

# Asymptotic behaviour and net force for the Navier-Stokes flows in exterior domains

Hi Jun Choe (Yonsei University, KOREA)  
Hideo Kozono (Tohoku University, JAPAN)

To the memory of Professor Tosio Kato

## 1 Introduction.

Let  $\Omega \subset \mathbb{R}^n (n \geq 3)$  be an *exterior* domain, i.e., a domain having a compact complement  $\mathbb{R}^n \setminus \Omega$  with the smooth boundary  $\partial\Omega$ . Consider the initial-boundary value problem of the Navier-Stokes equations in  $\Omega \times (0, \infty)$ :

$$(N-S) \quad \begin{cases} \frac{\partial u}{\partial t} - \Delta u + u \cdot \nabla u + \nabla p = 0 & \text{in } x \in \Omega, 0 < t < \infty, \\ \operatorname{div} u = 0 & \text{in } x \in \Omega, 0 < t < \infty, \\ u = 0 & \text{on } \partial\Omega, \quad u(x, t) \rightarrow 0 \text{ as } |x| \rightarrow \infty, \\ u|_{t=0} = a, \end{cases}$$

where  $u = u(x, t) = (u_1(x, t), \dots, u_n(x, t))$  and  $p = p(x, t)$  denote the unknown velocity vector and the pressure of the fluid at the point  $(x, t) \in \Omega \times (0, \infty)$ , while  $a = a(x) = (a_1(x), \dots, a_n(x))$  is the given initial velocity vector.

The global existence of strong solutions  $u$  to (N-S) for small data  $a$  had been investigated by many authors, Fujita-Kato [8], Solonnikov [25], Heywood [13], Giga-Miyakawa [11] and Kato [15]. In exterior domains, Iwashita [14] proved the most remarkable result together with the asymptotic behaviour. In [14], it turns out that for small  $a \in L^n(\Omega) \cap L^s(\Omega)$  with  $1 < s \leq n$  there exists a unique strong solution  $u$  with the following decay property

$$(1.1) \quad \begin{cases} \|u(t)\|_{L^r(\Omega)} = O(t^{-\frac{n}{2}(\frac{1}{s} - \frac{1}{r})}), & s \leq r \leq \infty, \\ \|\nabla u(t)\|_{L^r(\Omega)} = O(t^{-\frac{n}{2}(\frac{1}{s} - \frac{1}{r}) - \frac{1}{2}}), & s \leq r \leq n \end{cases}$$

as  $t \rightarrow \infty$ . The first purpose of this article is to consider whether or not it is possible to take  $s = 1$  in (1.1). Our problem is motivated by the fundamental question on the energy decay of solutions which was proposed by Leray [20]. For every  $a \in L^2(\Omega)$ , there exists at least one *weak* solution  $u$  to (N-S). In his famous paper [20], he had asked whether every weak solution does satisfy

$$\|u(t)\|_{L^2(\Omega)} \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

After 50 years of Leary's proposal, Masuda [21] and Kato [15] independently gave a positive answer to his question for all weak solutions  $u$  satisfying the energy inequality of the strong form. Then much effort had been made to obtain the decay rate of  $\|u(t)\|_{L^2(\Omega)}$  as  $t \rightarrow \infty$ . At the present, the best rate is given by Borchers-Miyakawa [3] who proved that if

$$\|e^{-tA}a\|_{L^2(\Omega)} = O(t^{-\alpha}) \quad \text{as } t \rightarrow \infty \quad (A; \text{ the Stokes operator}),$$

then there holds

$$\|u(t)\|_{L^2(\Omega)} = \begin{cases} O(t^{-\alpha}) & \text{for } 0 < \alpha \leq n/4, \\ O(t^{-\frac{n}{4}}) & \text{for } n/4 < \alpha < \infty \end{cases}$$

as  $t \rightarrow \infty$ . It should be noted that the decay rate  $t^{-n/4}$  can be obtained *formally* by taking  $r = 2$  and  $s = 1$  in (1.1). We shall show that if the initial data  $a \in L^1(\Omega) \cap L^n(\Omega)$  with certain regularity, then every strong solution  $u$  of (N-S) with (1.1) for  $s$  sufficiently close to 1 decays like

$$(1.2) \quad \|u(t)\|_{L^r(\Omega)} = O(t^{-\frac{n}{2}(1-\frac{1}{r})}) \quad \text{for all } 1 < r < \infty$$

as  $t \rightarrow \infty$ .

The second purpose of this article is to consider whether the above decay rate  $t^{-\frac{n}{2}(1-\frac{1}{r})}$  is optimal in the norm of  $L^r(\Omega)$ . In the whole space  $\mathbb{R}^n$ , Wiegner [28] showed that there exists a weak solution  $u$  such that

$$\|u(t)\|_{L^2(\mathbb{R}^n)} = O(t^{-\frac{n+2}{4}}) \quad \text{as } t \rightarrow \infty.$$

It was proven by Schonbek [23], [24] that this decay rate  $t^{-\frac{n+2}{4}}$  is optimal in  $L^2(\mathbb{R}^n)$ . In exterior domains  $\Omega$ , however, we shall show that the strong solution  $u$  decays like

$$(1.3) \quad \|u(t)\|_{L^r(\Omega)} = o(t^{-\frac{n}{2}(1-\frac{1}{r})}) \quad \text{for some } 1 < r < \infty$$

as  $t \rightarrow \infty$  if and only if

$$(1.4) \quad \int_0^\infty \int_{\partial\Omega} T[u, p](y, t) \cdot \nu dS_y dt = 0,$$

where  $T[u, p] = \{\partial u_i / \partial x_j + \partial u_j / \partial x_i - \delta_{ij} p\}_{i,j=1,\dots,n}$  denotes the stress tensor and  $\nu = (\nu_1, \dots, \nu_n)$  is the unit outward normal to  $\partial\Omega$ . This implies that the faster decay rate than  $t^{-\frac{n}{2}(1-\frac{1}{r})}$  in  $L^r(\Omega)$  of the velocity causes necessarily physical restriction on the net force exerted by the fluid to the obstacle. As a result, from a physical point of view, the decay rate like (1.2) seems to be optimal.

## 2 Results.

Before stating our results, we first introduce some function spaces. Let  $C_{0,\sigma}^\infty(\Omega)$  denote the set of all  $C^\infty$  vector functions  $\phi = (\phi_1, \dots, \phi_n)$  with compact support in  $\Omega$ , such that  $\operatorname{div} \phi = 0$ .  $L_\sigma^r(\Omega)$  is the closure of  $C_{0,\sigma}^\infty(\Omega)$  with respect to the  $L^r$ -norm  $\|\cdot\|_r \equiv \|\cdot\|_{L^r(\Omega)}$ ;  $(\cdot, \cdot)$  denotes the duality pairing between  $L^r(\Omega)$  and  $L^{r'}(\Omega)$ , where  $1/r + 1/r' = 1$ .  $L^r(\Omega)$  stands for the

usual (vector-valued)  $L^r$ -space over  $\Omega$ ,  $1 \leq r \leq \infty$ . It is known that for  $1 < r < \infty$ ,  $L_\sigma^r(\Omega)$  is characterized as

$$\begin{aligned} L_\sigma^r(\Omega) \\ = \{u \in L^r(\Omega); \operatorname{div} u = 0 \text{ in } \Omega, u \cdot \nu = 0 \text{ on } \partial\Omega \text{ in the sense } W^{1-1/r', r'}(\partial\Omega)^* \} \end{aligned}$$

and that there holds the Helmholtz decomposition

$$L^r(\Omega) = L_\sigma^r(\Omega) \oplus G^r(\Omega) \quad (\text{direct sum}), 1 < r < \infty,$$

where  $G^r(\Omega) = \{\nabla p \in L^r(\Omega); p \in L_{loc}^r(\bar{\Omega})\}$ . We denote by  $P_r$  the projection operator from  $L^r(\Omega)$  onto  $L_\sigma^r(\Omega)$  along  $G^r(\Omega)$ . Then the Stokes operator  $A_r$  is defined by  $A_r = -P_r \Delta$  with the domain  $D(A_r) = \{u \in W^{2,r}(\Omega) \cap L_\sigma^r(\Omega); u|_{\partial\Omega} = 0\}$ . It is proved by Giga [10] and Giga-Sohr [12] that  $-A_r$  generates a uniformly bounded holomorphic semigroup  $\{e^{-tA_r}\}_{t \geq 0}$  of class  $C_0$  in  $L_\sigma^r(\Omega)$  for  $1 < r < \infty$ . Hence one can define the fractional power  $A_r^\alpha$  for  $0 \leq \alpha \leq 1$ .

The class of solutions which we consider is as follows.

**Definition.** Let  $1 < s \leq n$  and let  $a \in L_\sigma^s(\Omega) \cap L_\sigma^n(\Omega)$ . A measurable function  $u$  on  $\Omega \times (0, \infty)$  is called a *strong solution* of (N-S) in the class  $CL_s(0, \infty)$  if

- (i)  $u \in C([0, \infty); L_\sigma^s(\Omega) \cap L_\sigma^n(\Omega));$
- (ii)  $Au, \partial u / \partial t \in C((0, \infty); L_\sigma^n(\Omega));$
- (iii)

$$(N-S') \quad \begin{cases} \frac{\partial u}{\partial t} + Au + P(u \cdot \nabla u) = 0 & \text{in } L_\sigma^n(\Omega), 0 < t < \infty, \\ u(0) = a, \end{cases}$$

**Remarks.** 1. It was shown by Kato [15] and Iwashita [14] that for  $1 < s \leq n$  there is a constant  $\lambda(s, n)$  such that for every  $a \in L_\sigma^s(\Omega) \cap L_\sigma^n(\Omega)$  with  $\|a\|_n \leq \lambda$ , there exists a unique strong solution  $u$  of (N-S) in the class  $CL_s(0, \infty)$ . Moreover, such a solution satisfies (1.1).

2. Every strong solution  $u$  in the class  $CL_s(0, \infty)$  satisfies (N-S') also in  $L_\sigma^s(\Omega)$  and there holds

$$\frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}, \quad \frac{\partial u}{\partial t} \in C(\bar{\Omega} \times (0, \infty))$$

for all multi-indices  $\alpha = (\alpha_1, \dots, \alpha_n)$ , where  $|\alpha| = \alpha_1 + \dots + \alpha_n$ . Moreover, there exists a unique (up to an additive function of  $t$ ) scalar function  $p \in C^1(\Omega \times (0, \infty))$  with

$$(2.1) \quad \nabla p \in C((0, \infty); L^s(\Omega) \cap L^n(\Omega))$$

such that the pair  $\{u, p\}$  satisfies (N-S) in the classical sense. We call such  $p$  the pressure associated with  $u$ .

3. If  $1 < s < n$ , by (2.1) and the Sobolev embedding ([12, Corollary 2.2]), we may take  $p$  as  $p \in C((0, \infty); L^{ns/(n-s)}(\Omega))$ .

Throughout this paper, we impose the following assumption on the initial data.

**Assumption.** For some  $\frac{n}{n-2} < q_* < \infty$  and  $\varepsilon > 0$  the initial data  $a$  satisfies

$$a \in L^1(\Omega) \cap L_\sigma^n(\Omega) \cap D(A_{q_*}^\varepsilon).$$

Our first result on the decay property of strong solutions now reads:

**Theorem 1.** *Let  $a$  be as in the Assumption. Suppose that  $u$  is the strong solution of (N-S) in the class  $CL_s(0, \infty)$  with (1.1) for  $1 < s < \min\{\frac{n}{n-1}, \frac{2n}{n+2}\}$ . Then  $u(t)$  decays like*

$$(2.2) \quad \|u(t)\|_r = O(t^{-\frac{n}{2}(1-\frac{1}{r})}) \quad \text{for all } 1 < r < \infty.$$

as  $t \rightarrow \infty$

**Remarks.** 1. Iwashita [14] showed the existence of the strong solution  $u$  in the class  $CL_s(0, \infty)$  with (1.1) for  $a \in L_\sigma^s(\Omega) \cap L_\sigma^n(\Omega)$  with  $1 < s \leq n$  provided  $\|a\|_n$  is small. Concerning the linear Stokes flows for  $s = 1$ , the author [19] proved

$$(2.3) \quad \|e^{-tA}a\|_r \leq Ct^{-\frac{n}{2}(1-\frac{1}{r})}(\|a\|_1 + \|a\|_{q_*} + \|A^\varepsilon a\|_{q_*}), \quad 1 < r \leq \infty,$$

$$(2.4) \quad \|\nabla e^{-tA}a\|_r \leq Ct^{-\frac{n}{2}(1-\frac{1}{r})-\frac{1}{2}}(\|a\|_1 + \|a\|_{q_*} + \|A^\varepsilon a\|_{q_*}), \quad 1 \leq r \leq n,$$

for all  $t > 1$  and for all  $a$  as in the Assumption.

2. In (2.2), we do not know whether  $r = 1$  is possible; the author [18] showed that  $u \in C([0, \infty); L^1(\Omega))$  with its associated pressure  $p \in C((0, \infty); L^{\frac{n}{n-1}}(\Omega))$  if and only if there holds

$$(2.5) \quad \int_{\partial\Omega} T[u, p](y, t) \cdot \nu dS_y = 0, \quad \text{for all } 0 < t < \infty,$$

where  $T[u, p] = \{\partial u_i / \partial x_j + \partial u_j / \partial x_i - \delta_{ij} p\}_{i,j=1,\dots,n}$  denotes the stress tensor and  $\nu = (\nu_1, \dots, \nu_n)$  is the unit outward normal to  $\partial\Omega$ . Hence, it seems to be difficult to take  $r = 1$  in (2.2) for all  $a$  satisfying the Assumption.

We next investigate the faster decay than (2.2):

**Theorem 2.** *Let  $a$  be as in the Assumption. Suppose that  $u$  is the strong solution as in Theorem 1. If*

$$(2.6) \quad \|u(t)\|_r = o(t^{-\frac{n}{2}(1-\frac{1}{r})}) \quad \text{for some } 1 < r \leq \infty$$

as  $t \rightarrow \infty$ , then there holds

$$(2.7) \quad \int_0^\infty dt \int_{\partial\Omega} T[u, p](y, t) \cdot \nu dS_y = 0.$$

Conversely, if (2.7) holds, then we have

$$(2.8) \quad \|u(t)\|_r = o(t^{-\frac{n}{2}(1-\frac{1}{r})}) \quad \text{for all } 1 < r \leq \infty$$

as  $t \rightarrow \infty$ .

**Remarks.** 1. In case  $\Omega = \mathbb{R}^n$ , the situation is quite different. Wiegner [28] showed existence of a weak solution  $u$  of (N-S) with the property that

$$\|u(t)\|_{L^2(\mathbb{R}^n)} = O(t^{-\frac{n}{4}-\frac{1}{2}}) \quad \text{as } t \rightarrow \infty.$$

Schonbek [23], [24] and Miyakawa-Schonbek [22] proved that there exist an initial data  $a \in L^1(\mathbb{R}^n) \cap L^2_\sigma(\mathbb{R}^n)$  and a weak solution  $u$  of (N-S) such that

$$\|u(t)\|_{L^2(\mathbb{R}^n)} \geq Ct^{-\frac{n}{4}-\frac{1}{2}} \quad \text{for large } t.$$

Fujigaki-Miyakawa [6] proved that there exist an initial data  $a \in L^1(\mathbb{R}^n) \cap L^2_\sigma(\mathbb{R}^n)$  and a strong solution  $u$  of (N-S) such that

$$\|u(t)\|_{L^r(\mathbb{R}^n)} \geq Ct^{-\frac{n}{2}(1-\frac{1}{r})-\frac{1}{2}} \quad \text{for large } t.$$

2. In case  $\Omega = \mathbb{R}^n_+$  (half space), based on the Ukai's formula [27] for  $e^{-tA}a$ , faster decay rates than in  $\mathbb{R}^n$  were obtained by Bae-Choe [1], Bae [2] and Fujigaki-Miyakawa [7].

3. The net force plays an important role also for the *spacial* decay at infinity of the solutions to the *stationary problem* in  $\Omega \subset \mathbb{R}^3$ :

$$(E) \quad \begin{cases} -\Delta w + w \cdot \nabla w + \nabla p = \operatorname{div} F, & \text{in } x \in \Omega \\ \operatorname{div} w = 0 & \text{in } x \in \Omega, \quad \text{in } x \in \Omega \\ w = 0 & \text{on } \partial\Omega, \quad w(x) \rightarrow w^\infty \quad \text{as } |x| \rightarrow \infty, \end{cases}$$

where  $F = F(x) = \{F_{ij}(x)\}_{i,j=1,2,3}$  denotes the given  $3 \times 3$  tensor, while  $w^\infty = (w_1^\infty, w_2^\infty, w_3^\infty)$  is the prescribed constant vector in  $\mathbb{R}^3$ . Finn [4], [5] treated the case when  $F \equiv 0$ ,  $w^\infty \neq 0$ . Introducing the notion of *physically reasonable solution*  $w$  of (E), i.e.,

$$|w(x) - w^\infty| = O(|x|^{-\frac{1}{2}-\varepsilon}) \quad (\varepsilon > 0) \quad \text{as } |x| \rightarrow \infty,$$

he proved that

$$|w(x) - w^\infty| = o(|x|^{-1}) \quad \text{as } |x| \rightarrow \infty$$

if and only if there holds

$$\int_{\partial\Omega} T[w, p](y) \cdot \nu dS_y = 0.$$

Kozono-Sohr-Yamazaki [17] considered the case when  $F \neq 0$ ,  $w^\infty = 0$ . They dealt with the  $D$ -solution  $w$ , i.e.,  $\int_{\Omega} |\nabla w(x)|^2 dx < \infty$  and showed that  $w \in L^3(\Omega)$  if and only if

$$\int_{\partial\Omega} (T[w, p](y) + F(y)) \cdot \nu dS_y = 0.$$

### 3 Outline of the proof of the theorems.

In this section, we shall give a sketch of the proof of Theorems 1 and 2. Let us first recall the fundamental tensor  $\{E_{ij}(x, t)\}_{i,j=1,\dots,n}$  to the linear Stokes system defined by

$$E_{ij}(x, t) = \Gamma(x, t)\delta_{ij} + \frac{\partial^2}{\partial x_i \partial x_j}(\Gamma(\cdot, t) * G)(x), \quad i, j = 1, \dots, n,$$

where

$$\Gamma(x, t) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}}, \quad G(x) = \frac{1}{n(n-2)\omega_n} |x|^{2-n}, \quad \omega_n = \text{vol.}(S^{n-1}).$$

We have the following representation formula of the strong solution.

**Lemma 3.1 (Representation formula)** *Let  $a$  be as in the Assumption. The strong solution  $u(t)$  to (N-S) in the class  $CL_s(0, \infty)$  for  $1 < s \leq n$  can be represented as*

$$\begin{aligned} u_i(x, t) &= \int_{\Omega} \Gamma(x - y, t) a_i(y) dy \\ &\quad + \int_0^t d\tau \int_{\partial\Omega} \sum_{j,k=1}^n E_{ij}(x - y, t - \tau) T_{jk}[u, p](y, \tau) \nu_k(y) dS_y \\ &\quad + \int_0^t d\tau \int_{\Omega} \sum_{j,k=1}^n \frac{\partial}{\partial y_k} E_{ij}(x - y, t - \tau) u_k \cdot u_j(y, \tau) dy \\ (3.1) \quad &\equiv U_i(x, t) + V_i(x, t) + W_i(x, t), \quad i = 1, \dots, n \end{aligned}$$

for all  $(x, t) \in \Omega \times (0, \infty)$ .

To make use of this representation formula, we need to investigate behaviour of the boundary integral

$$\int_{\partial\Omega} (|\nabla u(y, t)| + |p(y, t)|) dS_y \quad \text{for all } t \in (0, \infty).$$

**Lemma 3.2** *Let  $a$  be as in the Assumption. Let  $q \equiv nq_*/(n + q_*)$ .*

(i) *Every strong solution  $u$  of (N-S) in the class  $CL_s(0, \infty)$  for  $1 < s \leq n$  and its associated pressure  $p$  satisfy*

$$\int_{\partial\Omega} (|\nabla u(y, t)| + |p(y, t)|) dS_y \leq Ct^{\alpha-1} \quad \text{for all } 0 < t \leq 1$$

with  $\alpha \equiv (\frac{1-1/q}{1-1/q_*})\varepsilon$ , where  $C = C(n, q_*, \varepsilon)$ .

(ii) *Let  $u$  be a strong solution of (N-S) in the class  $CL_s(0, \infty)$  for  $1 < s < \min\{\frac{n}{n-1}, \frac{2n}{n+2}\}$  with the decay property (1.1). For every  $l$  with  $1 < s \leq l < n$ ,  $u$  and its associated pressure  $p$  are subject to the estimate*

$$\int_{\partial\Omega} (|\nabla u(y, t)| + |p(y, t)|) dS_y \leq Ct^{-\frac{n}{2}(\frac{1}{s}-\frac{1}{l})-\frac{1}{2}} \quad \text{for all } 1 < t < \infty,$$

where  $C = C(n, s, l)$ .

For the proof we need the trace theorem and the following estimate by Kozono-Ogawa [16]

$$\|\nabla^2 u\|_s \leq C(\|Au\|_s + \|\nabla u\|_s), \quad 1 < s < \infty$$

for all  $u \in D(A_s)$  together with the decay property

$$\|Au(t)\|_l = O(t^{-\frac{n}{2}(\frac{1}{s}-\frac{1}{l})-1}), \quad s \leq l < \infty, \quad \text{as } t \rightarrow \infty.$$

*Proof of Theorem 1:*

By Lemma 3.1, we may estimate  $U(t)$ ,  $V(t)$  and  $W(t)$  in  $L^r$ , respectively. First, let us consider the case  $1 < r < n/(n-1)$ . Recall

$$U_i(x, t) = \int_{\Omega} \Gamma(x - y, t) a_i(y) dy, \quad i = 1, \dots, n$$

Since

$$\int_{\Omega} |a(x)| dx < \infty \quad \text{with } \operatorname{div} a = 0 \text{ in } \Omega, \quad a \cdot \nu = 0 \text{ on } \partial\Omega,$$

there holds

$$\int_{\Omega} a_i(y) dy = 0, \quad i = 1, \dots, n.$$

Hence we have by the Hausdorff-Young inequality that

$$(3.2) \quad \|U(t)\|_r = o(t^{-\frac{n}{2}(1-\frac{1}{r})}) \quad \text{as } t \rightarrow \infty.$$

To deal with

$$V_i(x, t) = \sum_{j,k=1}^n \int_0^t d\tau \int_{\partial\Omega} E_{ij}(x - y, t - \tau) T_{jk}[u, p](y, \tau) \nu_k(y) dS_y, \quad i = 1, \dots, n,$$

we need to notice that  $\{E_{ij}\}_{i,j=1,\dots,n}$  can be expressed as

$$(3.3) \quad E_{ij}(\cdot, t) = (\delta_{ij} + R_i R_j) \Gamma(\cdot, t), \quad i, j = 1, \dots, n,$$

where  $R_i = \frac{\partial}{\partial x_i} (-\Delta)^{-\frac{1}{2}}$ ,  $i = 1, \dots, n$  denote the Riesz transforms. Since  $R_i : L^r(\mathbb{R}^n) \rightarrow L^r(\mathbb{R}^n)$  is bounded, we have

$$(3.4) \quad \|\partial_x^m \partial_t^k E_{ij}(\cdot, t)\|_r \leq C t^{-\frac{n}{2}(1-\frac{1}{r})-\frac{m}{2}-k}, \quad m, k = 0, 1, \forall t > 0,$$

which yields

$$\begin{aligned} \|V(t)\|_r &\leq \sum_{i,j,k=1}^n \int_0^t d\tau \int_{\partial\Omega} \|E_{ij}(\cdot - y, t - \tau) T_{jk}[u, p](y, \tau) \nu_k(y)\|_r dS_y \\ &\leq \sum_{i,j,k=1}^n \int_0^t d\tau \int_{\partial\Omega} |T_{jk}[u, p](y, \tau) \nu_k(y)| \|E_{ij}(\cdot - y, t - \tau)\|_r dS_y \\ (3.5) \quad &\leq C \int_0^t (t - \tau)^{-\frac{n}{2}(1-\frac{1}{r})} \left( \int_{\partial\Omega} (|\nabla u(y, \tau)| + |p(y, \tau)|) dS_y \right) d\tau. \end{aligned}$$

Applying Lemma 3.2 to the estimate of the R.H.S., we obtain

$$(3.6) \quad \|V(t)\|_r = O(t^{-\frac{n}{2}(1-\frac{1}{r})}) \quad \text{as } t \rightarrow \infty.$$

Finally, we treat the third term

$$W_i(x, t) = \int_0^t d\tau \int_{\Omega} \sum_{j,k=1}^n \frac{\partial}{\partial y_k} E_{ij}(x-y, t-\tau) u_k \cdot u_j(y, \tau) dy, \quad i = 1, \dots, n$$

By (3.4) and the Housdorff-Young inequality, we have

$$\begin{aligned} \|W(t)\|_r &\leq \int_0^t \|\nabla E(\cdot, t-\tau)\|_r \|u \otimes u(\tau)\|_1 d\tau \\ &\leq C \int_0^t (t-\tau)^{-\frac{n}{2}(1-\frac{1}{r})-\frac{1}{2}} \|u(\tau)\|_2^2 d\tau. \end{aligned}$$

Since  $\|u(t)\|_2 \leq Ct^{-\frac{n}{2}(\frac{1}{s}-\frac{1}{2})}$  (see (1.1)), we obtain from the above estimate

$$(3.7) \quad \|W(t)\|_r = O(t^{-\frac{n}{2}(1-\frac{1}{r})-\frac{1}{2}}) \quad \text{as } t \rightarrow \infty.$$

Notice that  $-\frac{n}{2}(1-\frac{1}{r})-\frac{1}{2} > -1 \iff r < n/(n-1)$ . Then by (3.2), (3.6) and (3.7), we have the desired decay for  $\|u(t)\|_r$  provided  $1 < r < n/(n-1)$ .

In case  $n/(n-1) \leq r < \infty$ , some skilful technique by duality is necessary. Here we omit the detail. This proves Theorem 1.

*Proof of Theorem 2:*

Without loss of generality, we may assume that

$$(3.8) \quad \|u(t)\|_r = o(t^{-\frac{n}{2}(1-\frac{1}{r})}) \quad \text{for some } r \text{ with } 1 < r < n/(n-1).$$

as  $t \rightarrow \infty$ . Indeed, if (2.6) holds for some  $n/(n-1) \leq r \leq \infty$ , then by choosing  $1 < r_0 < r_1 < n/(n-1)$  and  $0 < \theta < 1$  with  $1/r_1 = (1-\theta)/r_0 + \theta/r$ , we have

$$\begin{aligned} \|u(t)\|_{r_1} &\leq \|u(t)\|_{r_0}^{1-\theta} \|u(t)\|_r^\theta \\ &= O(t^{-\frac{n}{2}(1-\frac{1}{r_0})(1-\theta)}) \cdot o(t^{-\frac{n}{2}(1-\frac{1}{r})\theta}) \\ &= o(t^{-\frac{n}{2}(1-\frac{1}{r_1})}) \end{aligned}$$

as  $t \rightarrow \infty$ , which yields (3.8).

By Lemma 3.1, we have similarly to (3.1) that

$$\begin{aligned} (3.9) \quad u_i(x, t) &= U_i(x, t) + \tilde{V}_i(x, t) + W_i(x, t) + \sum_{j,k=1}^n E_{ij}(x, t) \int_0^t d\tau \int_{\partial\Omega} T_{jk}[u, p](y, \tau) \nu_k(y) dS_y, \\ &\quad i = 1, \dots, n, \end{aligned}$$

for all  $(x, t) \in \Omega \times (0, \infty)$ , where

$$\begin{aligned} U_i(x, t) &= \int_{\Omega} \Gamma(x - y, t) a_i(y) dy, \\ \tilde{V}_i(x, t) &= \sum_{j,k=1}^n \int_0^t d\tau \int_{\partial\Omega} \{E_{ij}(x - y, t - \tau) - E_{ij}(x, t)\} T_{jk}[u, p](y, \tau) \nu_k(y) dS_y, \\ W_i(x, t) &= \int_0^t d\tau \int_{\Omega} \sum_{j,k=1}^n \frac{\partial}{\partial y_k} E_{ij}(x - y, t - \tau) u_k \cdot u_j(y, \tau) dy \end{aligned}$$

for  $i = 1, \dots, n$ . Since  $1 < r < n/(n-1)$ , we have by (3.2) and (3.7) that

$$(3.10) \quad \|U(t)\|_r = o(t^{-\frac{n}{2}(1-\frac{1}{r})}), \quad \|W(t)\|_r = O(t^{-\frac{n}{2}(1-\frac{1}{r})-\frac{1}{2}})$$

as  $t \rightarrow \infty$ . Using the expression

$$\begin{aligned} &\tilde{V}_i(x, t) \\ &= \sum_{j,k=1}^n \int_0^t d\tau \int_{\partial\Omega} \left( \int_0^1 \frac{d}{d\theta} E_{ij}(x - \theta y, t - \theta\tau) d\theta \right) T_{jk}[u, p](y, \tau) \nu_k(y) dS_y \\ &= \sum_{j,k=1}^n \int_0^t d\tau \int_{\partial\Omega} \left( \int_0^1 \nabla E_{ij}(x - \theta y, t - \theta\tau) \cdot (-y) d\theta \right) T_{jk}[u, p](y, \tau) \nu_k(y) dS_y \\ &\quad + \sum_{j,k=1}^n \int_0^t d\tau \int_{\partial\Omega} \left( \int_0^1 \partial_t E_{ij}(x - \theta y, t - \theta\tau) (-\tau) d\theta \right) T_{jk}[u, p](y, \tau) \nu_k(y) dS_y, \end{aligned}$$

we can show, with the aid of some technical calculation, that

$$(3.11) \quad \|\tilde{V}(t)\|_r = o(t^{-\frac{n}{2}(1-\frac{1}{r})}) \quad \text{as } t \rightarrow \infty.$$

On the other hand, there holds

$$\begin{aligned} &\liminf_{t \rightarrow \infty} t^{\frac{n}{2}(1-\frac{1}{r})} \left\| \sum_{j=1}^n E_{ij}(\cdot, t) \int_0^t f_j(\tau) d\tau \right\|_r \\ (3.12) \quad &\geq \left( \int_{y \in \mathbb{R}^n} \left| \sum_{j=1}^n E_{ij}(y, 1) \int_0^\infty f_j(\tau) d\tau \right|^r dy \right)^{\frac{1}{r}}, \quad i = 1, \dots, n \end{aligned}$$

where

$$f_j(\tau) = \int_{\partial\Omega} \sum_{k=1}^n T_{jk}[u, p](y, \tau) \nu_k(y) dS_y, \quad j = 1, \dots, n.$$

Now, assume that

$$\|u(t)\|_r = o(t^{-\frac{n}{2}(1-\frac{1}{r})}) \quad \text{as } t \rightarrow \infty.$$

Then it follows from (3.10), (3.11) and (3.12) that

$$(3.13) \quad \sum_{j=1}^n E_{ij}(y, 1) \int_0^\infty f_j(\tau) d\tau = 0, \quad i = 1, \dots, n, \quad \text{for all } y \in \mathbb{R}^n.$$

Since  $\widehat{E}_{ij}(\xi, 1) = \left( \delta_{ij} - \frac{\xi_i \xi_j}{|\xi|^2} \right) e^{-|\xi|^2}$ ,  $i, j = 1, \dots, n$ , we have by (3.13) that

$$\sum_{j=1}^n (\delta_{ij} - \omega_i \omega_j) \int_0^\infty f_j(\tau) d\tau = 0, \quad i = 1, \dots, n$$

for all  $\omega = (\omega_1, \dots, \omega_n) \in \mathbb{R}^n$  with  $|\omega| = 1$ . Obviously, we conclude that

$$\int_0^\infty f_1(\tau) d\tau = \dots = \int_0^\infty f_n(\tau) d\tau = 0,$$

which implies

$$\int_0^\infty d\tau \int_{\partial\Omega} \sum_{k=1}^n T_{jk}[u, p](y, \tau) \nu_k(y) dS_y = 0, \quad j = 1, \dots, n.$$

This shows (2.7).

Conversely, if (2.7) holds, then we have by (3.9), (3.10) and (3.11) that

$$\begin{aligned} \|u(t)\|_r &\leq \|U(t)\|_r + \|\tilde{V}(t)\|_r + \|W(t)\|_r \\ &\quad + \sum_{i,j=1}^n \|E_{ij}(\cdot, t)\|_r \left| \int_0^t f_j(\tau) d\tau \right| \\ &= o(t^{-\frac{n}{2}(1-\frac{1}{r})}) \end{aligned}$$

for all  $1 < r < n/(n-1)$  as  $t \rightarrow \infty$ . By the same technique as before, we get (2.8). This proves Theorem 2.

## References

- [1] Bae, H.O., Choe, H. J., *Decay rate for the incompressible flows in half space*. to appear in Math. Z.
- [2] Bae, H.O., *Decays in  $L^1$  and  $L^\infty$  spaces for the Stokes flows in half spaces*. Preprint
- [3] Borchers, W., Miyakawa, T., *Algebraic  $L^2$  decay for Navier-Stokes flows in exterior domains. II*. Hiroshima Math. J. **21**, 621–640 (1991).
- [4] Finn, R., *Estimates at infinity for stationary solutions of the Navier-Stokes equations*. Bull. Math. Soc. Sci. Math. Phys. Répub. Populaire Roumaine **3**, 387–418 (1959).

- [5] Finn, R., *On exterior stationary problem for the Navier-Stokes equations and associated perturbation problems*. Arch. Rational Mech. Anal. **19**, 363–404 (1964).
- [6] Fujigaki, Y., Miyakawa, T., *Asymptotic profiles of nonstationary incompressible Navier-Stokes flows in  $\mathbb{R}^n$* . Preprint.
- [7] Fujigaki, Y., Miyakawa, T., *Asymptotic profiles of nonstationary incompressible Navier-Stokes flows in the half-space*. Preprint.
- [8] Fujita, H., Kato, T.: *On the Navier-Stokes initial value problem 1*. Arch. Rational Mech. Anal. **16**, 269–315 (1964).
- [9] Fujiwara, D., Morimoto, H., *An  $L_r$  theorem of the Helmholtz decomposition of vector fields*. J. Fac. Sci. Univ. Tokyo, Sec. IA **24**, 685–700 (1977).
- [10] Giga, Y., *Analyticity of the semigroup generated by the Stokes operator in  $L_r$  spaces*. Math. Z. **178**, 297–329 (1981).
- [11] Giga, Y., Miyakawa, T., *Solution in  $L_r$  of the Navier-Stokes initial value problem*. Arch. Rational Mech. Anal. **89**, 267–281 (1985).
- [12] Giga, Y., Sohr, H., *On the Stokes operator in exterior domains*. J. Fac. Sci. Univ. Tokyo, Sec. IA **36**, 103–130 (1989).
- [13] Heywood, J.G., *The Navier-Stokes equations: on the existence, regularity and decay of solutions*. Indiana Univ. Math. J. **29** (1980), 639–681.
- [14] Iwashita, H.,  *$L^q - L^r$  estimates for solutions of non-stationary Stokes equations in exterior domains and the Navier-Stokes initial value problems in  $L^q$  spaces*. Math. Ann. **285**, 265–288 (1989).
- [15] Kato, T., *Strong  $L^p$ -solution of the Navier-Stokes equation in  $\mathbb{R}^m$ , with applications to weak solutions*. Math. Z. **187**, 471–480 (1984).
- [16] Kozono, H., Ogawa, T., *Some  $L^p$  estimate for the exterior Stokes flow and an application to the nonstationary Navier-Stokes equations*. Indiana Univ. Math. J. **41**, 789–808 (1992).
- [17] Kozono, H., Sohr, H., Yamazaki, M., *Representation formula, net force and energy relation to the stationary Navier-Stokes equations in 3-dimensional exterior domains*. Kyushu J. Math. **51**, 239–260 (1997).
- [18] Kozono, H.,  *$L^1$ -solutions of the Navier-Stokes equations in exterior domains*. Math. Ann. **312** (1998), 319–340.
- [19] Kozono, H., *Rapid time-decay and net force to the obstacles by the Stokes flows in exterior domains*. Math. Ann. **320** (2001), 709–730.
- [20] Leray, J., *Sur le mouvement d'un liquide visqueux emplissant l'espace*. Acta. Math. **63**, 193–248 (1994).

- [21] Masuda, K., *Weak solutions of the Navier-Stokes equations*. Tohoku Math. J. **36**, 623–646 (1984).
- [22] Miyakawa, T., Schonbek, M.E., *On optimal decay rates for weak solutions to the Navier-Stokes equations in  $\mathbb{R}^n$* . Preprint.
- [23] Schonbek, M.E., *Lower bounds of rates of decay for solutions to the Navier-Stokes equations*. J. Amer. Math. Soc. **4**, 423–449 (1991).
- [24] Schonbek, M.E., *Asymptotic behavior of solutions to the three-dimensional Navier-Stokes equations*. Indiana Univ. Math. J. **41**, 809–823 (1992).
- [25] Solonnikov, V.A., *Estimates of the solutions of a nonstationary linearized system of the Navier-Stokes equations*. Amer. Math. Soc. Transl. **75**, 1–116 (1968).
- [26] Solonnikov, V.A., *Estimates for the solutions of a nonstationary Navier-Stokes equations*. J. Soviet Math. **8**, 1–116 (1977).
- [27] Ukai, S., *A solution formula for the Stokes equation in  $R_+^n$* . Comm. Pure Appl. Math. **40** (1987), 611–621.
- [28] Wiegner, M., *Decay results for weak solutions of the Navier-Stokes equations in  $\mathbb{R}^n$* . J. London Math. Soc.(2) **35**, 303–313 (1987).