

Chaotic Origin of Tsirelson Bound

Ken Umeno

Graduate School of Informatics, Kyoto University

**E-mail: umeno.ken.8z@kyoto.ac.jp*

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 We report a rigorous, deterministic origin for quantum non-locality and the Tsirelson bound ($2\sqrt{2}$) by revealing profound macroscopic emergent properties hidden within exactly solvable chaos towers under a strictly fixed invariant Cauchy measure ($\gamma = 1$). While evaluating joint correlations via conventional real-axis micro-sampling is universally imprisoned within a classical L_1 metric space—where the lack of exponential mode-damping ($e^0 = 1$) forces an infinite cascade of high-harmonic square-wave noises to remain unattenuated, collapsing the CHSH sum to the classical limit ($|S| \leq 1.5 < 2$)—we demonstrate that this fundamental restriction is completely bypassed at the macroscopic layer. By utilizing unimodular real-axis observation functions satisfying $|g_A(x)| = 1$, the physical joint correlation is strictly bounded by unity ($|\mathcal{E}| \leq 1$), naturally preserving physical unitariness. By generalizing these observations as expansions over the complete orthonormal Cauchy-Szegő basis $e_n(x) = \left(\frac{x-i}{x+i}\right)^n$ of the Hardy space $H^2(\mathbb{H}^+)$, we establish a rigorous signal characterization of emergent non-locality. While real-axis discrete point-sampling retains all odd harmonic modes, the continuous parameter tracking of the ergodic ensemble acts as a topological Szegő projection. Under this projection, the residue theorem filters out high-order complex poles ($n \geq 2$), isolating the fundamental mode $n = 1$ and instantaneously recovering the smooth quantum correlation $\cos(2\Delta\theta)$. Without invoking any quantum-mechanical postulates, this geometric phase transition transforms the optimization of the CHSH sum into a constraint of geodesic quadrilaterals within an emergent hyperbolic bulk, providing a novel realization of the Anti-de Sitter / Conformal Field Theory (AdS/CFT) correspondence and the Ryu-Takayanagi holographic paradigm. Crucially, by deriving the Tsirelson limit as a purely geometric consequence of tracking statistical metadata over the complex topology, this framework demands no superluminal signal propagation across spacetime, thereby harmonizing emergent quantum non-locality with strictly local relativistic causality.

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1 Introduction

The violation of the Bell and Clauser-Horne-Shimony-Holt (CHSH) inequalities stands as one of the most profound hallmarks of quantum mechanics, fundamentally separating the quantum universe from any local-realistic descriptions [1, 2]. While local hidden-variable theories strictly restrict the CHSH correlation measure S to the classical bound $|S| \leq 2$, quantum mechanics permits a maximum violation up to $2\sqrt{2}$, a fundamental ceiling discovered by Tsirelson [3]. Conventionally, the Tsirelson bound is understood as an intrinsic, rigid property embedded within the non-commutative algebraic structure of operator algebras or the geometric convex structures of quantum state spaces. However, a foundational question deeply persists: Why is the absolute ceiling of non-local correlations capped precisely at $2\sqrt{2}$, rather than the non-signaling algebraic maximum of 4 [4]? What is the underlying dynamical or information-theoretic infrastructure that dictates this exact limit without invoking quantum mechanics a priori?

Recently, a paradigm-shifting breakthrough has been established through the framework of the "Emergence of Maximal Quantum Chaos from Exactly Solvable Chaos Towers" [5], which demonstrates that physical spacetime geometry, non-vanishing quantum commutators, and fundamental measurement limitations can spontaneously emerge from a local deterministic infrastructure. By rigorously analyzing exactly solvable chaos towers spanned by multiple-angle formulas of $\tan$ and $\cot$, it was proven [5] that the universal Information Preservation Law governed by the Perron-Frobenius operator uniquely drives the boundary dynamical system to saturate the Maldacena-Shenker-Stanford (MSS) bound [6] as a quantized value

$$\lambda_L = \ln K \quad \text{for} \quad K = 2, 3, \dots \quad (1)$$

This microscopic maximization of information scrambling forces the macroscopic scale-parameter space of the resulting Cauchy invariant measures to form a statistical manifold endowed with a constant negative Fisher curvature $\kappa = -2$ [7]. This emergent hyperbolic space acts precisely as the anti-de Sitter (AdS) bulk, from which Ozawa's extended error-disturbance uncertainty relation [8] is naturally derived as a pure geometric consequence of the hyperbolic triangle inequality under the Cramér-Rao limit.

While the single-particle uncertainty relation is elegantly decoded via the underlying bulk geometry, the true essence of the holographic principle lies in multipartite correlation—the holographic entanglement that glues the spacetime bulk together [11]. To formulate this multipartite correlation and bridge the conceptual gap between the boundary dynamics and the emergent bulk, it is essential to construct a universal complex interface along the real axis. We define a unimodular complex observation function as $g_A(x) = \frac{x+i}{x-i}e^{i\theta_A}$. This

algebraic construction shares a unified mathematical infrastructure with the holographic spacetime emergence paradigm. Specifically, at the baseline orientation $\theta_A = \pi$, the complex observation function identically reduces to the fundamental form:

$$g_A(x)\Big|_{\theta_A=\pi} = -\frac{x+i}{x-i} \equiv Q(x), \quad (2)$$

where $Q(x)$ serves as the core operator driving the holographic emergence of the AdS bulk geometry and non-vanishing quantum commutators from the exactly solvable chaos towers. Therefore, the Cayley-like Möbius transformation from the real axis to the complex unitary circle S^1 acts as a universal common interface that bridges quantum non-locality and holographic quantum gravity, mapping hidden correlations that remain inaccessible within the conventional real-axis metric space. Furthermore, this dynamical mechanism exhibits strict universality for any general partition scale $K \geq 2$. When the Kolmogorov-Sinai entropy and the Lyapunov exponent strictly saturate the Maldacena-Shenker-Stanford (MSS) bound ($\lambda_L = \ln K$) within the cot and tan families of exactly solvable chaos, the geometric upper bound of the statistical correlation is uniquely locked at the Tsirelson bound ($2\sqrt{2}$) due to the unique algebraic stability of the Cauchy invariant measure. If the constant negative curvature of the Fisher information manifold, supported by this robust algebraic infrastructure, is indeed the dynamic foundation of quantum-like properties, it must inherently govern the behavior of composite systems and their collective non-local bounds.

In this paper, we bridge this vital conceptual gap by demonstrating that the Tsirelson bound possesses a purely deterministic, chaotic origin. By projecting the joint measurement configurations of two correlated observers (Alice and Bob) onto the $\kappa = -2$ Fisher information manifold, we show that the non-commutative parameter drives corresponding to their measurement baselines form a closed geodesic cycle—specifically, a hyperbolic quadrangle. We prove that the maximization of the CHSH correlation sum reduces entirely to a geometric optimization problem on this negative-curvature manifold. Without invoking any quantum-mechanical state vectors or operators, the strict hyperbolic triangle and polygon inequalities automatically restrict the maximum achievable value of S to exactly $2\sqrt{2}$. This result establishes the Exactly Solvable Chaos (ESC) towers as a comprehensive unified framework where both physical kinematics and multi-particle quantum boundaries are decoded as the inevitable geometric consequences of information preservation in discrete dynamical systems.

2 The Infrastructure of Exactly Solvable Chaos Towers: Conformal Attractors and Ergodicity

To establish a rigorous foundation for the emergence of macroscopic quantum-like correlations, we first define the deterministic microscopic infrastructure driving our combined system: the exactly solvable chaos towers constructed via the functional equations of trigonometric maps [12, 13].

Let $x_t \in \mathbb{R}$ be the state variable of a microscopic chaotic drive at discrete time t . The foundational dynamical system is governed by the K -th order mapping:

$$x_{t+1} = f_K(x_t), \quad (3)$$

which is exactly reduced from the K -multiplication formula of the tangent function. By expressing the state via the generalized phase variable ϕ_t as $x_t = \tan \phi_t$, the mapping $f_K(x)$ algebraically satisfies:

$$\tan(K\phi_t) = f_K(\tan \phi_t). \quad (4)$$

For instance, for $K = 2$ and $K = 3$, Eq. (4) explicitly yields the well-known exactly solvable chaotic maps [13]:

$$f_2(x) = \frac{2x}{1-x^2}, \quad f_3(x) = \frac{3x-x^3}{1-3x^2}. \quad (5)$$

The profound advantage of these tan-multiplication maps lies in their statistical tractability via the Perron-Frobenius operator. For any integer $K \geq 2$, the dynamical system possesses a unique, stationary invariant probability density function $P(x)$ that satisfies the fixed-point condition of the Perron-Frobenius equation. Remarkably, this invariant measure is uniquely locked onto the **standard Cauchy distribution with a fixed scale parameter $\gamma = 1$ **:

$$P(x) = \frac{1}{\pi(x^2 + 1)}. \quad (6)$$

Because the mapping $f_K(x)$ is topologically conjugate to the pure Bernoulli shift $\phi_{t+1} = K\phi_t \pmod{\pi}$ on the circle, the system is proven to be strictly mixing, which automatically guarantees its **mixing property** and consequently its strong **ergodicity** [12, 13]. Therefore, for any L_1 integrable observable $g(x)$, the temporal average over a single microscopic trajectory matches the ensemble average over the invariant measure with probability one, satisfying

the Birkhoff ergodic theorem:

$$\lim_{T \rightarrow \infty} \frac{1}{T} \sum_{t=0}^{T-1} g(x_t) = \int_{-\infty}^{\infty} g(x) P(x) dx = \int_{-\infty}^{\infty} g(x) \frac{1}{\pi(x^2 + 1)} dx. \quad (7)$$

We now extend this architecture to the concept of **Exactly Solvable Chaos Towers**. Consider a hierarchy of maps governed by different ranks K and K' . A crucial mathematical feature of this system is that the standard Cauchy distribution [Eq. (6)] acts as a universal invariant attractor across the entire tower. Whether the system executes a rapid K -multiplication transformation or a distinct K' -multiplication transformation, or even utilizes the cotangent dual map $x_t = \cot \phi_t$, **all components within the chaos tower share the identical invariant density with $\gamma = 1$** .

Furthermore, this algebraic robustness allows us to construct a generalized **Mixture of Exactly Solvable Chaos Towers** via the actions of the unimodular group $\text{SL}(2, \mathbb{R})$. Let $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{R})$ be a unimodular matrix satisfying $ad - bc = 1$. The corresponding fractional linear (Möbius) transformation is defined as:

$$\mathbf{M} \cdot x = \frac{ax + b}{cx + d}. \quad (8)$$

By utilizing the fact that the standard Cauchy measure is invariant under the specific conformal inversion $x \mapsto -1/x$ (corresponding to $\mathbf{M} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$), any composite dynamical sequence formed by a mixture of different ranks K_j interleaved with unimodular transformations preserves the exact same global attractor. As is easily proven, the macroscopic evolution of such a chaotic mixture can be exactly reduced from the Perron-Frobenius equations onto a closed statistical manifold without any information loss.

This universal invariant infrastructure establishes that Alice and Bob, by operating different ranks K, K' or utilizing unimodular transformations within their respective local chaos towers, are anchored to the exact same geometric background ($\gamma = 1$). It is this rigid statistical consensus that allows the macroscopic parameter metadata to emerge as a coherent topological space.

3 The Micro-Macro Duality: Cauchy-Szegő Basis Expansion and Topological Filtering

Based on the infrastructure of the solvable chaos towers established in Section 2, we now define the physical correlation interface between Alice and Bob. Crucially, we do not postulate

quantum mechanics from the outset; instead, the joint physical correlation is strictly defined as the long-term temporal average (ergodic average) of the co-evolving microscopic chaotic trajectories.

Let $g_A(x)$ and $g_B(x)$ be the microscopic observation functions determined by Alice and Bob's respective measurement angles θ_A, θ_B . Utilizing the Birkhoff ergodic theorem [Eq. (7)], the physical joint correlation $\mathcal{E}_{\text{phys}}$ is evaluated exactly via integration over the real axis $\mathbb{R}$ against the invariant Cauchy measure $P(x) dx$ [13]:

$$\mathcal{E}_{\text{phys}}(A, B) \equiv \lim_{T \rightarrow \infty} \frac{1}{T} \sum_{t=0}^{T-1} g_A(x_t) g_B^*(x_t) = \int_{-\infty}^{\infty} g_A(x) g_B^*(x) P(x) dx. \quad (9)$$

To eliminate any arbitrary choice of observation functions and formulate a universal theory, we expand $g_A(x)$ and $g_B(x)$ over the complete orthonormal Cauchy-Szegő basis (Takenaka-Malmquist basis [9, 10]) of the Hardy space $H^2(\mathbb{H}^+)$. Under the invariant Cauchy measure $P(x) = \frac{1}{\pi(x^2+1)}$, the basis functions are defined as:

$$e_n(x) = \left(\frac{x-i}{x+i} \right)^n, \quad n \in \mathbb{Z}. \quad (10)$$

Crucially, the basis functions $\{e_n(x)\}_{n \in \mathbb{Z}}$ constitute the exact spectral eigenfunctions (or the canonical basis for the infinite Lebesgue spectrum) of the Koopman operator $\mathcal{U}_K$ associated with the chaotic map $x_{t+1} = f_K(x_t)$. Because the map $f_K(x)$ is topologically conjugate to the Bernoulli shift $\phi_{t+1} = K\phi_t \pmod{\pi}$, the Koopman operator acts on this basis as a pure multiplication of the harmonic degree:

$$\mathcal{U}_K e_n(x) \equiv e_n(f_K(x)) = \left(\frac{f_K(x)-i}{f_K(x)+i} \right)^n = \left(\frac{x-i}{x+i} \right)^{nK} = e_{nK}(x), \quad (11)$$

because $e_n(x) = \left(\frac{x-i}{x+i} \right)^n = \exp(-2 \cdot i \cdot n \arccot x)$, $n \in \mathbb{Z}$. This exact covariance under the Koopman operator confirms that the Cauchy-Szegő basis $e_n(x)$ naturally diagonalizes the underlying ergodic dynamics, mapping the complex mixing of the chaos tower directly onto the shift dynamics of the Hilbert space $H^2(\mathbb{H}^+)$. Furthermore, these basis elements satisfy the strict orthogonality relation under the invariant measure $P(x)dx$:

$$\int_{-\infty}^{\infty} e_n(x) e_m^*(x) P(x) dx = \delta_{nm}. \quad (12)$$

Consequently, general unimodular observation functions for Alice and Bob are expanded as Fourier-Szegő series:

$$g_A(x) = \sum_{n=0}^{\infty} c_n e_n(x) e^{-inK\theta_A}, \quad g_B(x) = \sum_{m=0}^{\infty} d_m e_m(x) e^{-imK'\theta_B}, \quad (13)$$

where $\sum |c_n|^2 = \sum |d_m|^2 = 1$. Substituting Eq. (13) into the ergodic integration formula [Eq. (9)] and applying the orthogonality [Eq. (12)], all cross-harmonic terms ($n \neq m$) vanish

identically via the residue theorem. The joint physical correlation is thus generally reduced to:

$$\mathcal{E}_{\text{phys}}(A, B) = \sum_{n=0}^{\infty} c_n d_n^* e^{in(K'\theta_B - K\theta_A)}. \quad (14)$$

Under the canonical Szegő projection $\mathcal{P}_{H^2}$ onto the Hardy space $H^2(\mathbb{H}^+)$, the higher-order modes ($n \geq 2$), which possess higher-order complex poles at $z = i$, are topologically filtered out by the vanishing residue theorem over the complex parameter space. This isolates the fundamental harmonic mode ($n = 1$, $c_1 d_1^* = 1$ under $K = K' = 2$), yielding the pure quantum correlation:

$$\mathcal{E}(A, B) = \text{Re } \mathcal{E}_{\text{phys}}(A, B) = \cos(2\Delta\theta), \quad (15)$$

where $\Delta\theta = \theta_B - \theta_A$. Thus, the emergence of smooth quantum correlations and the Tsirelson bound $2\sqrt{2}$ is mathematically proven as a universal topological filtering of the Cauchy-Szegő basis cascade.

Statistical Boundedness and Unitarity—. At this juncture, we emphasize the rigorous physical soundness embedded in this definition: the intrinsic boundedness of the physical joint correlation. Since the observation functions satisfy $|g_A(x)| = 1$ and $|g_B(x)| = 1$ universally for all real $x \in \mathbb{R}$, the absolute value of the correlation is strictly bounded by unity via the exact normalization of the invariant Cauchy measure:

$$|\mathcal{E}_{\text{phys}}(A, B)| \leq \int_{-\infty}^{\infty} |g_A(x) g_B^*(x)| P(x) dx = \int_{-\infty}^{\infty} 1 \cdot P(x) dx = 1. \quad (16)$$

Equation (16) guarantees that our deterministic framework strictly preserves physical unitariness and never resorts to unphysical hyper-correlations. The emergence of the quantum correlation is achieved entirely within the mathematically sound, bounded probability space of the local classical attractor.

Vanishing Macroscopic Expectations—. Furthermore, for any mode $n \geq 1$ in the Cauchy-Szegő expansion, the macroscopic expectation value of Alice's observation vanishes identically:

$$\langle e_n(x) \rangle = \int_{-\infty}^{\infty} \left(\frac{x-i}{x+i} \right)^n \frac{1}{\pi(x-i)(x+i)} dx = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{(x-i)^{n-1}}{(x+i)^{n+1}} dx = 0. \quad (17)$$

Evaluating Eq. (17) by closing the contour in the upper half-plane $\mathbb{C}^+$, the integrand is purely analytic in $\mathbb{C}^+$ (having its pole of order $n+1$ strictly at $z = -i$ in the lower half-plane). Thus, by Cauchy's integral theorem, the residue at $z = i$ is identically zero. This proves $\langle g_A(x) \rangle = 0$ and $\langle g_B(x) \rangle = 0$ universally, establishing that the emergent macro-vacuum is a pure state completely uncompressed by classical statistical biases.

Residue Screening and Derivation of the Tsirelson Bound— For the fundamental mode $n = 1$ in Eq. (14), the joint macro-correlation reduces exactly to the smooth harmonic profile:

$$\mathcal{E}(A, B) = \text{Re } \mathcal{E}_{\text{phys}}(A, B) = \cos(2\Delta\theta). \quad (18)$$

Maximizing the CHSH sum constructed from Eq. (18) via trigonometric factorization yields the absolute mathematical ceiling:

$$S_{\text{max}} = 2\sqrt{2}. \quad (19)$$

1. The Microscopic Collapse (L_1 Prison):. If the observer attempts to evaluate the integration in Eq. (9) strictly along the real axis using discrete point-sampling, the absolute values of the phase-switching factors dominate. Because $\left|\frac{x-i}{x+i}\right| = 1$ holds universally for all real $x \in \mathbb{R}$, the micro-dynamics treat the integrand as a fractured cascade of unattenuated harmonic mode-couplings under $\gamma = 1$ ($e^0 = 1$). As proven in Section 4, this real-axis discrete integration forces a mathematical collapse into a non-differentiable triangle wave, strictly confining the CHSH sum to the classical local limit $|S| \leq 2$.

2. The Macroscopic Transition (Complex Hilbert Space):. Conversely, because the Cauchy-Szegő basis functions $e_n(x)$ are analytic in the upper half-plane $\mathbb{C}^+$, a macroscopic observer tracking the continuous evolution of the statistical manifold is mathematically justified in transitioning from the real-axis product to the continuous complex inner product $\mathcal{E} \equiv \langle \psi_A | \psi_B \rangle$. By factoring the invariant density $P(x)$ into the complex probability amplitudes, the overlapping pole terms cancel out inside the complex domain, allowing the residue theorem to act as a topological filter at the isolated pole $z = i$. As a direct consequence, the infinite high-harmonic square-wave noises are completely screened out, synthesizing the smooth quantum correlation $\cos 2(\theta_B - \theta_A)$ and achieving the absolute Tsirelson bound of $2\sqrt{2}$.

4. The Microscopic Trap: Emergence of the Triangle-Wave Barrier and Classical Confinement.

To firmly establish the barrier embedded within conventional frameworks, we provide a rigorous mathematical demonstration of how a naive microscopic correlation collapses into a linear geometric trap, rendering the violation of Bell-type limits strictly impossible. Let Alice and Bob monitor identical, synchronized solvable chaotic maps. Under the standard local-realistic paradigm, their joint correlation $\mathcal{E}_{\text{micro}}(A, B)$ is evaluated as the ergodic spatial average of the product of discrete sign variables ($A, B \in \{-1, +1\}$) along the real axis.

Crucially, by the ergodic hypothesis, this spatial average must be weighted by the unique, invariant density of the tan-map family, which is strictly fixed as the standard Cauchy distribution $P(x) = \frac{1}{\pi(x^2+1)}$ with a constant scale parameter $\gamma = 1$:

$$\mathcal{E}_{\text{micro}}(A, B) = \int_{-\infty}^{\infty} \text{sgn}[\sin(x - \theta_A)] \text{sgn}[\sin(x - \theta_B)] P(x) dx. \quad (20)$$

Because the sign function forms a discontinuous square wave, it must be decomposed into its infinite Fourier series of odd harmonics: $\text{sgn}(\sin \phi) = \frac{4}{\pi} \sum_{m=1,3,5,\dots}^{\infty} \frac{\sin(m\phi)}{m}$. Substituting this expansion for both observers converts the real-axis micro-correlation into a double infinite series:

$$\begin{aligned} \mathcal{E}_{\text{micro}}(A, B) &= \frac{16}{\pi^2} \sum_{m=1,3,5,\dots}^{\infty} \sum_{l=1,3,5,\dots}^{\infty} \frac{1}{ml} \\ &\quad \times \int_{-\infty}^{\infty} \sin[m(x - \theta_A)] \sin[l(x - \theta_B)] P(x) dx. \end{aligned} \quad (21)$$

Using the trigonometric product-to-sum identity, the integrand inside the spatial average expands to:

$$\begin{aligned} \sin[m(x - \theta_A)] \sin[l(x - \theta_B)] &= \frac{1}{2} \left[\cos((m - l)x - m\theta_A + l\theta_B) \right. \\ &\quad \left. - \cos((m + l)x - m\theta_A - l\theta_B) \right]. \end{aligned} \quad (22)$$

To evaluate the integration over the invariant standard Cauchy distribution, we invoke the exact integration identity $\int_{-\infty}^{\infty} \cos(ax + b) P(x) dx = e^{-|a|} \cos b$. Applying this relation directly transforms the joint correlation into a purely algebraic harmonic series:

$$\begin{aligned} \mathcal{E}_{\text{micro}}(A, B) &= \frac{8}{\pi^2} \sum_{m,l \in \text{odd}}^{\infty} \frac{1}{ml} \left[e^{-|m-l|} \cos(m\theta_A - l\theta_B) \right. \\ &\quad \left. - e^{-(m+l)} \cos(m\theta_A + l\theta_B) \right]. \end{aligned} \quad (23)$$

Here, a critical mathematical obstruction emerges from the invariant constraint $\gamma = 1$. When isolating the diagonal mode-coupling terms where $m = l$, the exponential damping factor in the first term simplifies to $e^{-|m-l|} = e^0 = 1$. Consequently, the exponential damping mechanism is completely deactivated for all diagonal components. While the off-diagonal terms ($m \neq l$) and the second term ($e^{-(m+l)} = e^{-2m}$) undergo rapid exponential decay, the remaining dominant diagonal cascade reduces to a rigid, non-vanishing infinite sum:

$$\mathcal{E}_{\text{micro}}(\Delta\theta) = \frac{8}{\pi^2} \sum_{m=1,3,5,\dots}^{\infty} \frac{\cos(m\Delta\theta)}{m^2}, \quad (24)$$

where $\Delta\theta = \theta_A - \theta_B$ represents the relative measurement angle.

Equation (24) represents the exact Fourier expansion of a continuous, periodic triangle wave possessing sharp coordinate cusps, rather than a smooth quantum cosinusoidal curve. For the fundamental domain $\Delta\theta \in [-\pi, \pi]$, this infinite harmonic sum synthesizes precisely into a linear absolute-value profile:

$$\mathcal{E}_{\text{micro}}(\Delta\theta) = 1 - \frac{2}{\pi}|\Delta\theta|. \quad (25)$$

This linear rigidity introduces a fatal geometric constraint when constructing multipartite quantum correlations. When evaluated under the standard CHSH correlation configuration at the benchmark relative angle $\Delta\theta = \pi/8$, the micro-local correlation components yield exactly $\mathcal{E}_{\text{micro}}(\pi/8) = 1 - 2/\pi(\pi/8) = 3/4$. Consequently, due to the sharp, non-differentiable coordinate cusps and the severe cancellation of high-harmonics, the collective CHSH sum is permanently suppressed. Specifically, under this un-elevated configuration, the destructive interference yields a heavily restricted outcome:

$$S_{\text{micro}} = \frac{3}{4} + \frac{3}{4} + \frac{3}{4} - \frac{3}{4} = \frac{3}{2} = 1.5 < 2. \quad (26)$$

As rigorously generalized in the subsequent sections, a correlation bounded by a linear triangle wave $\mathcal{E}_{\text{micro}}(\Delta\theta) = 1 - \frac{2}{\pi}|\Delta\theta|$ can never mathematically break the classical threshold $|S| \leq 2$ under any legitimate geometric configuration. This severe suppression definitively proves that the un-elevated real-axis micro-local infrastructure is strictly confined within the classical regime. This definitive mathematical trap shows that as long as one adheres to the dogma of real-axis discrete micro-sampling, the invariant constraint ($\gamma = 1$) of the physical attractor ensures that square-wave rigidities permanently suffocate any potential for quantum-like violations, necessitating a transition to a complex macro-interface.

5. Rigorous Geometric Proof of the Classical Bound Preservation

To prove that the microscopic correlation $\mathcal{E}_{\text{micro}}(\Delta\theta)$ defined in Eq. (25) strictly satisfies the Bell-CHSH inequality under any arbitrary choice of measurement angles, we unveil the underlying geometric metric space of the infinite harmonic cascade. Remarkably, the infinite Fourier series of quadratic odd denominators obtained via the diagonal mode-coupling,

$$\mathcal{E}_{\text{micro}}(\Delta\theta) = \frac{8}{\pi^2} \sum_{m=1,3,5,\dots}^{\infty} \frac{\cos(m\Delta\theta)}{m^2}, \quad (27)$$

is mathematically identical to the exact Fourier expansion of a continuous, periodic **triangle wave** possessing sharp coordinate cusps. For the fundamental domain $\Delta\theta \in [-\pi, \pi]$, this

infinite sum synthesizes precisely into a linear absolute-value profile:

$$\mathcal{E}_{\text{micro}}(\Delta\theta) = 1 - \frac{2}{\pi}|\Delta\theta|. \quad (28)$$

This linear reduction reveals a profound topological reality: the real-axis micro-sampling pins the correlation structure down to a flat, one-dimensional L_1 metric space (a circle). We can define the normalized geodesic distance between any two measurement orientations θ_i and θ_j along this circular manifold as $d(\theta_i, \theta_j) \equiv \frac{2}{\pi}|\theta_i - \theta_j|$. By construction, this distance function $d(\cdot, \cdot)$ satisfies all strict axioms of a formal mathematical metric, including the fundamental ****triangle inequality****.

Using this metric parametrization, the joint correlations for the four standard CHSH configurations between Alice (A, A') and Bob (B, B') are expressed as:

$$\begin{aligned} \mathcal{E}_{\text{micro}}(A, B) &= 1 - d(A, B), & \mathcal{E}_{\text{micro}}(A, B') &= 1 - d(A, B'), \\ \mathcal{E}_{\text{micro}}(A', B) &= 1 - d(A', B), & \mathcal{E}_{\text{micro}}(A', B') &= 1 - d(A', B'). \end{aligned} \quad (29)$$

We now construct the collective CHSH sum S_{micro} by substituting Eq. (29) into the standard definition:

$$\begin{aligned} S_{\text{micro}} &= \mathcal{E}_{\text{micro}}(A, B) - \mathcal{E}_{\text{micro}}(A, B') + \mathcal{E}_{\text{micro}}(A', B) + \mathcal{E}_{\text{micro}}(A', B') \\ &= [1 - d(A, B)] - [1 - d(A, B')] + [1 - d(A', B)] + [1 - d(A', B')] \\ &= 2 - [d(A, B) - d(A, B') + d(A', B) + d(A', B')]. \end{aligned} \quad (30)$$

To determine the absolute upper bound of S_{micro} across all possible angular configurations, we invoke the quadrilateral polygon inequality—a direct extension of the triangle inequality inherent to any valid metric space. For the four arbitrary points A, A', B, B' on the manifold, the shortest geodesic distance between A and B' is strictly bounded by the multi-step path traversing through B and A' :

$$d(A, B') \leq d(A, B) + d(B, A') + d(A', B'). \quad (31)$$

Noting that the distance metric is symmetric ($d(B, A') = d(A', B)$), we algebraically rearrange Eq. (31) by moving all terms to one side:

$$d(A, B') - d(A, B) - d(A', B) - d(A', B') \leq 0. \quad (32)$$

Substituting this strict geometric constraint [Eq. (32)] directly back into the CHSH brackets in Eq. (30) yields:

$$S_{\text{micro}} = 2 + \underbrace{[d(A, B') - d(A, B) - d(A', B) - d(A', B')]}_{\leq 0} \leq 2. \quad (33)$$

Symmetrically, by applying the alternative permutation of the metric triangle inequality, $d(A, B) \leq d(A, B') + d(B', A') + d(A', B)$, the lower bound is identically constrained to $S_{\text{micro}} \geq -2$.

Therefore, we have provided an absolute, non-perturbative proof that $|S_{\text{micro}}| \leq 2$ holds universally. No matter how one dynamically or externally manipulates the relative measurement angles $\Delta\theta$, the microscopic correlation is mathematically imprisoned within the flat geometry of the triangle inequality. This rigorous proof establishes that the failure to violate the Bell threshold in classical mechanics is not a failure of determinism itself, but an inescapable geometric consequence of real-axis discrete micro-sampling, which collapses the correlation into a rigid L_1 metric infrastructure.

6. The Macroscopic Resolution: Justification of the Complex Inner Product

We overcome this fundamental microscopic limitation by demonstrating that the true physical layer of observation is not the discrete micro-sampling along the real axis, but the geometric evolution of the macroscopic statistical ensemble itself. The justification for introducing the complex inner product stems directly from the algebraic topological structure of the invariant measure of the solvable chaos towers.

The invariant Cauchy distribution $P(x) = \frac{1}{\pi(x^2+1)}$ is uniquely characterized by its fixed scale parameter $\gamma = 1$, which mathematically encodes the global topology of the chaotic attractor in its stationary state. In the complex plane $\mathbb{C}$, this invariant density is completely governed by a unique pair of complex conjugate poles located precisely at $z = \pm i$. The ergodic mixing of the chaos tower induces a smooth, continuous conformal mapping of these complex parameters within the upper half-plane $\mathbb{C}^+$ via the Perron-Frobenius operator [5].

Therefore, a macroscopic observer monitoring the combined system is physically tracking the continuous transformation of the statistical manifold, specifically, the shift in the phase metadata. By factorizing the invariant density into complex probability amplitudes, the emergent macroscopic wave functions $\psi(x)$ arise not as an ad-hoc quantum postulate, but as a rigorous, deterministic descriptor of the global parameter state of the ergodic ensemble. Under the invariant constraint $\gamma = 1$, we define the localized wave functions for Alice and Bob as:

$$\psi_A(x) = \frac{1}{\sqrt{\pi}} \frac{e^{iK\theta_A}}{x - i}, \quad \psi_B(x) = \frac{1}{\sqrt{\pi}} \frac{e^{iK'\theta_B}}{x - i}, \quad (34)$$

where K, K' denote the integer modes (ranks) of the respective chaos towers, and θ_A, θ_B represent their spatial orientations. When Alice and Bob combine their statistical data, the joint physical interface is governed by the overlap of these parameter states. In information

geometry, the natural metric to measure the overlap between two continuous statistical states is defined via the inner product in the corresponding Hilbert space of amplitudes. Thus, the transition from the real-axis discrete product $\mathcal{E}_{\text{micro}}$ to the continuous complex inner product $\mathcal{E} \equiv \langle \psi_A | \psi_B \rangle$ is a mathematical necessity of analytic continuation to the Hilbert space of parameter amplitudes, shifting the domain of integration to the upper half-plane where the topological features of chaos reside.

The Macroscopic Resolution and Conformal Invariance of the Wave Function.— We overcome the microscopic L_1 geometric limitation by demonstrating that the true physical layer of observation is not the discrete real-axis sampling, but the geometric evolution of the macroscopic statistical ensemble itself. The justification for introducing the complex inner product stems directly from the algebraic topological structure of the invariant measure of the exactly solvable chaos towers.

The invariant Cauchy distribution $P(x) = \frac{1}{\pi(x^2+1)}$ is uniquely characterized by its fixed scale parameter $\gamma = 1$, which mathematically encodes the global topology of the chaotic attractor in its stationary state. Remarkably, this invariant density is strictly invariant on the conformal inversion map $x \mapsto -1/x$, reflecting the underlying $\text{SL}(2, \mathbb{R})$ möbius symmetry of the chaos tower. By factorizing this invariant density into a pair of complex probability amplitudes, we define the emergent macroscopic wave functions for Alice and Bob as:

$$\psi_A(x) = \frac{1}{\sqrt{\pi}} \frac{e^{i2\theta_A}}{x - i}, \quad \psi_B(x) = \frac{1}{\sqrt{\pi}} \frac{e^{i2\theta_B}}{x - i}, \quad (35)$$

where θ_A, θ_B represent their spatial orientations and the fundamental synchronized modes are fixed at $K = K' = 2$.

To unveil the deep mathematical legitimacy of this factorization, we evaluate how these emergent wave functions behave under the conformal boundary transformation $x = -1/y$. Treating the state as a geometric half-form $\psi(x)\sqrt{dx}$ weighted by the square root of the density metric, the transformation yields:

$$\begin{aligned} \psi_A(-1/y)\sqrt{dx} &= \left(\frac{1}{\sqrt{\pi}} \frac{e^{i2\theta_A}}{-\frac{1}{y} - i} \right) \sqrt{\frac{1}{y^2} dy} \\ &= \left(\frac{1}{\sqrt{\pi}} \frac{e^{i2\theta_A} \cdot y}{-1 - iy} \right) \frac{1}{y} \sqrt{dy} \\ &= \frac{1}{-i\sqrt{\pi}} \frac{e^{i2\theta_A}}{y - \frac{1}{i}} \sqrt{dy} = i \cdot \left(\frac{1}{\sqrt{\pi}} \frac{e^{i2\theta_A}}{y + i} \right) \sqrt{dy}. \end{aligned} \quad (36)$$

Equation (36) reveals a profound holographic symmetry: the real-axis inversion $x \mapsto -1/x$ physically maps the complex upper-half-plane pole $z = i$ directly to the lower-half-plane pole

$z = -i$, while simultaneously inducing a discrete $U(1)$ gauge phase rotation of $i = e^{i\pi/2}$. This exact conformal covariance establishes that the macro-wave functions are not arbitrary quantum constructs, but rigorous, weight-1 modular half-forms that perfectly inherit the structural symmetries of the chaotic ergodic invariant measure.

When Alice and Bob combine their statistical metadata, the joint physical interface is governed by the overlap of these parameter states. Under the conformal transformation, the gauge factors i and $-i$ (from the complex conjugate ψ_A^*) identically cancel out ($i \times (-i) = 1$), ensuring that the continuous complex inner product $\mathcal{E} \equiv \langle \psi_A | \psi_B \rangle$ is completely immune to boundary coordinate deformations.

7. Residue Screening and Derivation of the Tsirelson Bound

By substituting the $\gamma = 1$ fixed wave functions [Eq. (35)] into the inner product formula, the macroscopic joint correlation expands to the following complex contour integral along the real axis:

$$\langle \psi_A | \psi_B \rangle = \frac{1}{\pi} e^{i(K'\theta_B - K\theta_A)} \int_{-\infty}^{\infty} \frac{1}{(x+i)(x-i)} dx, \quad (37)$$

where the complex conjugate $\psi_A^*(x)$ naturally places a pole at $z = -i$ in the lower half-plane, while $\psi_B(x)$ places a pole at $z = i$ in the upper half-plane.

Unlike the microscopic real-axis integration [Eq. (20)], which was intrinsically fractured by an infinite cascade of overlapping harmonic modes, the macro-correlation integrand in Eq. (37) bypasses the infinite Fourier series entirely. The integrand possesses a single, isolated complex pole within the upper half-plane $\mathbb{C}^+$ located strictly at $z = i$. By enclosing the contour in the upper half-plane, the residue theorem effortlessly evaluates the continuous integration:

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{1}{(z+i)(z-i)} dz &= 2\pi i \operatorname{Res} \left(\frac{1}{(z+i)(z-i)}, i \right) \\ &= 2\pi i \left[\frac{1}{z+i} \right]_{z=i} = 2\pi i \left(\frac{1}{2i} \right) = \pi. \end{aligned} \quad (38)$$

Substituting this result back into Eq. (37), the geometric factor π originating from the residue calculation ****completely cancels out**** the normalization coefficient $\frac{1}{\pi}$ of the invariant density:

$$\langle \psi_A | \psi_B \rangle = \frac{1}{\pi} e^{i(K'\theta_B - K\theta_A)} \cdot \pi = e^{i(K'\theta_B - K\theta_A)}. \quad (39)$$

By choosing the fundamental synchronized modes ($K = K' = 2$) corresponding to the dual spatial projection of the tan-cot maps, and extracting the real physical projection, the

joint macro-correlation reduces exactly to the smooth harmonic profile:

$$\mathcal{E}(A, B) = \text{Re}\langle\psi_A|\psi_B\rangle = \cos(2\Delta\theta), \quad (40)$$

where $\Delta\theta = \theta_B - \theta_A$. Remarkably, the infinite cascade of high-harmonic square-wave noises that heavily penalized the classical micro-dynamics is 100% completely screened out by the topological filter of the residue theorem. The invariant constraint $\gamma = 1$, which acted as a permanent prison for the micro-states, now serves as the perfect geometric anchor for the macro-states.

We now construct the standard CHSH observable S utilizing four distinct combinations of measurement settings between Alice (A, A') and Bob (B, B'):

$$S = \mathcal{E}(A, B) - \mathcal{E}(A, B') + \mathcal{E}(A', B) + \mathcal{E}(A', B'). \quad (41)$$

Substituting the dynamically emerged cosinusoidal correlation [Eq. (40)] into Eq. (41), and defining the relative angles $\alpha = 2(\theta_B - \theta_A)$, $\beta = 2(\theta_{B'} - \theta_B)$, and $\alpha' = 2(\theta_{A'} - \theta_A)$, the four correlation terms can be geometrically mapped onto a closed configuration within the emergent hyperbolic bulk manifold ($\kappa = -2$) [7]. The CHSH sum transforms into the following trigonometric function of three independent angular variables:

$$S(\alpha, \beta, \alpha') = \cos \alpha - \cos(\alpha + \beta) + \cos(\alpha' - \alpha) + \cos(\alpha' - \alpha - \beta). \quad (42)$$

To determine the absolute mathematical ceiling of S , we find the extreme points by evaluating the gradient of Eq. (42) with respect to the internal configurations. By applying the standard trigonometric sum-to-product identities, the sum of the first two terms and the last two terms simplifies to:

$$\begin{aligned} S &= -2 \sin\left(\alpha + \frac{\beta}{2}\right) \sin\left(\frac{\beta}{2}\right) + 2 \cos\left(\alpha' - \alpha - \frac{\beta}{2}\right) \cos\left(\frac{\beta}{2}\right) \\ &= 2 \left[-\sin\left(\alpha + \frac{\beta}{2}\right) \sin\left(\frac{\beta}{2}\right) + \cos\left(\alpha' - \alpha - \frac{\beta}{2}\right) \cos\left(\frac{\beta}{2}\right) \right]. \end{aligned} \quad (43)$$

We maximize this expression by setting the configuration to the fundamental symmetric resonance of the dual tan-cot maps. Choosing the optimal relative angle $\beta = \pi/2$ locks the tracking components to $\cos(\beta/2) = \sin(\beta/2) = 1/\sqrt{2}$. Substituting these geometric invariants back into Eq. (43) yields:

$$S = \sqrt{2} \left[-\sin\left(\alpha + \frac{\pi}{4}\right) + \cos\left(\alpha' - \alpha - \frac{\pi}{4}\right) \right]. \quad (44)$$

Applying the cosine addition theorem and utilizing the phase shift $\sin(x) = \cos(\pi/2 - x)$, the brace in Eq. (44) can be rewritten as a single composite cosine profile via trigonometric

factorization:

$$\begin{aligned} S &= \sqrt{2} \left[\cos \left(\alpha + \frac{3\pi}{4} \right) + \cos \left(\alpha' - \alpha - \frac{\pi}{4} \right) \right] \\ &= 2\sqrt{2} \cos \left(\frac{\alpha'}{2} + \frac{\pi}{4} \right) \cos \left(\alpha - \frac{\alpha'}{2} + \frac{\pi}{2} \right). \end{aligned} \quad (45)$$

The absolute global maximum of Eq. (45) occurs when both independent modulating cosine envelopes simultaneously reach their peak magnitudes of ± 1 . This structural resonance uniquely fixes the remaining geometric configurations of the chaos towers to:

$$\alpha' = \frac{3\pi}{2}, \quad \alpha = \frac{5\pi}{4}. \quad (46)$$

Substituting these boundary constraints [Eq. (46)] directly into Eq. (45) yields the exact maximum valuation:

$$S_{\max} = 2\sqrt{2} \cdot \cos(\pi) \cdot \cos(\pi) = 2\sqrt{2}. \quad (47)$$

In terms of the original spatial orientation angles of Alice and Bob, this absolute upper ceiling is realized under the elegant geometric configuration $\theta_A = 0, \theta_{A'} = -3\pi/4, \theta_B = 5\pi/8$, and $\theta_{B'} = 7\pi/8$, matching the exact angular profile derived by Cirel'son [3].

Crucially, this absolute upper ceiling $2\sqrt{2}$ —the Tsirelson bound—is locked strictly as a geometric necessity of the constant negative curvature of the Fisher information manifold under the Cramér-Rao limit. Without invoking any quantum-mechanical state vectors, operators, or non-local action-at-a-distance, the strict topological filtering of local deterministic chaos naturally synthesizes the absolute boundary of quantum non-locality.

Generalization to Higher-Order Cauchy-Szegő (Takenaka-Malmquist) Modes [9, 10] —.

The recovery of the Tsirelson bound is not restricted to the fundamental mode $n = 1$, but naturally generalizes across the entire sequence of the Cauchy-Szegő (Takenaka-Malmquist) basis $\{e_n(x)\}_{n=1}^{\infty}$ of $H^2(\mathbb{H}^+)$. For a multi-pole configuration degenerating at $z_k = i$, the n -th order basis element takes the form $e_n(x) = \left(\frac{x-i}{x+i}\right)^n$. Expanding the macroscopic parameter states over the n -th harmonic subspace yields the scale-invariant correlation:

$$\mathcal{E}^{(n)}(A, B) = \text{Re} \left\langle \psi_A^{(n)} \middle| \psi_B^{(n)} \right\rangle = \cos \left(n(K'\theta_B - K\theta_A) \right). \quad (48)$$

Substituting $\mathcal{E}^{(n)}(A, B)$ into the CHSH functional, the structural optimization of $S^{(n)}$ reduces to an identical geometric constraint on the Fisher information manifold. Rescaling the angular configurations via $\alpha^{(n)} = nK\Delta\theta = nK\Delta(\theta_B - \theta_A)$ for $K = K'$, the global ceiling

satisfies:

$$S_{\max}^{(n)} = 2\sqrt{2}, \quad \forall n \in \mathbb{Z}^+, \quad (49)$$

realized at the rescaled optimal measurement baseline $\Delta\theta_{\text{optimal}} = \frac{\pi}{4nK}$. This invariance proves that the emergence of the Tsirelson bound is a robust, universal topological property of the Cauchy-Szegő (Takenaka-Malmquist) expansion over the complex boundary, holding independently of the microscopic harmonic order n . In the general case of distinct pole configurations $z_k = x_k + i\gamma_k \in \mathbb{C}^+$ (representing shifted chaotic invariant measure), the residue evaluation at each isolated pole combined with the canonical Szegő projection onto $H^2(\mathbb{H}^+)$ reduces the joint correlation either to a single dominant pole or to an orthogonal mode decomposition. Consequently, the universal attainment of the Tsirelson bound $2\sqrt{2}$ remains strictly preserved.

8. Conclusion and Discussion

In this paper, we have established a rigorous and entirely deterministic origin for quantum non-locality and the Tsirelson bound ($2\sqrt{2}$) by analyzing the macroscopic statistical behavior of exactly solvable chaos towers under a strictly fixed invariant measure ($\gamma = 1$). This framework resolves a long-standing paradox in the foundations of physics by exposing a profound dualism between microscopic real-axis sampling and macroscopic complex-amplitude inner products, effectively bridging classical determinism with emergent quantum boundaries.

Central to this synthesis is the complete signal characterization based on the Cauchy-Szegő basis expansion $e_n(x) = \left(\frac{x-i}{x+i}\right)^n$ over the Hardy space $H^2(\mathbb{H}^+)$. By mapping microscopic chaotic trajectories to this complex orthonormal basis, we demonstrate that the transition from a classical L_1 metric space ($|S| \leq 1.5$) to the quantum Tsirelson bound ($2\sqrt{2}$) is governed by a rigorous topological Szegő projection. Under the exact Szegő projection, the residue theorem acts as a topological filter at $z = i$, completely eliminating high-order noise modes ($n \geq 2$) and leaving the smooth sinusoidal correlation $\cos(2\Delta\theta)$ intact.

Crucially, this mechanism provides a profound mathematical counter-illumination to the celebrated Ryu-Takayanagi (RT) formula [11]. While the conventional RT formula brilliantly translates pre-existing quantum entanglements into macroscopic gravitational geometries as an ad-hoc, axiomatic postulate, the deterministic chaos tower paradigm completes and reverses this holographic dictionary. Under the stable bulk vacuum state ($\gamma = 1$), the conformal symmetry group $\text{SL}(2, \mathbb{R})$ inherent to the tan-cot chaotic maps acts on the real axis as a one-dimensional boundary CFT, while mapping the chaotic trajectories to the topological features of the complex upper half-plane. Consequently, the optimization of the CHSH sum is transformed into a pure geometric constraint of geodesic quadrilaterals within the emergent

hyperbolic bulk. The fact that the Tsirelson bound ($2\sqrt{2}$) emerges precisely at this stationary equilibrium limit proves that quantum non-locality is the inevitable holographic projection woven across the AdS bulk by the global parameter geometry of a local, deterministic, and information-preserving chaotic system [5].

This holographic breakthrough offers an entirely new perspective on the quantum incompleteness debate originally instigated by Einstein, Podolsky, and Rosen (EPR)[14]. In the foundational framework of conventional quantum mechanics, the Tsirelson bound has long been regarded as an absolute boundary reflecting an intrinsic "quantum non-locality" inaccessible by any classical system. However, by demonstrating that a deterministic chaos can fully capture and reproduce this Tsirelson bound, our results suggest the possibility of a novel interpretation via a new mathematical approach: that underlying quantum correlations, there may exist an alternative description rooted in deterministic background dynamics.

Furthermore, while conventional quantum foundations have long struggled with the apparent paradox that Bell-type violations require 'superluminal interactions' that violate relativistic causality, our framework entirely bypasses this conflict. By deriving the Tsirelson limit as a purely geometric consequence of tracking statistical metadata over the complex topology, our model demands no superluminal signal propagation across spacetime. Consequently, it harmonizes the emergent quantum non-locality with strictly local relativistic causality under a unified deterministic paradigm. By demonstrating that Bell-type violations, the vanishing of one-body expectations, and the Tsirelson threshold can be derived completely from a classical chaotic infrastructure, this work shatters the dogmatic chasm between classical determinism and quantum mechanics. Urging a historical reinterpretation of the boundary between classical and quantum mechanics, and ultimately, of the ongoing debates surrounding locality, determinism, and completeness in nature, this work opens a new interdisciplinary synthesis toward unifying chaotic dynamics, quantum foundations, and the deep, emergent geometric nature of spacetime.

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A. Szegő Projection and Harmonic Filtering of Cauchy-Szegő Modes

In this Appendix, we provide a self-contained proof of the orthogonality of the Cauchy-Szegő basis $e_n(x) = \left(\frac{x-i}{x+i}\right)^n$ and its topological filtering under the Szegő projection.

Consider the integral for $n, m \in \mathbb{Z}_{\geq 0}$:

$$I_{nm} = \int_{-\infty}^{\infty} e_n(x) e_m^*(x) P(x) dx = \frac{1}{\pi} \int_{-\infty}^{\infty} \left(\frac{x-i}{x+i}\right)^n \left(\frac{x+i}{x-i}\right)^m \frac{1}{(x-i)(x+i)} dx. \quad (\text{A1})$$

We classify the evaluation into three cases:

1. **Diagonal Case ($n = m$):**

$$I_{nn} = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{(x-i)(x+i)} dx = \frac{1}{\pi} \cdot \pi = 1. \quad (\text{A2})$$

2. **Upper-Off-Diagonal Case ($n > m$):** Let $k = n - m \geq 1$. The integrand becomes:

$$I_{nm} = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{(x-i)^{k-1}}{(x+i)^{k+1}} dx. \quad (\text{A3})$$

Enclosing the integration contour in the upper half-plane $\mathbb{C}^+$, the numerator $(z-i)^{k-1}$ and denominator $(z+i)^{k+1}$ contain no poles in $\mathbb{C}^+$ (the pole $z = -i$ lies strictly in the lower half-plane). Thus, by Cauchy's integral theorem, $I_{nm} = 0$.

3. **Lower-Off-Diagonal Case ($n < m$):** By taking the complex conjugate of Case 2, $I_{nm} = I_{mn}^* = 0$.

Combining these cases rigorously yields $I_{nm} = \delta_{nm}$. When macroscopic observation applies a Szegő projection $\mathcal{P}_{H^2}$ onto the fundamental boundary space, all higher-order poles ($n \geq 2$) undergo destructive interference, isolating $n = 1$ and producing the pristine cosinusoidal quantum correlation.

B. Lebesgue Spectrum of Dynamical Systems and Koopman Spectral Basis

In this Appendix, we provide a mathematically rigorous derivation establishing that the set of Cauchy-Szegő functions $\{e_n(x)\}_{n \in \mathbb{Z}}$ forms the canonical orthonormal basis corresponding to the countability of the Lebesgue spectrum for the Koopman operator associated with the chaotic maps.

B.1 Definitions of Koopman Operator and Lebesgue Spectrum

Let $(M, \mathcal{B}, \mu, T)$ be a measure-preserving dynamical system, where $M = \mathbb{R}$, $\mu(dx) = P(x)dx = \frac{1}{\pi(x^2+1)}dx$ is the standard Cauchy invariant measure, and $T(x) = f_K(x)$ represents the K -multiplication chaotic map defined via $f_K(\tan \phi) = \tan(K\phi)$.

The ****Koopman operator**** $\mathcal{U}_K : L^2(M, \mu) \rightarrow L^2(M, \mu)$ is a linear operator defined for any observable $g \in L^2(M, \mu)$ by composition:

$$(\mathcal{U}_K g)(x) \equiv g(f_K(x)). \quad (\text{A4})$$

Because $f_K(x)$ is a non-invertible, K -to-1 measure-preserving transformation, $\mathcal{U}_K$ preserves the Hilbert space inner product:

$$\langle \mathcal{U}_K g, \mathcal{U}_K h \rangle = \int_{-\infty}^{\infty} g(f_K(x)) h^*(f_K(x)) P(x) dx = \int_{-\infty}^{\infty} g(y) h^*(y) P(y) dy = \langle g, h \rangle. \quad (\text{A5})$$

Hence, $\mathcal{U}_K$ is a ****linear isometry**** (i.e., $\mathcal{U}_K^\dagger \mathcal{U}_K = I$, while $\mathcal{U}_K \mathcal{U}_K^\dagger \neq I$) on $L^2(\mathbb{R}, \mu)$.

A measure-preserving dynamical system is said to possess a ****countably infinite Lebesgue spectrum**** if there exists an orthonormal set of functions $\{\phi_0\} \cup \{\phi_{j,\ell}\}_{j \in \mathbb{Z}_{\geq 0}, \ell \in \mathcal{K}}$ in $L^2(M, \mu)$ such that:

- (1) $\phi_0(x) \equiv 1$ is the unique invariant eigenfunction corresponding to the simple eigenvalue $\lambda = 1$ ($\mathcal{U}_K \phi_0 = \phi_0$).
- (2) The subspace $H_0^\perp = \{g \in L^2(M, \mu) \mid \int g d\mu = 0\}$ is spanned by orthogonal forward orbits acting as isometric unilateral shifts:

$$\mathcal{U}_K \phi_{j,\ell} = \phi_{j+1,\ell}, \quad \forall j \in \mathbb{Z}_{\geq 0}, \quad \ell \in \mathcal{K}, \quad (\text{A6})$$

generating a purely continuous spectral resolution over the unit circle S^1 .

B.2 The Cauchy-Szegő Basis as the Koopman Spectral Basis

We now prove that the Cauchy-Szegő basis functions:

$$e_n(x) = \left(\frac{x-i}{x+i} \right)^n, \quad n \in \mathbb{Z}, \quad (\text{A7})$$

naturally realize this countable Lebesgue spectrum under the action of the Koopman operator $\mathcal{U}_K$.

First, using the isometric homeomorphism $x = \tan \phi$ ($\phi \in [-\pi/2, \pi/2]$), we observe the exact identity:

$$e_1(x) = \frac{\tan \phi - i}{\tan \phi + i} = \frac{\sin \phi - i \cos \phi}{\sin \phi + i \cos \phi} = -e^{-2i\phi} = e^{-2i(\phi - \pi/2)} = e^{-2i \operatorname{arccot}(x)}. \quad (\text{A8})$$

For any integer harmonic degree $n \in \mathbb{Z}$, the basis element simplifies to $e_n(x) = e^{-2in\phi}$.

Applying the Koopman operator $\mathcal{U}_K$ to $e_n(x)$ yields:

$$(\mathcal{U}_K e_n)(x) = e_n(f_K(x)) = \left(\frac{f_K(x) - i}{f_K(x) + i} \right)^n = \left(\frac{\tan(K\phi) - i}{\tan(K\phi) + i} \right)^n = e^{-2in(K\phi)} = e_{nK}(x). \quad (\text{A9})$$

Equation (B.6) demonstrates that the action of the Koopman operator on the Cauchy-Szegő basis is a multiplication of the spectral degree by rank K :

$$\mathcal{U}_K e_n(x) = e_{nK}(x). \quad (\text{A10})$$

For $n \neq 0$, the forward iterations $\mathcal{U}_K^j e_n = e_{nK^j}$ ($j \in \mathbb{Z}_{\geq 0}$) generate an infinite sequence of mutually orthogonal modes due to the orthogonality property proved in Appendix A:

$$\langle e_n, e_m \rangle = \int_{-\infty}^{\infty} e_n(x) e_m^*(x) P(x) dx = \delta_{nm}. \quad (\text{A11})$$

Because the forward orbit of any non-zero mode n under $\mathcal{U}_K$ acts as a unilateral isometric shift along the dyadic (or K -adic) tree of integer modes, the spectral measure restricted to $H_0^\perp$ is absolutely continuous with respect to the Lebesgue measure on S^1 .

Therefore, the Koopman isometry $\mathcal{U}_K$ associated with the exactly solvable chaos tower exhibits a **purely continuous Lebesgue spectrum** on $H_0^\perp \subset L^2(\mathbb{R}, P(x)dx)$. This establishes that the Cauchy-Szegő expansion is not merely an arbitrary set of functions, but the unique, canonical spectral basis that diagonalizes the underlying ergodic mixing dynamics of the chaotic state.