus_{\mathfrak{p} \in S(K)} K_{\mathfrak{p}}^\times / K_{\mathfrak{p}}^{\times p} \rightarrow H_{\mathcal{A}}^2(X_K, j_! \mu_p) \rightarrow H_{\mathcal{A}}^2((X \setminus S)_K, \mu_p).$$

As $\zeta_p \in k_{\mathfrak{p}}$ and $k_S(p)_{\mathfrak{p}} = k_{\mathfrak{p}}(p)$ for $\mathfrak{p} \in S$ by assumption, the left hand term of $(*)$ vanishes when passing to the limit over all K . We use the Kummer sequence to obtain an exact sequence

$$(**) \quad \text{Pic}((X \setminus S)_K)/p \longrightarrow H_{\mathcal{A}}^2((X \setminus S)_K, \mu_p) \longrightarrow {}_p \text{Br}((X \setminus S)_K).$$

The left hand term of $(**)$ vanishes in the limit by the principal ideal theorem. The Hasse principle for the Brauer group induces an injection

$${}_p\mathrm{Br}((X \setminus S)_K) \hookrightarrow \bigoplus_{\mathfrak{p} \in S(K)} {}_p\mathrm{Br}(K_{\mathfrak{p}}).$$

As $k_S(p)$ realizes the maximal unramified p -extension of $k_{\mathfrak{p}}$ for $\mathfrak{p} \in S$, the limit of the middle term in $(**)$, and hence also the limit of the middle term in $(*)$ vanishes. This shows that $G_S(p)$ is a duality group of dimension 2. $\square$

Remark: The dualizing module can be calculated to

$$D \cong \mathrm{tor}_p(C_S(k_S(p))),$$

i.e. D is isomorphic to the p -torsion subgroup in the S -idèle class group of $k_S(p)$. The proof is the same as in ([Sch1], Proof of Thm. 5.2), where we dealt with the tame case.

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