

A geometric flow for quadrature surfaces

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Abstract. A new geometric flow describing the motion of a closed surface is introduced. Moving surfaces evolving under the flow are shown to be a family of quadrature surfaces. It is proved that the geometric flow possesses a unique classical solution for any smooth initial surface with positive mean curvature.

1 Introduction

One of the classical problems in potential theory is to specify a closed surface Γ for a prescribed electric charge density μ in such a way that the uniform electric charge distribution on Γ induces the same potential in a neighborhood of the infinity as μ does. To formulate the problem mathematically, let F be the fundamental solution of $-\Delta$ in $\mathbb{R}^N$, i.e.,

$$(1.1) \quad F(x) := \begin{cases} -\frac{1}{2\pi} \log |x| & (N = 2), \\ \frac{1}{N(N-2)\omega_N |x|^{N-2}} & (N \geq 3), \end{cases}$$

where ω_N is the volume of the unit ball in $\mathbb{R}^N$, and let $\mathcal{H}^{N-1}|_{\Gamma}$ denote the $(N-1)$ -dimensional Hausdorff measure restricted to Γ . Then, the problem can be stated as follows: For a prescribed finite positive Radon measure μ with compact support in $\mathbb{R}^N$, find a $(N-1)$ -dimensional closed surface Γ enclosing a bounded domain Ω such that $F * \mu = F * \mathcal{H}^{N-1}|_{\Gamma}$ in $\mathbb{R}^N \setminus \overline{\Omega}$, i.e.,

$$(1.2) \quad \int F(x-y) d\mu(y) = \int_{\Gamma} F(x-y) d\mathcal{H}^{N-1}(y) \quad (x \in \mathbb{R}^N \setminus \overline{\Omega}).$$

In fact, (1.2) can be replaced by the equivalent condition that

$$(1.3) \quad \int h d\mu = \int_{\Gamma} h d\mathcal{H}^{N-1}$$

holds for all harmonic functions h defined in a neighborhood of $\overline{\Omega}$. Indeed, it is obvious that (1.3) implies (1.2). Conversely, if Γ satisfies (1.2), then by extending each harmonic function h to be smooth and have compact support in $\mathbb{R}^N$, we see that

$$\begin{aligned} \int h(y) d\mu(y) &= \int_{\mathbb{R}^N} \Delta h(x) \left(\int F(y-x) d\mu(y) \right) dx \\ &= \int_{\mathbb{R}^N} \Delta h(x) \left(\int_{\Gamma} F(y-x) d\mathcal{H}^{N-1}(y) \right) dx \\ &= \int_{\Gamma} h(y) d\mathcal{H}^{N-1}(y). \end{aligned}$$

Thus, (1.3) follows from (1.2).

The mean value property of harmonic functions implies that (1.3) holds when $\mu = N\omega_N\delta_0$ and $\Gamma = \partial B(0,1)$, where δ_0 is the Dirac measure supported at the origin and $B(0,1)$ is the unit ball in $\mathbb{R}^N$. Thus, the identity (1.3) can be seen as a generalization of the mean value formula for harmonic functions.

From this point of view, we also consider an analogous problem: For a prescribed measure μ , find a domain Ω such that

$$(1.4) \quad \int h d\mu = \int_{\Omega} h dx$$

holds for all harmonic functions h defined in a neighborhood of $\overline{\Omega}$. This problem also has a physical interpretation, and it is sometimes referred to as the “Potato Kugel” problem, especially when the uniqueness of a domain Ω is concerned.

Definition 1.1. *A closed surface Γ satisfying (1.3) is called a quadrature surface of μ for harmonic functions. Analogously, a domain Ω satisfying (1.4) is called a quadrature domain of μ for harmonic functions.*

The existence of a quadrature surface Γ of a prescribed μ has been studied by several authors with different approaches. Developing the idea of super/subsolutions of Beurling [4], Henrot [12] was able to prove that the existence of Γ is guaranteed when a supersolution and a subsolution are available. Gustafsson & Shahgholian [11] followed a variational approach developed by Alt & Caffarelli [1], namely, they consider the minimization problem for the functional

$$J(u) := \int_{\mathbb{R}^N} (|\nabla u|^2 - 2fu + \chi_{\{u>0\}}) dx,$$

and proved the existence and regularity of a minimizer u . Then, u is shown to satisfy the Euler-Lagrange equation

$$-\Delta u = f|_{\Omega} - \mathcal{H}^{N-1}|_{\partial\Omega}, \quad \Omega = \{u > 0\},$$

and thus $\Gamma = \partial\Omega$ is a quadrature surface of μ with $d\mu = f dx$.

Similarly, a quadrature domain has a variational characterization and can be obtained by solving an obstacle problem (see Sakai [18] and Gustafsson [10] for the detail). Moreover, the uniqueness of a quadrature domain follows from an argument based on the maximum principle. Indeed, it was shown by Sakai [17] that, if a quadrature domain Ω satisfies

$$F * (\mu - \chi_\Omega) > 0$$

everywhere in Ω , then there is no quadrature domain other than Ω . The above condition can be verified, in particular, when μ concentrates, relative to Ω .

However, as pointed out by Henrot [12], the uniqueness of a quadrature surface cannot be expected in general. He showed an example that the number of connected quadrature surfaces of $\mu(t) := t\delta_{(1,0)} + t\delta_{(-1,0)}$ in $\mathbb{R}^2$ changes according to the value of $t > 0$. The collapse of the uniqueness seems to indicate a bifurcation phenomenon of solutions to (1.3) with a parametrized measure $\mu = \mu(t)$. Hence, toward understanding of the uniqueness issue, we need to consider the corresponding family of surfaces $\Gamma = \Gamma(t)$. In this respect, it is natural to ask if there is a “flow” for surfaces $\{\Gamma(t)\}_{t>0}$ such that each $\Gamma(t)$ is a quadrature surface of a given parametrized measure $\mu(t)$. As a matter of fact, when $\mu(t) = t\delta_0 + \chi_{\Omega(0)}$ and $\Omega(t)$ is the corresponding quadrature domain, it is known that the Hele-Shaw flow, a model of interface dynamics in fluid mechanics, plays the desired role. This surprising connection between the two different physical problems was discovered by Richardson [16]. From this fact, the investigation of the evolution of quadrature domains is reduced to that of the Hele-Shaw flow, and the latter has been successfully proceeded by complex analysis and several methods in partial differential equations.

We are thus motivated to derive a flow having the corresponding property for quadrature surfaces, and eventually arrive at the following geometric flow:

$$(1.5) \quad \begin{aligned} &v_n = p \quad \text{for } x \in \partial\Omega(t), \\ &\text{where } \begin{cases} -\Delta p = \mu & \text{for } x \in \Omega(t), \\ (N-1)Hp + \frac{\partial p}{\partial n} = 0 & \text{for } x \in \partial\Omega(t), \end{cases} \end{aligned}$$

where v_n is the growing speed of $\partial\Omega(t)$ in the outer normal direction and H is the mean curvature of $\partial\Omega(t)$. Here and in what follows, μ denotes a finite positive Radon measure with compact support in $\Omega(0)$. Note that, for each fixed time $t > 0$, the maximum principle applied to the elliptic boundary problem in (1.5) yields that $p > 0$ everywhere on $\partial\Omega(t)$ if H is positive (see the proof of (2.2) in the next section). In other words, $\Omega(t)$ expands monotonically as long as the mean curvature of $\partial\Omega(t)$ is positive.

The following theorem shows that, as desired, for a given $\partial\Omega(0)$ as initial surface, the solution to (1.5) turns out to be a one-parameter family of quadrature surfaces. Moreover, we will see that (1.5) is the only possible flow having this property. Here,

we call $\{\partial\Omega(t)\}_{0 \leq t < T}$ a $C^{3+\alpha}$ family of surfaces if each $\partial\Omega(t)$ is of $C^{3+\alpha}$ and its time derivative is of $C^{2+\alpha}$, namely, $\partial\Omega(t)$ can be locally represented as a graph of a function in the Hölder space $C^{3+\alpha}$ and its time derivative is in $C^{2+\alpha}$ (see Section 3).

Theorem 1.2. *Let $\{\partial\Omega(t)\}_{0 \leq t < T}$ be a $C^{3+\alpha}$ family of surfaces, and assume that each $\partial\Omega(t)$ has positive mean curvature. Then, each $\partial\Omega(t)$ is a quadrature surface of $\mu(t) := t\mu + \mathcal{H}^{N-1}|_{\partial\Omega(0)}$, i.e.,*

$$(1.6) \quad \int_{\partial\Omega(0)} h d\mathcal{H}^{N-1} + t \int h d\mu = \int_{\partial\Omega(t)} h d\mathcal{H}^{N-1}$$

holds for all harmonic functions h defined in a neighborhood of $\overline{\Omega(t)}$, if and only if $\{\partial\Omega(t)\}_{0 \leq t < T}$ is a solution to (1.5).

Remark 1.3. The exponent $3 + \alpha$ naturally arises in the context of the Schauder theory for the oblique derivative problem (see Gilbarg & Trudinger [9]). Indeed, the regularity $H \in C^{1+\alpha}$ of the coefficient function H in the boundary condition is required for the existence of a solution $p \in C^{2+\alpha}(\overline{\Omega(t)})$ to the elliptic equation in (1.5). This implies that $\partial\Omega(t)$ is of $C^{3+\alpha}$. It is worth noting that, by taking appropriate coordinates, v_n can be regarded as the time derivative of a local function representation of $\partial\Omega(t)$. Hence, it is natural to impose the same regularity as $v_n = p \in C^{2+\alpha}$ on the time derivative of $\partial\Omega(t)$.

At this point, we are led to a fundamental question: Does the equation (1.5) really possess a unique smooth solution? The following theorem affirmatively answers this question. Here, $\{\partial\Omega(t)\}_{0 \leq t < T}$ is called a $h^{3+\alpha}$ solution if it is a $h^{3+\alpha}$ family of surfaces and satisfies (1.5), where $h^{3+\alpha}$ is the so-called little Hölder space and is defined as the closure of the Schwartz space $\mathcal{S}$ of rapidly decreasing functions in the topology of the Hölder space $C^{3+\alpha}$. Since our argument relies on the theory of maximal regularity of Da Prato and Grisvard [5], it is necessary to use $h^{3+\alpha}$, characterized as a continuous interpolation space, instead of $C^{3+\alpha}$.

Theorem 1.4. *There exists a unique $h^{3+\alpha}$ solution $\{\partial\Omega(t)\}_{0 \leq t < T}$ to (1.5) for any $h^{3+\alpha}$ initial surface $\partial\Omega(0)$ with positive mean curvature.*

Let us plot the points $(\Gamma, t) \in h^{3+\alpha} \times \mathbb{R}$ if Γ is a quadrature surface of $\mu(t)$. Theorem 1.4 shows that such points form a curve

$$t \mapsto (\partial\Omega(t), t) \quad (t \in [0, T))$$

in $h^{3+\alpha} \times \mathbb{R}$ starting from $(\partial\Omega(0), 0)$, if $\partial\Omega(0)$ has positive mean curvature. Moreover, as the parameter t increases, the curve does not split into two curves from any point $(\partial\Omega(t), t)$ unless $\partial\Omega(t)$ loses the positiveness of the mean curvature.

Corollary 1.5. *There is no curve*

$$s \mapsto (\Gamma(s), t(s)) \quad (s \in [0, \varepsilon))$$

of an $h^{3+\alpha}$ family of quadrature surfaces such that $(\Gamma(0), t(0)) = (\partial\Omega(0), 0)$, $\Gamma(s) \neq \partial\Omega(t(s))$ for $0 < s < \varepsilon$, and $t'(0) \geq 0$.

This paper is organized as follows. In Section 2 we prove Theorem 1.2, namely, we characterize (1.5) as a flow which produces a family of quadrature surfaces. Section 3 is devoted to proving Theorem 1.4. For this purpose, we reformulate the problem into an evolution equation in an infinite-dimensional Banach space, and proceed to the spectral analysis of the linearized operator. Finally, in section 4, we prove Corollary 1.5.

2 Generation of quadrature surfaces

In this section we show that the geometric flow (1.5) generates a family of quadrature surfaces.

We begin with a simple observation that the geometric flow remains unchanged by replacing the measure μ by the mollified measure $\tilde{\mu} := \eta_\varepsilon * \mu$, where η_ε is the standard symmetric mollifier supported on $\overline{B(0, \varepsilon)}$. Note that $\tilde{\mu}$ is then a smooth function supported in $\Omega(0)$ by taking $\varepsilon > 0$ small.

Lemma 2.1. *Let $\{\partial\Omega(t)\}_{0 \leq t < T}$ be a $C^{3+\alpha}$ solution to (1.5), and let $\{\widetilde{\partial\Omega(t)}\}_{0 \leq t < T}$ be a $C^{3+\alpha}$ solution to (1.5) with μ replaced by $\tilde{\mu}$ with the same initial surface $\partial\Omega(0) = \widetilde{\partial\Omega(0)}$. Assume moreover that $\partial\Omega(t)$ and $\widetilde{\partial\Omega(t)}$ have positive mean curvature. Then, $\partial\Omega(t) = \widetilde{\partial\Omega(t)}$ for all $0 < t < T$.*

Proof. It suffices to show that the boundary value of the solution p to the elliptic boundary problem

$$\begin{cases} -\Delta p = \mu & \text{for } x \in \Omega, \\ b_1(x)p + b_2(x)\frac{\partial p}{\partial n} = 0 & \text{for } x \in \partial\Omega \end{cases}$$

coincides with that of the solution $\tilde{p}$ to

$$\begin{cases} -\Delta \tilde{p} = \tilde{\mu} & \text{for } x \in \Omega, \\ b_1(x)\tilde{p} + b_2(x)\frac{\partial \tilde{p}}{\partial n} = 0 & \text{for } x \in \partial\Omega, \end{cases}$$

where $b_1(x)$, $b_2(x)$ are positive functions on $\partial\Omega$ and $\text{supp } \mu \subset \text{supp } \tilde{\mu} \subset \Omega$.

To this end, we prove that $q := p - \tilde{p}$ vanishes outside $\text{supp } \tilde{\mu}$. Let us decompose $q = F * (\mu - \tilde{\mu}) + h$, where F is the fundamental solution of $-\Delta$ (see (1.1)) and h is a harmonic function satisfying

$$(2.1) \quad \begin{cases} -\Delta h = 0 & \text{for } x \in \Omega, \\ b_1(x)h + b_2(x)\frac{\partial h}{\partial n} = -b_1(x)F * (\mu - \tilde{\mu}) - b_2(x)\frac{\partial F * (\mu - \tilde{\mu})}{\partial n} & \text{for } x \in \partial\Omega. \end{cases}$$

Then, it follows from the mean value property of harmonic functions that $F * (\mu - \tilde{\mu})$ vanishes outside $\text{supp } \tilde{\mu}$. Hence, the unique solvability of the oblique derivative problem (2.1) yields that $h \equiv 0$, which completes the proof. $\square$

We now proceed to the proof of Theorem 1.2.

Proof of Theorem 1.2. Let us first confirm that the positiveness of the mean curvature implies that

$$(2.2) \quad v_n = p > 0$$

everywhere on $\partial\Omega(t)$ for all $0 \leq t < T$. To see this, suppose that $p(\zeta_{\min}) = \min_{\zeta \in \partial\Omega(t)} p(\zeta) \leq 0$ for some $0 \leq t < T$ and $\zeta_{\min} \in \partial\Omega(t)$, and derive a contradiction. By the maximum principle applied to the elliptic equation in (1.5), we see that $p(\zeta_{\min}) < p(x)$ for all $x \in \Omega(t)$. Hence, from the Hopf boundary point lemma it follows that

$$(N-1)Hp(\zeta_{\min}) + \frac{\partial p}{\partial n}(\zeta_{\min}) < 0,$$

which violates the boundary condition. Note that (2.2) implies $\Omega(s) \subset \Omega(t)$ for $0 \leq s \leq t$.

Now recall that, by Lemma 2.1, we may replace the measure μ by $\tilde{\mu}$ in the equation (1.5). For each harmonic function h defined in a neighborhood of $\Omega(t)$, it follows from the well-known variational formulas for moving surfaces and domains that

$$\begin{aligned} \frac{d}{dt} \left[\int_{\partial\Omega(t)} h d\mathcal{H}^{N-1} \right] &= \int_{\partial\Omega(t)} \frac{\partial h}{\partial n} v_n d\mathcal{H}^{N-1} + (N-1) \int_{\partial\Omega(t)} h H v_n d\mathcal{H}^{N-1} \\ &= \int_{\partial\Omega(t)} \left\{ \frac{\partial h}{\partial n} p + (N-1)hHp \right\} d\mathcal{H}^{N-1} \\ &= \int_{\Omega(t)} (\Delta h p - h \Delta p) dx + \int_{\partial\Omega(t)} \left\{ h \frac{\partial p}{\partial n} + (N-1)hHp \right\} d\mathcal{H}^{N-1} \\ &= \int_{\Omega(t)} h \tilde{\mu} dx \\ &= \int h d\mu, \end{aligned}$$

where the last equality follows from the mean value property of harmonic functions. The integration with respect to t yields the identity (1.6).

Let us prove the converse statement. Differentiating the identity (1.6) with respect to t , we obtain that

$$\int h d\mu = \int_{\partial\Omega(t)} \left\{ \frac{\partial h}{\partial n} + (N-1)hH \right\} v_n d\mathcal{H}^{N-1}.$$

On the other hand, denoting p by a unique solution to the elliptic equation in (1.5), we have

$$\int h d\mu = \int_{\partial\Omega(t)} \left\{ \frac{\partial h}{\partial n} + (N-1)hH \right\} p d\mathcal{H}^{N-1}.$$

Hence,

$$(2.3) \quad \int_{\partial\Omega(t)} \left\{ \frac{\partial h}{\partial n} + (N-1)hH \right\} (v_n - p) d\mathcal{H}^{N-1} = 0$$

must hold for any harmonic function h defined in a neighborhood of $\overline{\Omega(t)}$. Let us denote by $h_0 \in C^{2+\alpha}(\overline{\Omega(t)})$ a unique solution to

$$\begin{cases} -\Delta h_0 = 0 & \text{for } x \in \Omega(t), \\ (N-1)Hh_0 + \frac{\partial h_0}{\partial n} = v_n - p & \text{for } x \in \partial\Omega(t). \end{cases}$$

If h_0 can be harmonically extended to a neighborhood of $\overline{\Omega(t)}$, then substituting $h = h_0$ into (2.3) deduces that $v_n = p$. But it is not the case in general, so let us take a sequence of solutions h_k to

$$\begin{cases} -\Delta h_k = 0 & \text{for } x \in \Omega_k, \\ (N-1)H_k h_k + \frac{\partial h_k}{\partial n} = q & \text{for } x \in \partial\Omega_k, \end{cases}$$

where $\Omega_k \supset \overline{\Omega(t)}$ is a sequence of bounded domains such that $\partial\Omega_k$ approaches $\partial\Omega(t)$ in the $C^{3+\alpha}$ sense, H_k is the mean curvature of $\partial\Omega_k$, and q is a $C^{1+\alpha}$ -extension of the function $v_n - p$ on $\partial\Omega(t)$ to $\mathbb{R}^N$, i.e., $q|_{\partial\Omega(t)} = v_n - p$. Then, the elliptic estimate

$$(2.4) \quad \|h_k\|_{C^{2+\alpha}(\overline{\Omega_k})} \leq C \left(\|h_k\|_{C^0(\overline{\Omega_k})} + \|q\|_{C^{1+\alpha}(\mathbb{R}^N)} \right) \leq C \|q\|_{C^{1+\alpha}(\mathbb{R}^N)}$$

holds uniformly in $k = 1, 2, \dots$, where the second inequality follows from the fact that

$$(2.5) \quad \|h_k\|_{C^0(\overline{\Omega_k})} \leq \max_{\partial\Omega_k} |h_k| \leq \frac{\max_{\partial\Omega_k} |q|}{(N-1) \min_{\partial\Omega_k} H_k}.$$

The proof of (2.5) is similar to that of (2.2). Now it can be shown by (2.4) together with the mean value theorem that

$$\sup_{\partial\Omega(t)} \left| \left\{ (N-1)Hh_k + \frac{\partial h_k}{\partial n} \right\} - (v_n - p) \right| \rightarrow 0.$$

Therefore, by taking $h = h_k$ with large k , we see that the identity (2.3) cannot hold unless $v_n = p$ on $\partial\Omega(t)$. $\square$

Remark 2.2. The identity (1.6) is still valid for subharmonic functions h by replacing equality with inequality $\leq$. Indeed, this follows from the positivity of p in $\Omega(t)$.

3 Existence of a solution to the geometric flow

In this section we describe the outline of the proof of Theorem 1.4. The complete proof can be found in Onodera [15], where a generalized flow which includes our flow (1.5) and the Hele-Shaw flow as special cases is studied. A direct method of the mathematical treatment of a geometric equation, which we will follow, is to reformulate the problem to a fixed boundary problem by using a time-dependent diffeomorphism such that the moving boundary transforms to a fixed reference boundary. Such a transformation makes clear the nonlinear nature of the original problem. Indeed, after the transformation, we encounter the situation where the evolution equation with fixed boundary turns out to be fully-nonlinear. The theory of maximal regularity of Da Prato and Grisvard [5] enables us to handle fully-nonlinear abstract parabolic equations by taking a continuous interpolation space as phase space. Thus, our effort will be made mainly to prove the “parabolicity” of the equation, namely, that the linearized operator is an infinitesimal generator of a strongly continuous analytic semigroup.

3.1 Reduction to an evolution equation

As a first step, let us reformulate the problem to an evolution equation in an abstract setting.

We fix a bounded reference domain Ω with smooth boundary Γ , and take a subdomain Ω_{sub} such that $\text{supp } \mu \subset \Omega_{\text{sub}} \subset \overline{\Omega_{\text{sub}}} \subset \Omega$. Let us recall that the little Hölder space $h^{k+\alpha}(\overline{\Omega})$ is defined as the closure of the Schwartz space $\mathcal{S}(\mathbb{R}^N)$ (restricted to Ω) in the topology of $C^{k+\alpha}(\overline{\Omega})$. The little Hölder space $h^{k+\alpha}(\Gamma)$ on the surface Γ can also be defined in the same manner in terms of its local coordinates. Let us define

$$\mathcal{U} = \mathcal{U}_a := \{\rho \in h^{3+\alpha}(\Gamma) \mid \|\rho\|_{C^1} < a\}$$

with $a > 0$ being sufficiently small such that $\theta(\zeta; r) := \zeta + rn_0(\zeta)$ defines a diffeomorphism between $\Gamma \times (-a, a)$ and its image though θ , where $n_0(\zeta)$ is the unit outer normal vector at $\zeta \in \Gamma$. In particular, for any $\rho \in \mathcal{U}$,

$$(3.1) \quad \Gamma_\rho := \{\zeta + \rho(\zeta)n_0(\zeta) \in \mathbb{R}^N \mid \zeta \in \Gamma\}$$

defines a $h^{3+\alpha}$ surface diffeomorphic to Γ though the diffeomorphism $\theta_\rho(\zeta) := \theta(\zeta, \rho(\zeta)) = \zeta + \rho(\zeta)n_0(\zeta)$ from Γ to Γ_ρ .

For the precise descriptions of the outer unit normal vector field n_ρ on Γ_ρ and a diffeomorphism from Ω to Ω_ρ , where Ω_ρ is the domain enclosed by Γ_ρ , we will use a level set representation of the surface Γ_ρ . Let us denote by ζ_0 and r_0 the components of the inverse map θ^{-1} such that $\theta^{-1}(x) = (\zeta_0(x), r_0(x))$. Note that $\zeta_0(x)$ is the nearest point on Γ to the point x , and $r_0(x)$ is the signed distance from Γ to x . It is then easy to see that

$$L_\rho(x) := r_0(x) - \rho(\zeta_0(x)) \quad (x \in \theta(\Gamma \times (-a, a)))$$

defines Γ_ρ as its 0-level set. This representation is now used to define the normal vector field $n_\rho \in h^{3+\alpha}(\Gamma, \mathbb{R}^N)$ and a diffeomorphism from Ω to Ω_ρ , which we denote again by θ_ρ , as follows:

$$n_\rho(\zeta) := \frac{\nabla L_\rho(\theta_\rho(\zeta))}{|\nabla L_\rho(\theta_\rho(\zeta))|},$$

$$\theta_\rho(x) := \begin{cases} \theta(\zeta_0(x), r_0(x) + \varphi(r_0(x))\rho(\zeta_0(x))) & (x \in \theta(\Gamma \times (-a, a))), \\ x & (x \notin \theta(\Gamma \times (-a, a))), \end{cases}$$

where φ is a smooth cut-off function satisfying

$$\varphi(r) := \begin{cases} 1 & (|r| \leq a/4), \\ 0 & (|r| \geq 3a/4) \end{cases} \quad \text{and} \quad \left| \frac{d\varphi}{dr}(r) \right| < \frac{4}{a}.$$

We also note that the speed v_n of the moving boundary at $\theta_\rho(\zeta) \in \Gamma_\rho$ can be represented by $(\partial\rho/\partial t)(\zeta)/|\nabla L_\rho(\theta_\rho(\zeta))|$.

The pull-back and push-forward operators induced by θ_ρ are defined by

$$\theta_\rho^* u := u \circ \theta_\rho, \quad \theta_\rho^\rho v := v \circ \theta_\rho^{-1}$$

for $u \in h^{k+\alpha}(\overline{\Omega}_\rho)$, $v \in h^{k+\alpha}(\overline{\Omega})$, respectively. Then it can be shown that θ_ρ^* , θ_ρ^ρ are isomorphisms between $h^{k+\alpha}(\overline{\Omega}_\rho)$ and $h^{k+\alpha}(\overline{\Omega})$, and $(\theta_\rho^*)^{-1} = \theta_\rho^\rho$. In the same fashion, θ_ρ^* , θ_ρ^ρ also denote isomorphisms between $h^{k+\alpha}(\Gamma_\rho)$ and $h^{k+\alpha}(\Gamma)$.

Given $\rho \in \mathcal{U}$, we now define transformed operators $A(\rho)$, $B(\rho)$ and $R(\rho)$ by

$$\begin{aligned} A(\rho) &:= \theta_\rho^*(-\Delta)\theta_\rho^\rho, \\ B(\rho)v &:= \text{Tr } \theta_\rho^* \langle \nabla \theta_\rho^\rho v, n_\rho \rangle, \\ R(\rho)v &:= (N-1)M_{H(\rho)} \text{Tr } v + B(\rho)v, \end{aligned}$$

where Tr and M_ψ are the trace operator and the pointwise multiplication operator defined by

$$\text{Tr } v(\zeta) := v(\zeta), \quad (M_\psi \psi)(\zeta) := \varphi(\zeta)\psi(\zeta) \quad (\zeta \in \Gamma)$$

for $v \in h^{k+\alpha}(\overline{\Omega})$ and $\varphi, \psi \in h^{k+\alpha}(\Gamma)$, respectively, and $H(\rho) \in h^{1+\alpha}(\Gamma)$ assigns the mean curvature of Γ_ρ at $\theta_\rho(\zeta)$ to the point $\zeta \in \Gamma$. Note also that here we have used the notation $\langle \cdot, \cdot \rangle$ to denote the pointwise inner product. It can be shown (see Escher & Simonett [7, 8]) that

$$\begin{aligned} A &\in C^\omega(\mathcal{U}, \mathcal{L}(h^{2+\alpha}(\overline{\Omega}), h^\alpha(\overline{\Omega}))), \\ B &\in C^\omega(\mathcal{U}, \mathcal{L}(h^{2+\alpha}(\overline{\Omega}), h^{1+\alpha}(\Gamma))), \\ R &\in C^\omega(\mathcal{U}, \mathcal{L}(h^{2+\alpha}(\overline{\Omega} \setminus \Omega_{\text{sub}}), h^{1+\alpha}(\Gamma))). \end{aligned}$$

In view of (3.1), the moving surface $\partial\Omega(t)$ can be represented by $\rho(t) = \rho(\cdot, t)$ which is a real-valued function defined on the fixed reference surface Γ . Hence, the

problem can be reduced to the following system of differential equations, in which unknowns are the functions ρ and u :

$$(3.2) \quad \partial_t \rho = M_{|\theta_\rho^*(\nabla L_\rho)|} \text{Tr} (\theta_\rho^* E + u)$$

$$(3.3) \quad \text{where} \quad \begin{cases} A(\rho)u = 0, \\ R(\rho)u = -R(\rho)\theta_\rho^* E. \end{cases}$$

Here, E is defined by

$$E(x) = E_\mu(x) := (F * \mu)(x),$$

and hence $-\Delta E = \mu$.

Furthermore, since u is determined only by ρ by virtue of the unique solvability of the elliptic equation (3.3) (see Gilbarg and Trudinger [9, Theorem 6.31]), the problem becomes a non-local evolution equation. To make it precise, let us define

$$\begin{aligned} S : \mathcal{U} &\rightarrow \mathcal{L}(h^\alpha(\bar{\Omega}), h^{2+\alpha}(\bar{\Omega})), & S(\rho)v &:= (A(\rho), R(\rho))^{-1}(v, 0), \\ T : \mathcal{U} &\rightarrow \mathcal{L}(h^{1+\alpha}(\Gamma), h^{2+\alpha}(\bar{\Omega})), & T(\rho)\varphi &:= (A(\rho), R(\rho))^{-1}(0, \varphi). \end{aligned}$$

Then, we see that $u = -T(\rho)R(\rho)\theta_\rho^* E$. Therefore, our problem is to solve the following evolution equation:

$$(3.4) \quad \partial_t \rho + \Phi(\rho) = 0,$$

where

$$\Phi : \mathcal{U} \rightarrow h^{1+\alpha}(\Gamma), \quad \Phi(\rho) := M_{|\theta_\rho^*(\nabla L_\rho)|} \text{Tr} (T(\rho)R(\rho) - I) \theta_\rho^* E.$$

Here, I is the identity map.

3.2 Linearized operator and its principal part

The theory of abstract evolution equations enables us to reduce the existence of a solution of (3.4) to the spectral properties of the linearized operator $\partial\Phi(\rho)$ of Φ at $\rho \in \mathcal{U}$. Indeed, once $\partial\Phi(\rho)$ is shown to be a sectorial operator, i.e., an infinitesimal generator of an analytic semigroup, then it follows from the theory of maximal regularity of Da Prato and Grisvard [5] that the equation (3.4) is uniquely solvable for initial data in a certain function space characterized as a continuous interpolation space.

By the implicit function theorem, we have the representation of the linearized operator $\partial T(\rho)$ of T at $\rho \in \mathcal{U}$ as follows.

Lemma 3.1. *For $\rho \in \mathcal{U}$ and $\varphi \in h^{1+\alpha}(\Gamma)$, let us set $v = v(\rho) := T(\rho)\varphi$, i.e., v satisfies*

$$\begin{cases} A(\rho)v = 0 & \text{in } \Omega, \\ R(\rho)v = \varphi & \text{on } \partial\Omega. \end{cases}$$

Then, the linearized operator $\partial v(\rho) \in \mathcal{L}(h^{3+\alpha}(\Gamma), h^{2+\alpha}(\overline{\Omega}))$ of v at ρ is given by

$$\partial v(\rho)[\tilde{\rho}] = \partial(T(\rho)\varphi)[\tilde{\rho}] = -S(\rho)\partial A(\rho)[\tilde{\rho}]T(\rho)\varphi - T(\rho)\partial R(\rho)[\tilde{\rho}]T(\rho)\varphi.$$

Moreover, $T \in C^\omega(\mathcal{U}, \mathcal{L}(h^{1+\alpha}(\Gamma), h^{2+\alpha}(\overline{\Omega})))$.

From the above lemma, we see that

$$\partial\Phi(\rho)[\tilde{\rho}] = M_{|\theta_\rho^*(\nabla L_\rho)|} \text{Tr } T(\rho)\partial R(\rho)[\tilde{\rho}] (I - T(\rho)R(\rho)) \theta_\rho^* E + F_1(\rho)[\tilde{\rho}] + F_2(\rho)[\tilde{\rho}] + F_3(\rho)[\tilde{\rho}],$$

where the linear operators

$$F_1(\rho)[\tilde{\rho}] := -M_{|\theta_\rho^*(\nabla L_\rho)|} \text{Tr } S(\rho)\partial A(\rho)[\tilde{\rho}]T(\rho)R(\rho)\theta_\rho^* E,$$

$$F_2(\rho)[\tilde{\rho}] := \partial M_{|\theta_\rho^*(\nabla L_\rho)|}[\tilde{\rho}] \text{Tr } (T(\rho)R(\rho) - I) \theta_\rho^* E,$$

$$F_3(\rho)[\tilde{\rho}] := M_{|\theta_\rho^*(\nabla L_\rho)|} \text{Tr } (T(\rho)R(\rho) - I) \partial(\theta_\rho^* E)[\tilde{\rho}]$$

can be thought of as perturbations in the sense that

$$\|F_j(\rho)[\tilde{\rho}]\|_{h^{2+\alpha}(\Gamma)} \leq C\|\tilde{\rho}\|_{h^{2+\alpha}(\Gamma)} \quad (j = 1, 2, 3),$$

where the constant C depends on $\rho \in \mathcal{U}$, but not on $\tilde{\rho} \in h^{3+\alpha}(\Gamma)$.

Moreover, the operator $\partial R(\rho)$ can also be decomposed further into the principal part and its perturbation. For this purpose, let us recall that the mean curvature operator $H = H(\rho)$ has a useful representation as in the following lemma. Here we take γ such that $\alpha < \gamma < 1$ and set

$$\mathcal{V} = \mathcal{V}_a := \{\rho \in h^{2+\gamma}(\Gamma) \mid \|\rho\|_{C^1} < a\}.$$

Lemma 3.2 (Escher & Simonett [7, Lemma 3.1]). *For each $\rho \in \mathcal{U}$, the mean curvature operator $H(\rho)$ can be decomposed as*

$$H(\rho) = P(\rho)\rho + K(\rho),$$

where $P \in C^\omega(\mathcal{V}, \mathcal{L}(h^{3+\alpha}(\Gamma), h^{1+\alpha}(\Gamma)))$ and $K \in C^\omega(\mathcal{V}, h^{1+\gamma}(\Gamma))$.

Hence, for $v \in h^{2+\alpha}(\overline{\Omega} \setminus \Omega_{\text{sub}})$, we have

$$\partial(R(\rho)v)[\tilde{\rho}] = (N-1)M_v P(\rho)[\tilde{\rho}] + F_4(\rho, v)[\tilde{\rho}],$$

where

$$\|F_4(\rho, v)[\tilde{\rho}]\|_{h^{1+\alpha}(\Gamma)} \leq C\|v\|_{h^{2+\alpha}(\Gamma)}\|\tilde{\rho}\|_{h^{2+\gamma}(\Gamma)}$$

with C being a constant independent of $\tilde{\rho}$. Therefore, the linearized operator $\partial\Phi(\rho)$ can now be represented in the following form:

$$\partial\Phi(\rho)[\tilde{\rho}] = (N-1)M_1(\rho)\text{Tr } T(\rho)M_2(\rho)P(\rho)[\tilde{\rho}] + F(\rho)[\tilde{\rho}],$$

where

$$M_1(\rho) := M_{|\theta_\rho^*(\nabla L_\rho)|} \in \mathcal{L}(h^{2+\alpha}(\Gamma)),$$

$$M_2(\rho) := M_{(I-T(\rho)R(\rho))\theta_\rho^* E} \in \mathcal{L}(h^{1+\alpha}(\Gamma)),$$

$$F(\rho) \in \mathcal{L}(h^{2+\gamma}(\Gamma), h^{2+\alpha}(\Gamma)).$$

3.3 The generation property of the linearized operator

Our task is now to prove that the linear operator

$$W = W(\rho) := -M_1(\rho)\mathrm{Tr} T(\rho)M_2(\rho)P(\rho) \in \mathcal{L}(h^{3+\alpha}(\Gamma), h^{2+\alpha}(\Gamma))$$

is sectorial in $h^{2+\alpha}(\Gamma)$, i.e., it generates an analytic semigroup on $h^{2+\alpha}(\Gamma)$. Indeed, a standard perturbation result of sectorial operators implies that, if W is sectorial, then $-\partial\Phi(\rho)$ is also sectorial. The following theorem is the main assertion in this section.

Theorem 3.3. $W \in \mathcal{L}(h^{3+\alpha}(\Gamma), h^{2+\alpha}(\Gamma))$ is sectorial in $h^{3+\alpha}(\Gamma)$.

Corollary 3.4. $-\partial\Phi(\rho) \in \mathcal{L}(h^{3+\alpha}(\Gamma), h^{2+\alpha}(\Gamma))$ is sectorial in $h^{3+\alpha}(\Gamma)$.

To prove Theorem 3.3, it is well-known (see Amann [2]) that W is sectorial if there exist positive constants λ_* and C such that

- (i) $\lambda_* I - W \in \mathcal{L}(h^{3+\alpha}(\Gamma), h^{2+\alpha}(\Gamma))$ is bijective, i.e., λ_* is in the resolvent set.
- (ii) $|\lambda| \|\tilde{\rho}\|_{h^{2+\alpha}(\Gamma)} + \|\tilde{\rho}\|_{h^{3+\alpha}(\Gamma)} \leq C \|(\lambda I - W)\tilde{\rho}\|_{h^{2+\alpha}(\Gamma)}$ holds for $\tilde{\rho} \in h^{3+\alpha}(\Gamma)$ and $\lambda \in \{z \in \mathbb{C} \mid \mathrm{Re} z \geq \lambda_*\}$.

Let us first confirm the condition (i) by assuming (ii). Since (ii) implies that $\lambda_* I - W$ is injective, we only need to prove that it is also surjective. Note that $\mathcal{U}$ is star-shaped with respect to 0 in $h^{3+\alpha}(\Gamma)$ and $\mathcal{K} := \{t\rho \in \mathcal{U} \mid 0 \leq t \leq 1\}$ is a compact subset in $\mathcal{U}$. Hence, from the continuity of the map $\rho \mapsto W = W(\rho)$ it follows that the constant C in the resolvent estimate (ii) can be chosen uniformly in $\rho \in \mathcal{K}$. Therefore, by the continuity method (see Gilbarg & Trudinger [9, Theorem 5.2]) together with the uniform resolvent estimate (ii), it is sufficient to show that $\lambda_* I - W$ is surjective in the case $\rho = 0$.

Then, it is known that

$$(3.5) \quad P(0) = -\frac{1}{N-1} \Delta_\pi^\Gamma,$$

where Δ_π^Γ is the principal part of the Laplace-Beltrami operator with respect to Γ . Moreover, we have

$$(3.6) \quad v := (I - T(0)R(0))E > 0$$

everywhere on Γ . This can be verified in the same way as (2.2), since v satisfies

$$\begin{cases} -\Delta v = \mu, \\ R(0)v = 0. \end{cases}$$

Now (3.5) and (3.6) imply that

$$I + M_2(0)P(0) = I + M_{(I-T(0)R(0))E}P(0) \in \mathcal{L}(h^{3+\alpha}(\Gamma), h^{1+\alpha}(\Gamma))$$

is a bijective operator having bounded inverse.

Note also that

$$M_1(0)\mathrm{Tr} T(0) = M_{|\nabla L_0|}\mathrm{Tr} T(0) \in \mathcal{L}(h^{1+\alpha}(\Gamma), h^{2+\alpha}(\Gamma))$$

is bijective. This follows from $|\nabla L_0| > 0$ and the unique solvability of the oblique derivative problem in the Hölder spaces (see Gilbarg & Trudinger [9, Theorem 6.31]).

In the expression

$$\lambda_* I - W = M_1(0)\mathrm{Tr} T(0) \{I + M_2(0)P(0)\} + \lambda_* I - M_1(0)\mathrm{Tr} T(0),$$

the second and third operators in the right hand side are compact perturbations, since the embedding $h^{3+\alpha}(\Gamma) \hookrightarrow h^{2+\alpha}(\Gamma)$ is compact. Furthermore, as we have already seen, the first one is a bijective operator from $h^{3+\alpha}(\Gamma)$ to $h^{2+\alpha}(\Gamma)$. Therefore, $\lambda_* I - W$ is a Fredholm operator of index 0. Now the assertion follows from the fact that $\lambda_* I - W$ is injective.

We will establish the remaining resolvent estimate (ii) in the following sections.

3.4 Fourier multiplier operators associated with localized operators

Let us take an atlas $\{U_l, \psi_l\}_{1 \leq l \leq m}$ of $R_d := \theta(\Gamma \times (-d, 0])$ for small $0 < d < a/4$ such that $\mathrm{diam} U_l < d$ and that ψ_l maps $Q := (-d, d)^{N-1} \times [0, d)$, $Q_0 := (-d, d)^{N-1} \times \{0\}$ onto U_l , $U_l \cap \Gamma$, respectively. Note that the number of local coordinates m depends on d .

Localizing the operator W to each U_l , and choosing an appropriate constant coefficient operator on $\mathbb{R}^{N-1}$ which approximates W in that localized region U_l , we will show that this constant coefficient operator has a representation as a Fourier multiplier operator, and moreover that it generates an analytic semigroup in an appropriate Banach space, namely, the little Hölder space $h^{2+\alpha}(\mathbb{R}^{N-1})$. The latter will be established by applying a general result due to H. Amann, which states that, for given $\sigma \in \mathcal{EL}S_1^\infty(\gamma_*)$, $\gamma_* > 0$ and $\eta_0 > 0$, it follows that

$$\Sigma_{\eta_0} := -\mathcal{F}^{-1} \mathcal{M}_{\sigma(\cdot, \eta_0)} \mathcal{F} \in \mathcal{L}(h^{3+\alpha}(\mathbb{R}^{N-1}), h^{2+\alpha}(\mathbb{R}^{N-1}))$$

is sectorial, i.e., it generates a strongly continuous analytic semigroup on $h^{2+\alpha}(\mathbb{R}^{N-1})$. Here, $\sigma \in \mathcal{EL}S_1^\infty(\gamma_*)$ if $\sigma = \sigma(\xi, \eta) \in C^\infty(\mathbb{R}^{N-1} \times (0, \infty))$ is positively homogeneous of degree one and its all derivatives are bounded on the set $\{|\xi|^2 + \eta^2 = 1\}$ and if

$$(3.7) \quad \mathrm{Re} \sigma(\xi, \eta) \geq \gamma_* \sqrt{|\xi|^2 + \eta^2} \quad ((\xi, \eta) \in \mathbb{R}^{N-1} \times (0, \infty))$$

holds. The linear operator $\mathcal{M}_\phi$ with a given function ϕ on $\mathbb{R}^{N-1}$ is the localized version of the pointwise multiplication operator induced by ϕ .

Let us fix $\rho \in \mathcal{U}$ and $(U, \psi) = (U_l, \psi_l)$ for some $l = 1, \dots, m$, and define the pull-back and push-forward operators induced by ψ by

$$\psi^* u := u \circ \psi, \quad \psi_* v := v \circ \psi^{-1}$$

for $u \in h^{k+\alpha}(\overline{U})$, $v \in h^{k+\alpha}(\overline{Q})$, respectively. We then introduce local representations $\mathcal{A}$, $\mathcal{R}$ and $\mathcal{P}$ of the operators $A(\rho)$, $R(\rho)$ and $P(\rho)$ defined by

$$\mathcal{A} := \psi^* A(\rho) \psi_*, \quad \mathcal{R} := \psi^* R(\rho) \psi_*, \quad \mathcal{P} := \psi^* P(\rho) \psi_*.$$

In what follows, for simplicity, we write

$$\partial_j := \frac{\partial}{\partial \omega_j} \quad (j = 1, \dots, N-1), \quad \partial_N := \frac{\partial}{\partial r}.$$

As shown in Escher & Simonett [7, Lemma 3.2] and [8, Lemma 3.1], we have

$$\begin{aligned} \mathcal{A} &= - \sum_{j,k=1}^N a_{jk}(\rho) \partial_j \partial_k + \sum_{j=1}^N a_j(\rho) \partial_j, \\ \mathcal{R} &= b_0(\rho) \text{Tr} - \sum_{j=1}^N b_j(\rho) \text{Tr} \partial_j, \\ \mathcal{P} &= - \sum_{j,k=1}^{N-1} p_{jk}(\rho) \partial_j \partial_k \end{aligned}$$

where $a_{jk} \in C^\omega(\mathcal{U}, h^{2+\alpha}(Q))$, $a_j \in C^\omega(\mathcal{U}, h^{1+\alpha}(Q))$, $b_j \in C^\omega(\mathcal{U}, h^{2+\alpha}(Q_0))$ and $p_{jk} \in C^\omega(\mathcal{U}, h^{2+\alpha}(Q_0))$, and we used the same notation Tr to denote the trace operator on Q_0 . Moreover, the matrices $(a_{jk}(\rho)(\omega, r))$, $(p_{jk}(\rho)(\omega))$ are symmetric and uniformly positive definite on Q , Q_0 , respectively, and $b_0(\rho)$, $b_N(\rho)$ are uniformly positive on Q_0 . Here, we may further assume that

$$b_j(\rho) = 0 \quad (j = 1, \dots, N-1).$$

Indeed, the validity of this assumption is guaranteed by taking the diffeomorphisms ψ_l so that each $\theta_\rho \circ \psi_l$ preserves the normal directions to the corresponding boundaries, namely,

$$\partial_N(\theta_\rho \circ \psi_l) = D(\theta_\rho \circ \psi_l) e_N = -s(n_\rho \circ \psi_l)$$

holds with some positive number s at each point on Q_0 , where $e_N := {}^t(0, \dots, 0, 1)$. For the construction of such a diffeomorphism, we refer to Ni & Takagi [14].

We are now in a position to introduce associated constant coefficient operators. By setting

$$a_{jk}^0 := a_{jk}(\rho)(0, 0), \quad b_j^0 := b_j(\rho)(0), \quad p_{jk}^0 = p_{jk}(\rho)(0),$$

let us define

$$\begin{aligned} \mathcal{A}_0 &:= - \sum_{j,k=1}^N a_{jk}^0 \partial_j \partial_k, \\ \mathcal{R}_0 &:= b_0^0 \text{Tr} - b_N^0 \text{Tr} \partial_N, \\ \mathcal{P}_0 &:= I - \sum_{j,k=1}^{N-1} p_{jk}^0 \partial_j \partial_k. \end{aligned}$$

The constant coefficient operator $\mathcal{T}_0$ associated with $T(\rho)$ will be defined such that, for $\varphi \in h^{1+\alpha}(\mathbb{R}^{N-1})$, $v := \mathcal{T}_0\varphi \in h^{2+\alpha}(\mathbb{R}^{N-1} \times [0, \infty))$ and v satisfies

$$(3.8) \quad \begin{cases} (I + \mathcal{A}_0)v = 0 & \text{in } \mathbb{R}^{N-1} \times (0, \infty), \\ \mathcal{R}_0 v = \varphi & \text{on } \mathbb{R}^{N-1} \simeq \mathbb{R}^{N-1} \times \{0\}. \end{cases}$$

To derive an explicit representation of $\mathcal{T}_0$, we set

$$z(\xi) := \frac{i}{a_{NN}^0} \sum_{j=1}^{N-1} a_{jN}^0 \xi_j + \frac{1}{a_{NN}^0} \sqrt{a_{NN}^0 \left(1 + \sum_{j,k=1}^{N-1} a_{jk}^0 \xi_j \xi_k \right) - \left(\sum_{j=1}^{N-1} a_{jN}^0 \xi_j \right)^2},$$

where $i := \sqrt{-1}$. Then, $z = z(\xi)$ is a solution to the quadratic equation

$$1 + \sum_{j,k=1}^{N-1} a_{jk}^0 \xi_j \xi_k + 2i \left(\sum_{j=1}^{N-1} a_{jN}^0 \xi_j \right) z - a_{NN}^0 z^2 = 0$$

and satisfies $\operatorname{Re} z(\xi) > 0$ by the ellipticity of (a_{jk}^0) . Denoting by $\mathcal{F}$ and $\mathcal{F}^{-1}$ the (partial) Fourier transform and the inverse (partial) Fourier transform on $\mathbb{R}^{N-1}$, respectively, we have an explicit representation formula of the solution operator $\mathcal{T}_0$ as the following lemma shows.

Lemma 3.5. *Let $\mathcal{T}_0$ be defined by*

$$\begin{aligned} \mathcal{T}_0\varphi(\omega, r) &:= [\mathcal{F}^{-1} \mathcal{M}_{\sigma_1(\cdot, r)} \mathcal{F} \varphi](\omega), \\ \sigma_1(\xi, r) &:= \frac{e^{-z(\xi)r}}{b_0^0 + b_N^0 z(\xi)}. \end{aligned}$$

Then, $\mathcal{T}_0 \in \mathcal{L}(h^{1+\alpha}(\mathbb{R}^{N-1}), h^{2+\alpha}(\mathbb{R}^{N-1} \times [0, \infty)))$ and, for any $\varphi \in h^{1+\alpha}(\mathbb{R}^{N-1})$, $v := \mathcal{T}_0\varphi$ is the unique solution to (3.8) in $h^{2+\alpha}(\mathbb{R}^{N-1} \times [0, \infty))$.

Proof. By a direct computation, it is easy to see that $v := \mathcal{T}_0\varphi$ satisfies (3.8) for smooth φ . Moreover, $\mathcal{T}_0 \in \mathcal{L}(h^{1+\alpha}(\mathbb{R}^{N-1}), h^{2+\alpha}(\mathbb{R}^{N-1} \times [0, \infty)))$ follows from the decomposition

$$\mathcal{T}_0\varphi(\omega, r) = [(\mathcal{F}^{-1} \mathcal{M}_{\sigma_{1,1}(\cdot, r)} \mathcal{F})(\mathcal{F}^{-1} \mathcal{M}_{\sigma_{1,2}} \mathcal{F})](\omega),$$

where

$$\sigma_{1,1}(\xi, r) := e^{-z(\xi)r}, \quad \sigma_{1,2}(\xi) := (b_0^0 + b_N^0 z(\xi))^{-1}.$$

Indeed, $\mathcal{F}^{-1} \mathcal{M}_{\sigma_{1,1}(\cdot, r)} \mathcal{F} \in \mathcal{L}(h^{2+\alpha}(\mathbb{R}^{N-1}), h^{2+\alpha}(\mathbb{R}^{N-1} \times [0, \infty)))$ can be checked as in Escher & Simonett [6, Lemma B.2], and also it is easy to prove that $\mathcal{F}^{-1} \mathcal{M}_{\sigma_{1,2}} \mathcal{F} \in \mathcal{L}(h^{1+\alpha}(\mathbb{R}^{N-1}), h^{2+\alpha}(\mathbb{R}^{N-1}))$ in view of Escher & Simonett [6, Theorem A.1]. For the uniqueness of a solution, it suffices to show that any solution $v \in h^{2+\alpha}(\mathbb{R}^{N-1} \times [0, \infty))$ of

$$\begin{cases} (I + \mathcal{A}_0)v = 0 & \text{in } \mathbb{R}^{N-1} \times (0, \infty), \\ \mathcal{R}_0 v = 0 & \text{on } \mathbb{R}^{N-1}. \end{cases}$$

must be identical with the trivial solution $v \equiv 0$. By virtue of the Phragmén-Lindelöf principle, this can be reduced to showing that $v = 0$ on the boundary $\mathbb{R}^{N-1}$. Let us prove that $v \leq 0$ on $\mathbb{R}^{N-1}$ by assuming

$$c := \sup_{\omega \in \mathbb{R}^{N-1}} v(\omega, 0) > 0$$

and deriving a contradiction. For any $\omega \in \mathbb{R}^{N-1}$ and $r > 0$, observe that

$$\begin{aligned} v(\omega, 0) + \frac{b_0^0}{b_N^0} r v(\omega, 0) - v(\omega, r) &= v(\omega, 0) + r \partial_N v(\omega, 0) - v(\omega, r) \\ &= \int_0^r (\partial_N v(\omega, 0) - \partial_N v(\omega, s)) ds \\ &\leq \frac{r^2}{2} \|v\|_{h^{2+\alpha}(\mathbb{R}^{N-1} \times [0, \infty))}. \end{aligned}$$

Thus, by choosing a sufficiently small $\varepsilon > 0$ and $\omega \in \mathbb{R}^{N-1}$ such that $v(\omega, 0) > c - \varepsilon$, we see that

$$\begin{aligned} v(\omega, r) &\geq v(\omega, 0) + \frac{b_0^0}{b_N^0} r v(\omega, 0) - \frac{r^2}{2} \|v\|_{h^{2+\alpha}(\mathbb{R}^{N-1} \times [0, \infty))} \\ &> c - \varepsilon + \frac{b_0^0}{b_N^0} r (c - \varepsilon) - \frac{r^2}{2} \|v\|_{h^{2+\alpha}(\mathbb{R}^{N-1} \times [0, \infty))} \\ &> c, \end{aligned}$$

where the last inequality is valid for $\varepsilon > 0$ and $r \in (0, 1)$ such that

$$r \left(\frac{b_0^0}{b_N^0} c - \frac{r}{2} \|v\|_{h^{2+\alpha}(\mathbb{R}^{N-1} \times [0, \infty))} \right) > \varepsilon \left(1 + \frac{b_0^0}{b_N^0} r \right).$$

and the existence of such a pair of ε and r can be easily checked. However, recalling that the Phragmén-Lindelöf principle yields $v(\omega, r) < c$ for all $\omega \in \mathbb{R}^{N-1}$ and $r > 0$, we are now arriving at a contradiction and thus $v \leq 0$ is proved. The inequality $v \geq 0$ can be proved by a similar argument. $\square$

For later use, we also provide the solution operator $\mathcal{S}_0$ of the following boundary value problem:

$$(3.9) \quad \begin{cases} (I + \mathcal{A}_0)v = f & \text{in } \mathbb{R}^{N-1} \times (0, \infty), \\ \mathcal{R}_0 v = 0 & \text{on } \mathbb{R}^{N-1}. \end{cases}$$

In what follows, we write $\mathcal{F}_N$ and $\mathcal{F}_N^{-1}$ for the Fourier transform and the inverse Fourier transform on $\mathbb{R}^N$, respectively, and $\mathcal{E} \in \mathcal{L}(h^\alpha(\mathbb{R}^{N-1} \times [0, \infty)), h^\alpha(\mathbb{R}^N))$ denotes an extension operator, i.e., $\mathcal{E}f = f$ on $\mathbb{R}^{N-1} \times [0, \infty)$.

Lemma 3.6. *Let $\mathcal{S}_0$ be defined by*

$$\begin{aligned}\mathcal{S}_0 f(\omega, r) &:= (I - \mathcal{T}_0 \mathcal{R}_0) \{ \mathcal{F}_N^{-1} \mathcal{M}_{\sigma_2} \mathcal{F}_N \mathcal{E} f \} \lfloor_{\mathbb{R}^{N-1} \times [0,1]}, \\ \sigma_2(\xi) &:= \left(1 + \sum_{j,k=1}^N a_{jk}^0 \xi_j \xi_k \right)^{-1}.\end{aligned}$$

Then, $\mathcal{S}_0 \in \mathcal{L}(h^\alpha(\mathbb{R}^{N-1} \times [0, \infty)), h^{2+\alpha}(\mathbb{R}^{N-1} \times [0, \infty)))$ and, for any $f \in h^\alpha(\mathbb{R}^{N-1} \times [0, \infty))$, $v := \mathcal{S}_0 f$ is the unique solution to (3.9) in $h^{2+\alpha}(\mathbb{R}^{N-1} \times [0, \infty))$.

Proof. A direct computation shows that $v := \mathcal{S}_0 f$ satisfies (3.9) for smooth f . Moreover, Lemma 3.5 and the facts that

$$\begin{aligned}\mathcal{R}_0 &\in \mathcal{L}(h^{2+\alpha}(\mathbb{R}^{N-1} \times [0, \infty)), h^{1+\alpha}(\mathbb{R}^{N-1})), \\ \mathcal{F}_N^{-1} \mathcal{M}_{\sigma_2} \mathcal{F}_N &\in \mathcal{L}(h^\alpha(\mathbb{R}^N), h^{2+\alpha}(\mathbb{R}^N))\end{aligned}$$

yield the desired conclusion $\mathcal{S}_0 \in \mathcal{L}(h^\alpha(\mathbb{R}^{N-1} \times [0, \infty)), h^{2+\alpha}(\mathbb{R}^{N-1} \times [0, \infty)))$. The uniqueness of a solution again follows from the Phragmén-Lindelöf principle. $\square$

Finally, by setting

$$\begin{aligned}m_1 &:= \psi^* |\theta_\rho^*(\nabla L_\rho)| (0, 0) > 0, \\ m_2 &:= \psi^* \{ (I - T(\rho) R(\rho)) \theta_\rho^* E \} (0, 0) > 0,\end{aligned}$$

we define $\mathcal{W}_0$ by

$$\begin{aligned}\mathcal{W}_0 &:= -m_1 m_2 \text{Tr } \mathcal{T}_0 \mathcal{P}_0 \\ &= -\mathcal{F}^{-1} \mathcal{M}_\sigma \mathcal{F},\end{aligned}$$

where

$$\sigma(\xi) := \frac{m_1 m_2 \left(1 + \sum_{j,k=1}^{N-1} p_{jk}^0 \xi_j \xi_k \right)}{b_0^0 + b_N^0 z(\xi)}.$$

Then, we have the following proposition.

Proposition 3.7. $\mathcal{W}_0 \in \mathcal{L}(h^{3+\alpha}(\mathbb{R}^{N-1}), h^{2+\alpha}(\mathbb{R}^{N-1}))$ is sectorial.

Proof. Let us define the parametrized symbol $\tilde{\sigma}$ by

$$\tilde{\sigma}(\xi, \eta) := \frac{m_1 m_2 \left(\eta^2 + \sum_{j,k=1}^{N-1} p_{jk}^0 \xi_j \xi_k \right)}{b_0^0 \eta + b_N^0 \tilde{z}(\xi, \eta)},$$

where

$$\tilde{z}(\xi, \eta) := \frac{i}{a_{NN}^0} \sum_{j=1}^{N-1} a_{jN}^0 \xi_j + \frac{1}{a_{NN}^0} \sqrt{a_{NN}^0 \left(\eta^2 + \sum_{j,k=1}^{N-1} a_{jk}^0 \xi_j \xi_k \right) - \left(\sum_{j=1}^{N-1} a_{jN}^0 \xi_j \right)^2}.$$

Note that $\tilde{z}(\xi, 1) = z(\xi)$ and hence $\tilde{\sigma}(\xi, 1) = \sigma(\xi)$. We show that $\tilde{\sigma} \in \mathcal{EllS}_1^\infty(\gamma_*)$ with some positive number γ_* . Indeed, it is easy to see that $\tilde{\sigma} \in C^\infty(\mathbb{R}^{N-1} \times (0, \infty))$, and it is positively homogeneous of degree one, and its all derivatives are bounded on $\{|\xi|^2 + \eta^2 = 1\}$. To check the condition (3.7), let a_* , p_* denote the ellipticity constants for $\mathcal{A}_0$, $\mathcal{P}_0$, i.e.,

$$(3.10) \quad \sum_{j,k=1}^{N-1} a_{jk}^0 \xi_j \xi_k + 2\tilde{\eta} \sum_{j=1}^{N-1} a_{jN}^0 \xi_j + a_{NN}^0 \tilde{\eta}^2 \geq a_* (|\xi|^2 + \tilde{\eta}^2),$$

$$(3.11) \quad \sum_{j,k=1}^{N-1} p_{jk}^0 \xi_j \xi_k \geq p_* |\xi|^2.$$

Then, in particular, by taking $\tilde{\eta} = -(a_{NN}^0)^{-1} \sum_{j=1}^{N-1} a_{jN}^0 \xi_j$ in (3.10), we have

$$\sum_{j,k=1}^{N-1} a_{jk}^0 \xi_j \xi_k - \frac{1}{a_{NN}^0} \left(\sum_{j=1}^{N-1} a_{jN}^0 \xi_j \right)^2 \geq a_* |\xi|^2,$$

and hence

$$(3.12) \quad \begin{aligned} \operatorname{Re} \tilde{z}(\xi, \eta) &\geq \frac{1}{a_{NN}^0} \sqrt{a_{NN}^0 (\eta^2 + a_* |\xi|^2)} \\ &\geq \sqrt{\frac{\min\{1, a_*\}}{a_{NN}^0}} \sqrt{|\xi|^2 + \eta^2} \end{aligned}$$

We also observe that

$$(3.13) \quad \begin{aligned} |b_0^0 \eta + b_N^0 \tilde{z}(\xi, \eta)|^2 &\leq 2b_0^0 \eta^2 + 2b_N^0 |\tilde{z}(\xi, \eta)|^2 \\ &\leq 2b_N^0 \left(\sum_{j,k=1}^{N-1} a_{jk}^0 \right) |\xi|^2 + 2(b_0^0 + b_N^0) \eta^2. \end{aligned}$$

Therefore, combining (3.11), (3.12) and (3.13), we deduce that

$$\begin{aligned} \operatorname{Re} \tilde{\sigma}(\xi, \eta) &= \frac{m_1 m_2 \left(\eta^2 + \sum_{j,k=1}^{N-1} p_{j,k}^0 \xi_j \xi_k \right) (b_0^0 \eta + b_N^0 \operatorname{Re} \tilde{z}(\xi, \eta))}{|b_0^0 \eta + b_N^0 \tilde{z}(\xi, \eta)|^2} \\ &\geq \frac{m_1 m_2 (\eta^2 + p_* |\xi|^2) \left(b_0^0 \eta + b_N^0 \sqrt{\frac{\min\{1, a_*\}}{a_{NN}^0}} \sqrt{|\xi|^2 + \eta^2} \right)}{2b_N^0 \left(\sum_{j,k=1}^{N-1} a_{jk}^0 \right) |\xi|^2 + 2(b_0^0 + b_N^0) \eta^2} \\ &\geq \gamma_* \sqrt{|\xi|^2 + \eta^2}, \end{aligned}$$

where

$$\gamma_* := \frac{m_1 m_2 b_N^0 \min\{1, p_*\} \sqrt{\min\{1, a_*\}}}{2\sqrt{a_{NN}^0} \max \left\{ b_N^0 \left(\sum_{j,k=1}^{N-1} a_{jk}^0 \right), b_0^0 + b_N^0 \right\}} > 0.$$

Therefore, $\tilde{\sigma} \in \mathcal{E}ll\mathcal{S}_1^\infty(\gamma_*)$, and hence

$$\mathcal{W}_0 = -\mathcal{F}^{-1} \mathcal{M}_{\tilde{\sigma}(\cdot, 1)} \mathcal{F}$$

is a sectorial operator on $h^{2+\alpha}(\mathbb{R}^{N-1})$. $\square$

3.5 Resolvent estimate by a perturbation argument

Proposition 3.7 implies that the operator $\mathcal{W}_0^{(l)} = \mathcal{W}_0$, which approximates W in the localized region U_l , satisfies the resolvent estimate

$$(3.14) \quad |\lambda| \|\tilde{\rho}\|_{h^{2+\alpha}(\mathbb{R}^{N-1})} + \|\tilde{\rho}\|_{h^{3+\alpha}(\mathbb{R}^{N-1})} \leq C \|(\lambda I - \mathcal{W}_0^{(l)})\tilde{\rho}\|_{h^{2+\alpha}(\mathbb{R}^{N-1})}$$

for any $\tilde{\rho} \in h^{3+\alpha}(\mathbb{R}^{N-1})$ and $\lambda \in \{z \in \mathbb{C} \mid \operatorname{Re} z \geq \lambda_0\}$, by taking $\lambda_0 > 0$ and $C > 0$ appropriately.

We will show that $\mathcal{W}_0^{(l)}$ indeed approximates W by taking $d > 0$ so small that the atlas $\{U_l, \psi_l\}_{1 \leq l \leq m}$ of R_d becomes fine enough (see the beginning of Section 3.4) in the sense that the desired resolvent estimate

$$(3.15) \quad |\lambda| \|\tilde{\rho}\|_{h^{2+\alpha}(\Gamma)} + \|\tilde{\rho}\|_{h^{3+\alpha}(\Gamma)} \leq C \|(\lambda I - W)\tilde{\rho}\|_{h^{2+\alpha}(\Gamma)}$$

holds after patching all the local estimates together. This estimate completes the proof of Theorem 3.3.

For this purpose, we take a partition of unity $\{\phi_l\}_{l=1}^m$ associated with $\{U_l\}_{l=1}^m$ such that $\operatorname{supp} \phi_l \subset U_l$ and $\bigcup_{l=1}^m \phi_l = 1$ on $R_{d/2}$. Combining the atlas and the partition of unity, we call such a pair a localization sequence of R_d . Note that, we can choose a family of smooth cut-off functions $\{\chi_l\}_{l=1}^m$ as well as a localization sequence of R_d such that $\operatorname{supp} \chi_l \subset U_l$, $\chi_l = 1$ on $\operatorname{supp} \phi_l$ and

$$(3.16) \quad \|\chi_l\|_{0, U_l} + d^\alpha [\chi_l]_{\alpha, U_l} \leq C$$

with a positive constant C which is independent of d . Here and in what follows, we use the notation

$$\begin{aligned} \|v\|_{k+\alpha, U} &:= \|v\|_{h^{k+\alpha}(U)}, \quad [v]_{\alpha, U} := \sup_{\substack{x, y \in U \\ x \neq y}} \frac{|v(x) - v(y)|}{|x - y|^\alpha}, \\ \|v\|_{k+\alpha} &:= \|v\|_{k+\alpha, \mathbb{R}^{N-1}}, \quad [v]_\alpha := [v]_{\alpha, \mathbb{R}^{N-1}}. \end{aligned}$$

Now we state the following perturbation result.

Lemma 3.8. *For any $\varepsilon > 0$, $0 < \beta < \alpha$ and $\rho \in \mathcal{U}$, there are $d > 0$, a localization sequence of R_d , and a constant $C = C(\varepsilon, \beta, \rho, d)$ such that*

$$\left\| \psi_l^* (\phi_l W \tilde{\rho}) - \mathcal{W}_0^{(l)} \psi_l^* (\phi_l \tilde{\rho}) \right\|_{2+\alpha} \leq \varepsilon \|\psi_l^* (\phi_l \tilde{\rho})\|_{3+\alpha} + C \|\tilde{\rho}\|_{3+\beta, \Gamma}$$

holds for $\tilde{\rho} \in h^{3+\alpha}(\Gamma)$ and $1 \leq l \leq m$.

The proof is straightforward, but lengthy. The detail can be found in Onodera [15]. Let us now complete the proof of Theorem 3.3.

Proof of Theorem 3.3. We only need to prove the resolvent estimate (3.15). For simplicity, we will denote $C > 0$ a generic constant. Combining (3.14) and Lemma 3.8 with sufficiently small $\varepsilon > 0$, we see that

$$\begin{aligned} |\lambda| \|\psi_l^* (\phi_l \tilde{\rho})\|_{2+\alpha} + \|\psi_l^* (\phi_l \tilde{\rho})\|_{3+\alpha} &\leq C \|(\lambda I - \mathcal{W}_0^{(l)}) \psi_l^* (\phi_l \tilde{\rho})\|_{2+\alpha} \\ &\leq C (\|\psi_l^* (\phi_l (\lambda I - W) \tilde{\rho})\|_{2+\alpha} + \|\tilde{\rho}\|_{3+\beta, \Gamma}) \end{aligned}$$

holds for any $\tilde{\rho} \in h^{3+\alpha}(\Gamma)$, $\lambda \in \{z \in \mathbb{C} \mid \operatorname{Re} z \geq \lambda_0\}$, and $1 \leq l \leq m$. Since

$$\tilde{\rho} \mapsto \max_{1 \leq l \leq m} \|\psi_l^* (\phi_l \tilde{\rho})\|_{k+\alpha}$$

defines an equivalent norm on $h^{k+\alpha}(\Gamma)$ ($k = 2, 3$), the above inequality implies

$$|\lambda| \|\tilde{\rho}\|_{2+\alpha, \Gamma} + \|\tilde{\rho}\|_{3+\alpha, \Gamma} \leq C (\|(\lambda I - W) \tilde{\rho}\|_{2+\alpha, \Gamma} + \|\tilde{\rho}\|_{3+\beta, \Gamma}).$$

Then, using the interpolation inequality

$$\|\tilde{\rho}\|_{3+\beta, \Gamma} \leq \varepsilon \|\tilde{\rho}\|_{3+\alpha, \Gamma} + C \|\tilde{\rho}\|_{2+\alpha, \Gamma},$$

we deduce that

$$|\lambda| \|\tilde{\rho}\|_{2+\alpha, \Gamma} + \|\tilde{\rho}\|_{3+\alpha, \Gamma} \leq C \|(\lambda I - W) \tilde{\rho}\|_{2+\alpha, \Gamma}$$

holds for any $\tilde{\rho} \in h^{3+\alpha}(\Gamma)$ and $\lambda \in \{z \in \mathbb{C} \mid \operatorname{Re} z \geq \lambda_*\}$ with sufficiently large $\lambda_* > \lambda_0$. This is nothing but (3.15). $\square$

Theorem 1.4 now follows from Theorem 3.3 and the theory of maximal regularity of Da Prato and Grisvard [5], since $h^{2+\alpha}(\Gamma)$ is characterized as a continuous interpolation space between $h^{3+\alpha'}(\Gamma)$ and $h^{2+\alpha'}(\Gamma)$ with $0 < \alpha' < \alpha < 1$. For the proof of the solvability of fully-nonlinear equations in continuous interpolation spaces, we refer to Angenent [3, Theorem 2.7] and Lunardi [13].

4 Bifurcation criterion for quadrature surfaces

Theorems 1.2 and 1.4 immediately deduce Corollary 1.5.

Proof of Corollary 1.5. Assuming the existence of a curve $s \mapsto (\Gamma(s), t(s))$, let us derive a contradiction. We divide the proof into two cases: (i) $t'(0) > 0$ and (ii) $t'(0) = 0$.

In the case (i), we can take the inverse function t^{-1} of $t = t(s)$ at least in a neighborhood of $s = 0$. Setting

$$\tilde{\Gamma}(\tau) := \Gamma(t^{-1}(\tau)),$$

we see that $\{\tilde{\Gamma}(\tau)\}_{0 \leq \tau < \tilde{\varepsilon}}$ with small $\tilde{\varepsilon}$ is an $h^{3+\alpha}$ family of surfaces satisfying

$$\int_{\partial\Omega(0)} h d\mathcal{H}^{N-1} + \tau \int h d\mu = \int_{\tilde{\Gamma}(\tau)} h d\mathcal{H}^{N-1}$$

for harmonic functions h . Then, it follows from Theorem 1.2 that $\{\tilde{\Gamma}(\tau)\}_{0 \leq \tau < \tilde{\varepsilon}}$ is a solution to (1.5). However, the uniqueness assertion in Theorem 1.4 implies that $\tilde{\Gamma}(\tau) = \partial\Omega(\tau)$, or $\Gamma(s) = \partial\Omega(t(s))$. This is a contradiction.

In the case (ii), by differentiating the identity

$$\int_{\partial\Omega(0)} h d\mathcal{H}^{N-1} + t(s) \int h d\mu = \int_{\Gamma(s)} h d\mathcal{H}^{N-1}$$

with respect to s at $s = 0$, we have a nonzero function $v_n \in h^{2+\alpha}(\partial\Omega(0))$ satisfying

$$0 = \int_{\partial\Omega(0)} \left\{ \frac{\partial h}{\partial n} + (N-1)hH \right\} v_n d\mathcal{H}^{N-1}$$

for all harmonic functions h defined in a neighborhood of $\overline{\Omega(0)}$. Therefore, by an argument similar to the last part of the proof of Theorem 1.2, we deduce that $v_n = 0$ on $\partial\Omega(0)$, which is again a contradiction. $\square$

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