

Surjectivity of a Gluing for Stable T^2 -cones in Special Lagrangian Geometry

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Abstract. We suppose that M is a Calabi–Yau (or more generally almost Calabi–Yau) manifold of dimension 3 over $\mathbb{C}$. We define a moduli space $\mathcal{V}$ of special Lagrangian varifolds (or currents) in M , using geometric measure theory. We suppose that X is a compact special Lagrangian 3-fold in M with singularity at one point and modelled on a stable T^2 -cone of multiplicity 1. We regard X as a point of $\mathcal{V}$, and determine a neighbourhood of X in $\mathcal{V}$.

1 Introduction

Gluing is a method of constructing blowing-up solutions to partial differential equations, and we shall say that the gluing is surjective if all blowing-up solutions may be re-constructed by the gluing. In Yang–Mills gauge theory, for example, Taubes [25] constructs some instantons by the method of gluing, and Donaldson [6] proves its surjectivity; for the details we refer also to Freed and Uhlenbeck [8]. In special Lagrangian geometry, by a theory of Joyce [13, 14, 15, 16, 17], we can smooth some isolated conical singularities by the method of gluing, and we wish to prove its surjectivity.

We can formulate the surjectivity problems in terms of moduli spaces. In the situation of Donaldson’s result [6] indeed one has the moduli space of Yang–Mills instantons modulo gauge equivalence; using a result of Uhlenbeck [26] one can compactify it by adding some singular instantons; one defines the boundary of the moduli space as the set of all the singular instantons added for the compactification; using the surjectivity result one can determine a neighbourhood of the boundary in the compactified space.

We wish to do a similar thing in special Lagrangian geometry. We suppose that M is a Calabi–Yau (or more generally almost Calabi–Yau) manifold of dimension m over $\mathbb{C}$. We take $a \in H_m(M; \mathbb{Z})$. We denote by $\mathcal{N}$ the set of all compact special Lagrangian submanifolds of M with homology class a . We give $\mathcal{N}$ the C^∞ -topology. Mclean [19, Theorem 3.6] proves that $\mathcal{N}$ is a manifold of dimension finite.

For compactifying $\mathcal{N}$ we shall use varifolds and currents in geometric measure theory. As special Lagrangian varifolds are calibrated in the sense of Harvey and Lawson [9] they may be oriented and regarded as currents. By a fundamental theorem of Harvey and Lawson special Lagrangian currents are homologically

area-minimizing, and so their area will not be cancelled; for the details we refer e.g. to Federer [7, 5.4.2] or Simon [24, Theorem 34.5]. In effect we may use varifolds in place of currents and vice versa. We shall mainly use varifolds rather than currents.

We denote by $\mathcal{V}$ the set of all special Lagrangian integral varifolds (or currents) in M with support compact, boundary 0 and homology class a . We give $\mathcal{V}$ the weak topology of varifolds. From an integral compactness theorem of Allard [1, Theorem 6.4] we see that $\mathcal{V}$ is compact. We may regard $\mathcal{N}$ as a subset of $\mathcal{V}$. From a regularity theorem of Allard [1, Theorem 8.19] we see that $\mathcal{N}$ is an open subset of $\mathcal{V}$, and the C^∞ -topology on $\mathcal{N}$ is the same as the one induced from $\mathcal{V}$. We denote by $\bar{\mathcal{N}}$ the closure of $\mathcal{N}$ in $\mathcal{V}$. We put $\partial\mathcal{N} = \bar{\mathcal{N}} \setminus \mathcal{N}$.

In the situation of Donaldson's result [6] the boundary of the moduli space consists of instantons with isolated singularities, to which one can apply the gluing method of Taubes. In our situation, however, $\partial\mathcal{N}$ consists of special Lagrangian varifolds whose singularities may be non-isolated or branched, and it will be difficult therefore to determine a neighbourhood of $\partial\mathcal{N}$ in $\mathcal{V}$. What we shall do is to take a certain point X of $\partial\mathcal{N}$ and determine a neighbourhood of X in $\mathcal{V}$.

We have mainly two things to do: the analysis of bubbling-off and the classification of local models. In the proof of Donaldson's surjectivity theorem one uses a technique of Uhlenbeck [26] for the analysis of bubbling-off, and a result of Atiyah, Hitchin and Singer [3] for the classification of local models.

We suppose that X has singularity only at one point and modelled on Jacobi-integrable smooth cones of multiplicity 1. We can then analyse the bubbling-off near X , using a technique of Allard–Almgren [2], Simon [22, 23] and the author [12]. On the other hand it seems difficult in general to classify local models for smoothing isolated conical singularities.

For stable T^2 -cones in the sense of Joyce [14, Definition 3.6], however, we can classify the local models. We can describe the stable T^2 -cones as follows. We put

$$C = \{(z_1, z_2, z_3) \in \mathbb{C}^3 \setminus \{0\} : |z_1| = |z_2| = |z_3|, z_1 z_2 z_3 \in (0, \infty)\}. \quad (1)$$

Harvey and Lawson [9, Chapter III.3.A, Theorem 3.1] prove that C is a special Lagrangian submanifold of $\mathbb{C}^3 \setminus \{0\}$. We put $\Sigma = C \cap S^5$. It is easy to see that Σ is diffeomorphic to T^2 . Joyce proves that C is a stable cone. Haskins [10] proves that every stable T^2 -cone may be expressed as $a(C)$ for some $a \in SU_3$.

We suppose now that M is of dimension 3 over $\mathbb{C}$. We suppose also that X has singularity only at one point and modelled on C with multiplicity 1. We prove that we have a local model L compatible with X in the situation of Joyce [17, Theorem 10.4]; L is a special Lagrangian submanifold of $\mathbb{C}^3$ asymptotic at infinity to C with multiplicity 1. For each $t > 0$ we can define tL by re-scaling L by t . We can find a real number $\tau > 0$ such that for each $t \in (0, \tau)$ we can glue tL to X by the method of gluing. Hence we get a smooth embedding $G : (0, \tau) \rightarrow \mathcal{N}$ with $G(t)$ tending to X as $t \rightarrow +0$.

We shall parametrize $G : (0, \tau) \rightarrow \mathcal{N}$ by a family of special Lagrangian 3-folds with isolated conical singularities modelled on C . We denote by $\mathcal{X}$ the set

of all elements of $\mathcal{V}$ with supported connected, singularity only at one point and modelled on C with multiplicity 1. Joyce [14, Definition 5.6] defines a topology on $\mathcal{X}$. We prove that it is actually the same as the one induced from $\mathcal{V}$. By a result of Joyce [14, Corollary 6.11] we can take a neighbourhood $\mathcal{Y}$ of X in $\mathcal{X}$ which has a structure of manifold. Joyce proves [17, Proposition 10.3] also $\dim \mathcal{Y} = \dim \mathcal{N} - 1$. Making $\tau > 0$ and $\mathcal{Y}$ smaller if necessary we can construct a one-to-one local diffeomorphism $G : (0, \tau) \times \mathcal{Y} \rightarrow \mathcal{N}$ with $G(t, Y)$ tending to Y as $t \mapsto +0$ for each $Y \in \mathcal{Y}$.

We can extend G continuously to $[0, \tau) \times \mathcal{Y}$ by setting $G(0, Y) = Y$ for each $Y \in \mathcal{Y}$. We note that $G : [0, \tau) \times \mathcal{Y} \rightarrow \mathcal{N}$ is a one-to-one open mapping. Our surjectivity result is:

Theorem 1. *G maps $[0, \tau) \times \mathcal{Y}$ onto a neighbourhood of X in $\mathcal{V}$.*

We begin in §2 with a review of special Lagrangian geometry and geometric measure theory. In §3 we analyse the bubbling-off of special Lagrangian varifolds near Jacobi-integrable smooth cones of multiplicity 1 (which need not be stable nor T^2 -cones). In §4 we classify the local models for smoothing C . In §5 we prove Theorem 1.

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2 Review of SL Geometry and GMT

In this section we shall give a review of special Lagrangian geometry and geometric measure theory. First of all we shall define almost Calabi–Yau manifolds and their special Lagrangian submanifolds.

Let (M, ω) be a symplectic manifold of dimension $2m$, and let J be a complex structure on M such that if we put $\hat{g}(v, w) = \omega(v, Jw)$ then $\hat{g}$ will be a Kähler metric on (M, J) . Let Ω be a holomorphic $(m, 0)$ -form on (M, J) with $\Omega|_x \neq 0$ for every $x \in M$. Then we shall call (M, ω, J, Ω) an almost Calabi–Yau manifold, and (ω, J, Ω) an almost Calabi–Yau structure on M . We can define a smooth function $\psi : M \rightarrow (0, \infty)$ such that

$$\frac{\psi^{2m}}{m!} \omega^{\wedge m} = (-1)^s \left(\frac{i}{2} \right)^m \Omega \wedge \bar{\Omega} \text{ where } s = \frac{m(m-1)}{2}. \quad (2)$$

We put $g = \psi^2 \hat{g}$. We shall call g the almost Calabi–Yau metric on (M, ω, J, Ω) . Here g need not be a Kähler metric on (M, J) . If we have $\psi(x) = 1$ for every $x \in M$ then g will be a Kähler metric of Ricci curvature 0. In that case we shall call (ω, J, Ω) a Calabi–Yau structure on M , and g the Calabi–Yau metric on (M, ω, J, Ω) .

We define a Calabi–Yau structure on $\mathbb{R}^{2m} = \mathbb{C}^m$ as follows. Let $(z^1, \dots, z^m)$ be the co-ordinates of $\mathbb{C}^m$. Let $\omega_0 = \frac{i}{2}(dz^1 \wedge d\bar{z}^1 + \dots + dz^m \wedge d\bar{z}^m)$, $g_0 =$

$dz^1 \otimes \overline{dz^1} + \cdots + dz^m \otimes \overline{dz^m}$ and $\Omega_0 = dz^1 \wedge \cdots \wedge dz^m$. Let J_0 be the complex structure of $\mathbb{C}^m$. Then $(\omega_0, J_0, \Omega_0)$ is a Calabi–Yau structure on $\mathbb{R}^{2m}$, and g is the almost Calabi–Yau metric on $(\mathbb{R}^{2m}, \omega_0, J_0, \Omega_0)$.

Let (M, ω, J, Ω) be an almost Calabi–Yau manifold, and let g be the almost Calabi–Yau metric on (M, ω, J, Ω) . Then $\text{Re } \Omega$ will be a calibration on (M, g) in the sense of Harvey and Lawson [9], and so we can define $\text{Re } \Omega$ -submanifolds of (M, g) , which we shall call special Lagrangian submanifolds of (M, ω, J, Ω) . Special Lagrangian submanifolds of (M, ω, J, Ω) will be Lagrangian submanifolds of (M, ω) .

We can also define $\text{Re } \Omega$ -varifolds and currents in (M, g) , which we shall call special Lagrangian varifolds and currents in (M, ω, J, Ω) , respectively. In the remainder of this section we shall consider calibrated geometry not limited to special Lagrangian geometry.

We suppose that M is a manifold. For each $x \in M$ we denote by $G_p(T_x M)$ the Grassmann manifold of all vector subspaces of $T_x M$ of dimension p . We put $G_p(TM) = \bigcup_{x \in M} G_p(T_x M)$. By a varifold of dimension p in M we shall mean a Radon measure on $G_p(TM)$.

We suppose that g is a Riemannian metric on M , and ϕ is a calibration of degree p on M . For each $x \in M$ we put $G_\phi(T_x M) = \{S \in G_p(T_x M) : |\phi|_S|_g = 1\}$. We also put $G_\phi(TM) = \bigcup_{x \in M} G_\phi(T_x M)$. By a ϕ -varifold in M we shall mean a Radon measure on $G_\phi(TM)$.

For each $x \in M$ and $S \in G_\phi(T_x M)$ we define $\vec{S} \in \bigwedge^p T_x M$ as follows. We take an orthonormal basis $(e_1, \dots, e_p)$ for $T_x M$ with respect to $g|_x$ such that $\langle \phi|_x, e_1 \wedge \cdots \wedge e_p \rangle = 1$. We set $\vec{S} = e_1 \wedge \cdots \wedge e_p$. It is easy to see that $\vec{S}$ is independent of the choice of $(e_1, \dots, e_p)$, and so well-defined.

Let V be a ϕ -varifold in M . Then we can define a p -current $\vec{V}$ in M by setting

$$\vec{V}(\chi) = \int_{G_\phi(TM)} \langle \chi|_x, \vec{S} \rangle dV(x, S)$$

for every p -form χ on M with support compact.

Harvey and Lawson [9, Chapter II.1, Definition 1.4] define positive ϕ -currents in M , which we shall explain next. First of all we recall a definition of Harvey and Lawson [9, Chapter II.A, Definition A.1]: by a ϕ -non-negative p -form on M we shall mean a p -form χ on M with $\langle \chi|_x, \vec{S} \rangle \geq 0$ for every $x \in M$ and $S \in G_\phi(T_x M)$. Harvey and Lawson [9, Chapter II.A, Proposition A.2] prove that a p -current T in M is a positive ϕ -current if (and only if) we have $T(\chi) \geq 0$ for every ϕ -non-negative p -form on M with support compact. We have:

Theorem. *Let V be a ϕ -varifold in M . Then $\vec{V}$ is a positive ϕ -current in M in the sense of Harvey and Lawson.*

Proof. We have only to prove that if χ is a ϕ -non-negative p -form on M with support compact then we have $\vec{V}(\chi) \geq 0$. By the definition of ϕ -non-negative p -forms we have $\langle \chi|_x, \vec{S} \rangle \geq 0$ for every $x \in M$ and $S \in G_\phi(T_x M)$. Hence we get $\vec{V}(\chi) = \int \langle \chi|_x, \vec{S} \rangle dV(x, S) \geq 0$, completing the proof. $\square$

We suppose now that M is compact. We take $a \in H_p(M; \mathbb{R})$. We denote by $\mathcal{V}$ the set of all integral ϕ -varifolds V in M with support compact, $\partial \vec{V} = 0$ and $[\vec{V}] = a$. We give $\mathcal{V}$ the weak topology in the sense of Allard [1, Definition 2.6(2)], i.e. the topology of the Radon measures on $G_p(TM)$. We have then:

Theorem. $\mathcal{V}$ is compact.

Proof. It is easy to see that $\mathcal{V}$ is a metrizable space. It suffices therefore to prove that if $V_1, V_2, V_3, \dots \in \mathcal{V}$ then there exists a subsequence of $(V_n)_{n=1}^\infty$ converging in $\mathcal{V}$. Since M is compact it is clear that ϕ is compactly-supported, and for each $n = 1, 2, 3, \dots$ therefore we have $\text{area } V_n = \vec{V}_n(\phi) = a \cdot [\phi]$ where $[\phi]$ denotes the de Rham cohomology class of ϕ . This implies that $\text{area } V_n$ is bounded with respect to n . By an integral compactness theorem of Allard [1, Theorem 6.4], therefore, we can find a subsequence of $(V_n)_{n=1}^\infty$ converging as Radon measures on $G_p(TM)$. We may identify $(V_n)_{n=1}^\infty$ with the subsequence. We denote its limit by V . It suffices then to prove $V \in \mathcal{V}$. Since V_n tends to V as varifolds we see that $\vec{V}_n$ tends to $\vec{V}$ as p -currents in M . As $\partial \vec{V}_n = 0$ for every $n = 1, 2, 3, \dots$ so we have $\partial \vec{V} = 0$. As $[\vec{V}_n] = a$ for every $n = 1, 2, 3, \dots$ so we have $[\vec{V}] = a$. We have therefore $V \in \mathcal{V}$ as we want. $\square$

Let V be a varifold of dimension p in M . Then we shall denote by $\|V\|$ the Radon measure on M defined by setting $\|V\|(f) = V(\pi \circ f)$ for every $f \in C_c(M; \mathbb{R})$, where π denotes the projection of $G_p(TM)$ onto M , and $C_c(M; \mathbb{R})$ denotes the set of all continuous functions on M with support compact.

We denote by $\mathcal{R}$ the set of all Radon measures on M which may be expressed as $\|V\|$ for some $V \in \mathcal{V}$. We give $\mathcal{R}$ the topology of the Radon measures on M . We have then:

Theorem. The mapping $V \mapsto \|V\|$ is a homeomorphism of $\mathcal{V}$ onto $\mathcal{R}$.

Proof. From the definition of $\mathcal{R}$ it is clear that $V \mapsto \|V\|$ maps $\mathcal{V}$ onto $\mathcal{R}$. Notice also that each $V \in \mathcal{V}$ is rectifiable by a result of Allard [1, Theorem 5.5]. Then we see that $V \mapsto \|V\|$ is one-to-one. From the definition of $\|V\|$, moreover, we see that $V \mapsto \|V\|$ is continuous. We have thus proved that $V \mapsto \|V\|$ is a continuous bijection of $\mathcal{V}$ onto $\mathcal{R}$. Notice also that $\mathcal{V}$ is compact and $\mathcal{R}$ is Hausdorff. Then we see that $V \mapsto \|V\|$ is a homeomorphism of $\mathcal{V}$ onto $\mathcal{R}$. $\square$

We denote by $\mathcal{T}$ the set of all positive ϕ -currents in M which may be expressed as $\vec{V}$ for some $V \in \mathcal{V}$. We give $\mathcal{T}$ the topology of p -currents in M . We have then:

Theorem. The mapping $V \mapsto \vec{V}$ is a homeomorphism of $\mathcal{V}$ into $\mathcal{T}$.

Proof. From the definition of $\mathcal{T}$ it is clear that $V \mapsto \vec{V}$ maps $\mathcal{V}$ onto $\mathcal{T}$. Notice also that $V \mapsto \vec{V}$ is one-to-one by the previous proposition. Then we see that $V \mapsto \vec{V}$ is one-to-one. It is also easy to see that $V \mapsto \vec{V}$ is continuous. We

have thus proved that $V \mapsto \vec{V}$ is a continuous bijection of $\mathcal{V}$ onto $\mathcal{T}$. Notice also that $\mathcal{V}$ is compact and $\mathcal{T}$ is Hausdorff. Then we see that $V \mapsto \vec{V}$ is a homeomorphism of $\mathcal{V}$ onto $\mathcal{T}$. $\square$

3 Analysis of Bubbling-off

In this section we shall analyse bubbling-off near special Lagrangian Jacobi-integrable smooth cones of multiplicity 1. We can summarize our results as follows.

We take a special Lagrangian varifold X with singularity only at one point x and asymptotic at x to a multiplicity 1 special Lagrangian Jacobi-integrable smooth cone C . We consider special Lagrangian varifolds V close to X . We define an energy functional for V . We prove that if the energy of V is small then V has singularity only at one point y and asymptotic at y to a multiplicity 1 smooth cone close to C , and if the energy of V is large then V bubbles off.

We begin in §3.1 with the definition of an energy functional and a discussion about related results. In §3.2 we discuss an important property of special Lagrangian Jacobi-integrable cones. In §3.3 we analyse the bubbling-off.

3.1 Energy Functional

We define an energy functional as follows. For each varifold V of dimension m in $\mathbb{R}^n$ we put

$$\mathcal{E}(V) = \int_{\mathbb{R}^n \times G_m(\mathbb{R}^n)} \frac{|S^\perp y|^2}{|y|^{m+2}} dV(y, S) \in [0, \infty]$$

where $S^\perp$ denotes the orthogonal complement to S in $\mathbb{R}^n$, and $S^\perp y$ denotes the projection of y onto $S^\perp$.

We shall recall a monotonicity formula for stationary varifolds. We put $B_\rho = \{y \in \mathbb{R}^n : |y| < \rho\}$ for each $\rho > 0$ and $A_{\sigma, \rho} = B_\rho \setminus \overline{B_\sigma}$ for each $\rho > \sigma > 0$. For each $Z \subset \mathbb{R}^n$ we put $\tilde{Z} = Z \times G_m(\mathbb{R}^n)$. If V is a stationary varifold of dimension m in (B_1, g_0) and if $0 < \sigma < \rho < 1$ then we have

$$\frac{\|V\|(B_\rho)}{\rho^m} - \frac{\|V\|(B_\sigma)}{\sigma^m} = \mathcal{E}(V \llcorner \tilde{A}_{\sigma, \rho}); \quad (3)$$

for the proof we refer e.g. to Allard [1, Theorem 5.1(1)] or Simon [24, Equation 17.4]. It is easy to extend (3) to Riemannian metrics in place of g_0 :

Proposition 1. *Let g be a Riemannian metric on B_1 with $\epsilon = |g - g_0|_{C^1(B_1)} < \infty$. Then there exists $k > 0$ such that if V is a stationary varifold of dimension m in (B_1, g) and if $0 < \sigma < \rho < 1$ then we have*

$$e^{k\epsilon\rho} \frac{\|V\|(B_\rho)}{\rho^m} - e^{k\epsilon\sigma} \frac{\|V\|(B_\sigma)}{\sigma^m} \geq \mathcal{E}(V \llcorner \tilde{A}_{\sigma, \rho}).$$

We suppose now that $n = 2m$ and (ω, J, Ω) is an almost Calabi–Yau structure on B_1 . We denote by $\mathcal{V}_0$ the space of all special Lagrangian integral varifolds V with $\partial \vec{V} = 0$ in (B_1, ω, J, Ω) .

We define $r : \mathbb{R}^{2m} \rightarrow [0, \infty)$ by setting $r(y) = |y|$ for each $y \in \mathbb{R}^{2m}$. We put $g_{\text{cyl}} = r^{-2}g_0$, which we shall call the cylindrical metric on $\mathbb{R}^{2m} \setminus \{0\}$.

We regard $(0, \infty)$ as a multiplicative group acting upon $\mathbb{R}^{2m}$ as re-scaling. By a smooth cone in $\mathbb{R}^{2m}$ we shall mean a closed submanifold of $\mathbb{R}^{2m} \setminus \{0\}$ invariant under the re-scaling by $(0, \infty)$. We denote by $\mathcal{C}$ the set of all special Lagrangian smooth cones in $(\mathbb{R}^{2m}, \omega_0, J_0, \Omega_0)$.

We suppose $C \in \mathcal{C}$. We denote by NC the normal bundle to C in $\mathbb{R}^{2m}$ with respect g_0 . We get the same bundle even if we use g_{cyl} in place of g_0 .

For each $u \in C^k(C; NC)$ and $\rho < \sigma$ we define $|u|_{[\rho, \sigma]}^{k, \text{cyl}}$ as follows. We put $t = -\log r$ and $\partial_t = \frac{\partial}{\partial t}$. We put $\Sigma = C \cap S^{2m-1}$. By the definition of smooth cones Σ is a compact submanifold of S^{2m-1} . We denote by ∇_Σ the Levi-Civita connexion over Σ induced from g_0 . We put

$$|u|_{[\rho, \sigma]}^{k, \text{cyl}} = \sup_{C \cap A_{\sigma, \rho}} \sum_{i, j \geq 0, i+j \leq k} |\partial_t^i \nabla_\Sigma^j u|.$$

If $|u|_{[\rho, \sigma]}^{1, \text{cyl}}$ is sufficient small, then we can define the exponential map $\exp u : C \cap A_{\sigma, \rho} \rightarrow A_{\sigma, \rho}$ with respect to the metric g_{cyl} , and the image of $\exp u$ will be a submanifold of $A_{\sigma, \rho}$, which we shall denote by $\text{Graph}_{\text{cyl}} u$. We put $|\text{Graph}_{\text{cyl}} u| = \mathcal{H}^m \llcorner \text{Graph}_{\text{cyl}} u$ where $\mathcal{H}^m$ denotes the Hausdorff m -dimensional measure with respect to the almost Calabi–Yau metric g .

We suppose that (ω, J, Ω) is an almost Calabi–Yau structure on B_1 , and g is the almost Calabi–Yau structure on B_1 . From the proof of the author [12, Theorem 2.2] we get:

Theorem 2. *There exists $\epsilon > 0$ depending only on m and C such that if we have*

$$|\Omega - \Omega_0|_{C^0(B_1)} + |g - g_0|_{C^1(B_1)} < \epsilon, \quad V \in \mathcal{V}_0, \quad 0 < \rho < 1, \quad \mathcal{E}(V \llcorner \tilde{A}_{\rho, 1}) < \epsilon$$

$$\text{and } v \in C^\infty(C \cap A_{1/2, 1}; NC), \quad |v|_{[1/2, 1]}^{2, \text{cyl}} < \epsilon, \quad |\text{Graph}_{\text{cyl}} v| = \|V\| \llcorner A_{1/2, 1}$$

then we can extend v to $C \cap A_{\rho, 1}$ so that $\|V\| \llcorner A_{\rho, 1} = |\text{Graph}_{\text{cyl}} v|$ and $|v|_{[\rho, 1]}^{2, \text{cyl}} < k\epsilon^\alpha$ for some $k > 0$ and $\alpha \in (0, 1)$ depending only on m and C .

Remark. For the proof one has to use a result of Łojasiewicz [18], following Simon [22].

We shall give a corollary to the theorem above. We give $\mathcal{C}$ the C^∞ -topology. Let $\mathcal{C}'$ be a neighbourhood of C in $\mathcal{C}$. Then we have:

Corollary 1. *There exists $\epsilon > 0$ depending only on m , C and $\mathcal{C}'$ such that if we have*

$$|\Omega - \Omega_0|_{C^0(B_1)} + |g - g_0|_{C^1(B_1)} < \epsilon, \quad V \in \mathcal{V}_0, \quad \mathcal{E}(V) < \epsilon$$

and $v \in C^\infty(C \cap A_{1/2,1}; NC)$, $|v|_{[1/2,1]}^{2,\text{cyl}} < \epsilon$, $|\text{Graph}_{\text{cyl}} v| = \|V\|_{\mathcal{L}A_{1/2,1}}$

then V is singular only at 0 and asymptotic at 0 to some $C' \in \mathcal{C}'$ with multiplicity 1.

Remark. By a result of Simon [22, Theorem 5] C' will be a unique tangent cone to V at 0.

Proof. Let C' be a tangent cone to V at 0. Then we can take $\delta_1 > \delta_2 > \delta_3 > \dots$ tending to 0 with $\delta_n^{-1}V$ tending to C' as $n \rightarrow \infty$. We can also take $\eta > 0$ such that if $v \in C^\infty(\Sigma; NC)$ and $|v|_{C^1(\Sigma)} \leq \eta$ then we have $\text{Graph}_{\text{cyl}} v \in \mathcal{C}'$. By Theorem 2 we can find $v_n \in C^\infty(C \cap A_{\delta_n,1}; NC)$ such that $\|V\|_{\mathcal{L}A_{\delta_n,1}} = |\text{Graph}_{\text{cyl}} v_n|$ with $|v_n|_{[\delta_n,1]}^{2,\text{cyl}} \leq \eta$, and so we can find a subsequence of $(\|\delta_n^{-1}V\|_{\mathcal{L}A_{1/2,1}})_{n=1}^\infty$ converging to $|\text{Graph}_{\text{cyl}} v_\infty|$ for some $w \in C^\infty(C \cap A_{1/2,1}; NC)$ with $|w|_{[1/2,1]}^{1,\text{cyl}} \leq \eta$. This implies that C' is smooth of multiplicity 1 and we have $C' \in \mathcal{C}'$ as we want. $\square$

3.2 Special Lagrangian Jacobi-integrable Cones

We shall discuss an important property of special Lagrangian Jacobi-integrable cones. For a moment we shall consider Jacobi-integrable cones which need not be special Lagrangian.

We denote by $\mathcal{K}$ the set of all smooth cones in $\mathbb{R}^n$. We define a functional $F : \mathcal{K} \rightarrow (0, \infty)$ by setting $F(K) = \text{area}(K \cap B_1)$ for each $K \in \mathcal{K}$. We denote by $\text{Crit } F$ the set of all critical points of F .

Let $K \in \mathcal{K}$. Then we have $T_K \mathcal{K} = C^\infty(K; NK)$. We can also define the hessian of F at K as a linear operator $\text{hess}_K F : T_K \mathcal{K} \rightarrow T_K \mathcal{K}$. We shall say that K is Jacobi-integrable if for every $v \in \text{Ker } \text{hess}_K F$ there exists a one parameter family $(K_t)_{t \in \mathbb{R}}$ in $\text{Crit } F$ with

$$K_0 = K \text{ and } \left. \frac{dK_t}{dt} \right|_{t=0} = v.$$

We suppose that K is Jacobi-integrable. Adams and Simon [4, Lemma 1] prove:

Proposition 2. *There exists a neighbourhood of K in $\mathcal{K}$ on which F is constant.*

For the proof one has to use the real-analyticity of F . Adams and Simon also prove that $\text{Crit } F$ is a non-degenerate critical manifold in the sense of Bott [5].

We shall do a similar thing for Lagrangian cones. We denote by $\widehat{\mathcal{C}}$ the set of all Lagrangian smooth cones in $(\mathbb{R}^{2m}, \omega_0)$. We have $\widehat{\mathcal{C}} \subset \mathcal{K}$. We put $\widehat{F} = F|_{\widehat{\mathcal{C}}}$. We denote by $\widehat{\text{Crit } \widehat{F}}$ the set of all critical points of $\widehat{F}$.

We suppose $\widehat{C} \in \widehat{\mathcal{C}}$. By a result of Weinstein [27, Corollary 6.2], then, $T_{\widehat{C}} \widehat{\mathcal{C}}$ may be identified with the set of all closed 1-forms on $\widehat{C}$. By a result of Oh [21, Theorem 3.5] the hessian of $\widehat{F}$ at $\widehat{C}$ will be the Laplacian acting upon closed 1-forms on $\widehat{C}$. We shall denote the Laplacian by Δ , and say that $\widehat{C}$ is Lagrangian

Jacobi-integrable if for every closed 1-form ξ on $\widehat{C}$ with $\Delta\xi = 0$ there exists a one parameter family $(\widehat{C}_t)_{t \in \mathbb{R}}$ in $\text{Crit } \widehat{F}$ with

$$\widehat{C}_0 = \widehat{C} \text{ and } -\frac{d\widehat{C}_t}{dt}\Big|_{t=0} \lrcorner \omega_0 = \xi.$$

In a way similar to the proof of Proposition 2 we can prove:

Proposition 3. *If $\widehat{C}$ is Lagrangian Jacobi-integrable then there exists a neighbourhood of $\widehat{C}$ in $\widehat{\mathcal{C}}$ on which $\widehat{F}$ is constant.*

Joyce [13, Definition 6.7] defines special Lagrangian Jacobi-integrable cones, which we shall recall next. As in §3.1 we denote by $\mathcal{C}$ the set of all special Lagrangian smooth cones in $(\mathbb{R}^{2m}, \omega_0, J_0, \Omega_0)$. We take $C \in \mathcal{C}$. We shall say that C is special Lagrangian Jacobi-integrable if for every closed 1-form ξ on C with $d*\xi = 0$ there exists a one parameter family $(C_t)_{t \in \mathbb{R}}$ in $\mathcal{C}$ with

$$C_0 = C \text{ and } -\frac{dC_t}{dt}\Big|_{t=0} \lrcorner \omega_0 = \xi.$$

We have:

Lemma. *If C is special Lagrangian Jacobi-integrable then C is Lagrangian Jacobi-integrable.*

Proof. Let ξ be a closed 1-form on C with $\Delta\xi = 0$. Then we have $d*d*\xi = 0$ and so

$$*d*\xi = a \tag{4}$$

for some $a \in \mathbb{R}$. We define a map $f_t^a : \mathbb{C}^m \rightarrow \mathbb{C}^m$ by setting $f_t^a(z_1, z_2, \dots, z_m) = (e^{iat}z_1, z_2, \dots, z_m)$. We put $C_t^a = f_t^a(C)$. We have then

$$\Omega_0|_{C_t^a} = (f_t^{a*}\Omega_0)|_C = e^{iat}\Omega_0|_C = e^{iat} * 1. \tag{5}$$

We put

$$\xi^a = -\frac{dC_t^a}{dt}\Big|_{t=0} \lrcorner \omega_0.$$

Since C_t^a is Lagrangian we see that ξ^a is a closed 1-form on C . We may also write $\xi^a = g_0(\tilde{\xi}^a, \bullet)$ for some $\tilde{\xi}^a$. We have then $\frac{dC_t^a}{dt}\Big|_{t=0} = J_0\tilde{\xi}^a$ and so

$$\frac{d}{dt}\Omega_0|_{C_t^a}\Big|_{t=0} = d(J_0\tilde{\xi}^a \lrcorner \Omega_0|_C) = d(i\tilde{\xi}^a \lrcorner \Omega_0|_C) = d(i\tilde{\xi}^a \lrcorner * 1) = id*\xi^a.$$

This combined with (5) implies that on C we have

$$id*\xi^a = ia*1$$

and so $*d*\xi^a = a$. This combined with (4) implies $*d*(\xi - \xi^a) = 0$. By the special Lagrangian Jacobi-integrability of C , therefore, we can find a one-parameter family $(C_t)_{t \in \mathbb{R}}$ in $\mathcal{C}$ with

$$C_0 = C \text{ and } -\frac{dC_t}{dt}\Big|_{t=0} \lrcorner \omega_0 = \xi - \xi^a.$$

Putting $\widehat{C}_t = f_t^a(C_t)$ we get

$$\widehat{C}_t \in \widehat{\mathcal{C}}, \widehat{C}_0 = C \text{ and } -\frac{d\widehat{C}_t}{dt}\Big|_{t=0} \lrcorner \omega_0 = \xi,$$

completing the proof. $\square$

By Proposition 3 and the lemma above we have:

Proposition 4. *Let C be special Lagrangian Jacobi-integrable. Then there exists a neighbourhood $\mathcal{C}'$ of C in $\mathcal{C}$ on which the area functional is constant; i.e. for each $C' \in \mathcal{C}'$ we have $\text{area}(C' \cap B_1) = \text{area}(C \cap B_1)$.*

3.3 Bubbling-off

We suppose that (ω, J, Ω) is an almost Calabi–Yau structure on B_1 with $J|_0 = J_0$ and $\Omega|_0 = \Omega_0$. We denote by g the almost Calabi–Yau metric on (B_1, ω, J, Ω) . We write $g = \sum_{i,j=1}^{2m} g_{ij} dy^i dy^j$ and suppose

$$g_{ij}(0) = \delta_{ij} \text{ and } \frac{\partial g_{ij}}{\partial y^k}(0) = 0 \text{ for each } i, j, k = 1, \dots, 2m.$$

If W is a varifold in $\mathbb{R}^n$ and if $\delta > 0$ then we can define $\delta^{-1}W$ by re-scaling W by δ^{-1} . If $b \in \mathbb{R}^n$ then we can also define $W - b$ by translating W by $-b$.

We take $C \in \mathcal{C}$ and a neighbourhood $\mathcal{C}'$ of C in $\mathcal{C}$. $X \in \mathcal{V}_0$. We denote by $\mathcal{X}'$ the space of all elements of $\mathcal{V}_0$ with support connected, with singularity only at one point y and asymptotic at y to some element of $\mathcal{C}'$ with multiplicity 1. We take $X \in \mathcal{V}_0$ and suppose that 0 is the singular point of X and C is the multiplicity 1 smooth tangent cone to X at 0. We have then:

Theorem 3. *Let $(X_n)_{n=1}^\infty$ be a sequence in $\mathcal{X}'$ converging to X . For each $n = 1, 2, 3, \dots$ let x_n be the singular point of X_n , and let C_n be the multiplicity 1 smooth tangent cone to X_n at x_n . Then x_n tends to 0 and C_n tends to C as $n \rightarrow \infty$.*

Proof. By Allard's regularity theorem x_n tends to 0 as $n \rightarrow \infty$. By Proposition 1 we have

$$\mathcal{E}(X_n - x_n) \leq e^{k\epsilon\rho} \frac{\|X_n - x_n\|(B_\rho)}{\rho^m} - e^{k\epsilon\sigma} \frac{\|X_n - x_n\|(B_\sigma)}{\sigma^m}$$

where $0 < \sigma < \rho < 1$. Letting $\sigma \rightarrow 0$ we get

$$\mathcal{E}(X_n - x_n) \leq e^{k\epsilon\rho} \frac{\|X_n - x_n\|(B_\rho)}{\rho^m} - \text{area}(C_n \cap B_1).$$

This combined with Proposition 4 implies

$$\mathcal{E}(X_n - x_n) \leq e^{k\epsilon\rho} \frac{\|X_n - x_n\|(B_\rho)}{\rho^m} - \text{area}(C \cap B_1). \quad (6)$$

Making ρ smaller if necessary we may suppose

$$e^{k\epsilon\rho} \frac{\|X\|(B_\rho)}{\rho^m} - \text{area}(C \cap B_1) < \epsilon.$$

For n sufficiently large, therefore, we have

$$e^{k\epsilon\rho} \frac{\|X_n - x_n\|(B_\rho)}{\rho^m} - \text{area}(C \cap B_1) < \epsilon.$$

This combined with (6) implies

$$\mathcal{E}(X_n - x_n) < \epsilon.$$

By Corollary 1, therefore, C_n tends to C as $n \rightarrow \infty$. $\square$

We have also:

Theorem 4. *Let $(V_n)_{n=1}^\infty$ be a sequence in $\mathcal{V}_0 \setminus \mathcal{X}'$ converging to X . Then there exists a sequence $(\delta_n)_{n=1}^\infty$ of positive real numbers converging to 0, a sequence $(y_n)_{n=1}^\infty$ in B_1 converging to 0, and a subsequence of $(\delta_n^{-1}(V_n - y_n))_{n=1}^\infty$ converging to some varifold W in $\mathbb{R}^{2m}$ asymptotic at infinity to some element of $\mathcal{C}'$ with multiplicity 1 and satisfying $\mathcal{E}(W - b) > 0$ for every $b \in \mathbb{R}^{2m}$.*

Remark. W will automatically be a special Lagrangian integral varifold with $\vec{\partial\tilde{W}} = 0$ in $(\mathbb{R}^{2m}, \omega_0, J_0, \Omega_0)$.

Proof. We take positive real numbers ρ and ϵ , which we shall make smaller if necessary. By assumption we have $V_n \in \mathcal{V}_0 \setminus \mathcal{X}'$. By Corollary 1 therefore we may suppose $\mathcal{E}(V_n - y) \geq \epsilon$ for each $y \in B_{\rho\epsilon}$; otherwise V_n will be singular only at y and asymptotic at y to some element of $\mathcal{C}'$ with multiplicity 1, which contradicts $V_n \in \mathcal{V}_0 \setminus \mathcal{X}'$. We put

$$\delta_n(y) = \inf \left\{ \delta \in (0, \rho) : \mathcal{E}((V_n - y) \llcorner \tilde{A}_{\delta, \rho}) = \frac{\epsilon}{2} \right\}.$$

Since $\mathcal{E}(V_n - y) \geq \epsilon$ we see $\delta_n(y) > 0$. It is also easy to see that $y \mapsto \delta_n(y)$ is lower semi-continuous. Hence we can find $y_n \in B_{\epsilon\rho}$ with $\delta_n(y_n) = \inf_{y \in B_\epsilon} \delta_n(y)$. We put $\delta_n = \delta_n(y_n) > 0$. We have then:

Lemma. δ_n tends to 0 as $n \rightarrow \infty$.

Proof. Making ρ smaller if necessary we may suppose $\mathcal{E}(X \llcorner \tilde{B}_\rho) < \epsilon/2$. We take $\delta > 0$. We have then $\mathcal{E}(X \llcorner \tilde{A}_{\delta, \rho}) < \epsilon/2$. Since V_n tends to X as $n \rightarrow \infty$ we see that for n sufficiently large we have $\mathcal{E}(V_n \llcorner \tilde{A}_{\delta, \rho}) < \epsilon/2$. This implies $\delta_n(0) < \delta$ and so $\delta_n(0)$ tends to 0 as $n \rightarrow \infty$. Since $\delta_n \leq \delta_n(0)$ we see that δ_n also tends to 0 as $n \rightarrow \infty$. $\square$

We have also:

Lemma. $(\delta_n^{-1}(V_n - y_n))_{n=1}^\infty$ has a subsequence converging to some varifold W in $\mathbb{R}^{2m}$.

Proof. Let $R > 0$. Then we have only to prove

$$\sup_{n=1,2,3,\dots} \|\delta_n^{-1}(V_n - y_n)\|(B_R) < \infty. \quad (7)$$

Notice that $\|\delta_n^{-1}(V_n - y_n)\|(B_R) = \delta_n^{-m} \|V_n - y_n\|(B_{\delta_n R})$. Notice also that by Proposition 1 we have

$$e^{k\epsilon_n \delta_n R} \frac{\|V_n - y_n\|(B_{\delta_n R})}{(\delta_n R)^m} \leq e^{k\epsilon_n} \|V_n - y_n\|(B_1)$$

which tends to $\|X\|(B_1)$ as $n \rightarrow \infty$. Then we get (7). $\square$

We take W as above. From the definition of δ_n it is easy to see $\mathcal{E}(W \llcorner \tilde{A}_{1,\infty}) \leq \epsilon/2$. In a way similar to the proof of Corollary 1, therefore, we can prove that W is asymptotic at infinity to a multiplicity 1 smooth cone C' . Making ϵ smaller if necessary we may suppose $C' \in \mathcal{C}'$.

It remains to prove $\mathcal{E}(W - b) > 0$ for every $b \in \mathbb{R}^{2m}$. We put $V'_n = V_n - y_n - \delta_n b$. By Proposition 1 we have

$$\exp(k\epsilon_n \rho) \frac{\|V'_n\|(B_\rho)}{\rho^m} - \exp(k\epsilon_n \delta_n) \frac{\|V'_n\|(B_{\delta_n})}{\delta_n^m} \geq \mathcal{E}(V'_n \llcorner \tilde{A}_{\delta_n, \rho}) \geq \frac{\epsilon}{2}. \quad (8)$$

We note that V'_n tends to X and $\delta_n^{-1} V'_n$ tends to $W - b$ as $n \rightarrow \infty$. By (8) therefore we have

$$\frac{\|X\|(B_\rho)}{\rho^m} - \|W - b\|(B_1) \geq \frac{\epsilon}{2}. \quad (9)$$

Making $\rho > 0$ smaller if necessary we may suppose

$$\frac{\|X\|(B_\rho)}{\rho^m} \leq \text{area}(C \cap B_1) + \frac{\epsilon}{4}.$$

This combined with (9) implies

$$\text{area}(C \cap B_1) - \|W - b\|(B_1) > \frac{\epsilon}{4}.$$

This combined with Proposition 4 implies

$$\text{area}(C' \cap B_1) - \|W - b\|(B_1) > \frac{\epsilon}{4}. \quad (10)$$

On the other hand by (3) we have

$$\frac{\|W - b\|(B_R)}{R^m} - \|W - b\|(B_1) = \mathcal{E}((W - b) \llcorner \tilde{A}_{\lambda, R})$$

for each $R > 1$. Letting $R \rightarrow \infty$ we get

$$\text{area}(C' \cap B_1) - \|W - b\|(B_1) = \mathcal{E}((W - b) \llcorner \tilde{A}_{\lambda, \infty}).$$

This combined with (10) implies $\mathcal{E}((W - b) \llcorner \tilde{A}_{1, \infty}) > \epsilon/4$ and so $\mathcal{E}(W - b) > 0$, completing the proof. $\square$

4 Classification of Local Models

We define C by (1) in §1. In this section we shall classify local models for smoothing C . We can describe the local models as follows. We put

$$\begin{aligned} L_1 &= \{(z_1, z_2, z_3) \in \mathbb{C}^3 : |z_1|^2 - 1 = |z_2|^2 = |z_3|^2, z_1 z_2 z_3 \in [0, \infty)\}, \\ L_2 &= \{(z_1, z_2, z_3) \in \mathbb{C}^3 : |z_1|^2 = |z_2|^2 - 1 = |z_3|^2, z_1 z_2 z_3 \in [0, \infty)\}, \\ L_3 &= \{(z_1, z_2, z_3) \in \mathbb{C}^3 : |z_1|^2 = |z_2|^2 = |z_3|^2 - 1, z_1 z_2 z_3 \in [0, \infty)\}. \end{aligned} \quad (11)$$

If $L \in \{L_1, L_2, L_3\}$ then L will be a submanifold of $\mathbb{C}^3$ of dimension 3 over $\mathbb{R}$. We note that L is non-singular in spite of the condition $z_1 z_2 z_3 \in [0, \infty)$. Harvey and Lawson [9, Chapter III.3.A, Theorem 3.1] prove that L is a special Lagrangian submanifold of $\mathbb{C}^3$. It is easy to see that L is asymptotic at infinity to C with multiplicity 1. Our classification result is:

Theorem 5. *Let W be a special Lagrangian integral varifold with $\vec{\partial W} = 0$ in $(\mathbb{R}^6, \omega_0, J_0, \Omega_0)$ asymptotic at infinity to C with multiplicity 1. Then we have $\|W\| = |sL + b|$ for some $s > 0$, $L \in \{C, L_1, L_2, L_3\}$ and $b \in \mathbb{R}^6$.*

Here $|sL + b|$ denotes the Radon measure on $\mathbb{R}^6$ associated to $sL + b$ with multiplicity 1 with respect to the Euclidean metric g_0 .

In what follows we give a sketch of the proof of Theorem 5.

First of all we analyse the asymptotic behaviour of W at infinity. From the stability of C we find $b \in \mathbb{R}^6$ such that $W - b$ will decay rapidly to C at infinity. For the sake of simplicity, therefore, we may suppose that W decays rapidly to C at infinity.

We define a T^2 -action on $\mathbb{R}^6 = \mathbb{C}^3$ by setting

$$(e^{i\theta}, e^{i\phi}) \cdot (z_1, z_2, z_3) = (e^{i\theta} z_1, e^{i\phi} z_2, e^{-i\theta - i\phi} z_3).$$

We note that ω_0, J_0, Ω_0 and C are invariant under the T^2 -action. We take a moment map $\mu : \mathbb{C}^3 \rightarrow \mathbb{R}^2$. We prove that μ is constant on $\text{Spt } \|W\|$. The idea of the proof is as follows.

We suppose for a moment that W is a submanifold of $\mathbb{C}^3$. By the T^2 -action, then, we can deform W as a special Lagrangian submanifold of $\mathbb{C}^3$. The infinitesimal deformation may be identified with $d\mu|_W$, and so $d\mu|_W$ will be a harmonic 1-form on W , by the theory of Mclean. Therefore $\mu|_W$ will be a harmonic function on W . Since W decays rapidly to C at infinity we shall see that $\mu|_W$ decays rapidly at infinity to some constant. By the maximum principle, therefore, $\mu|_W$ will be a constant function.

Actually W need not be a submanifold of $\mathbb{R}^6$. Using some basic results on varifolds, however, we can modify the argument above. Thus we see that μ is constant on $\text{Spt } \|W\|$.

Harvey and Lawson [9, Chapter III.3.A, Theorem 3.1] construct a special Lagrangian fibration $F : \mathbb{C}^3 \rightarrow \mathbb{R}^3$ invariant under the T^2 -action. From the proof of Harvey and Lawson we see that every T^2 -invariant special Lagrangian submanifold of $\mathbb{C}^3$ is contained in a fibre of F .

We suppose again that W is a submanifold of $\mathbb{C}^3$. Since μ is constant on W we see that W is then invariant under the T^2 -action, and so contained in a fibre of F . Actually W need to be a submanifold of $\mathbb{C}^3$, but we can again modify the argument and so $\text{Spt } \|W\|$ is contained in a fibre of F .

For each fibre of F we have an explicit description of the topology and asymptotic behaviour at infinity, and so we shall be able to complete the proof by an elementary argument.

We begin now with a review of some basic properties of Laplacians over cones.

We suppose that Σ is a compact manifold of dimension $m - 1$. We put $C = (0, \infty) \times \Sigma$. We denote by r the projection of C onto $(0, \infty)$. We suppose that g_Σ is a Riemannian metric on Σ . We put $g_C = dr^2 + r^2 g_\Sigma$. With respect to g_C we can define the Laplacian $\Delta_C : C^\infty(C; \mathbb{R}) \rightarrow C^\infty(C; \mathbb{R})$.

We have also the Laplacian $\Delta_\Sigma : C^\infty(\Sigma; \mathbb{R}) \rightarrow C^\infty(\Sigma; \mathbb{R})$ with respect to g_Σ . We put $t = \log r : C \rightarrow \mathbb{R}$ and $\partial_t = \frac{\partial}{\partial t}$. It is easy then to see $-e^{2t} \Delta_C = \partial_t^2 + (m - 2)\partial_t - \Delta_\Sigma$. We denote by $0 = \gamma_0 \leq \gamma_1 \leq \gamma_2 \leq \dots$ the eigenvalues of Δ_Σ . For each integer $i \geq 0$ we consider the equation $x^2 + (m - 2)x - \gamma_i = 0$ in x . We denote by α_i and β_i the two solutions with $\alpha_i \geq \beta_i$. We suppose $m \geq 2$. Since $\gamma_0 = 0$ we get $\beta_0 = 2 - m \leq 0 = \alpha_0$. Since $\gamma_0 \leq \gamma_1 \leq \gamma_2 \dots$ we get $\alpha_0 \leq \alpha_1 \leq \alpha_2 \leq \dots$ and $\beta_0 \geq \beta_1 \geq \beta_2 \geq \dots$. Since γ_i tends to ∞ as $i \rightarrow \infty$ we see that α_i tends to ∞ and β_i tends to $-\infty$ as $i \rightarrow \infty$. We also put $\Lambda = \{\alpha_0, \beta_0, \alpha_1, \beta_1, \alpha_2, \beta_2, \dots\}$. For each $\lambda \in \mathbb{R}$ we put

$$E_\lambda = \{f \in C^\infty(\Sigma; \mathbb{R}) : \Delta_\Sigma f = \lambda(\lambda + m - 2)f\}.$$

By the definition of the eigenvalues we have $E_\lambda \neq \{0\}$ if and only if $\lambda \in \Lambda$.

We can also take a complete orthonormal basis $\{v_0, v_1, v_2, \dots\}$ for $L^2(\Sigma; \mathbb{R})$ such that $\Delta_\Sigma v_i = \gamma_i v_i$ for each integer $i \geq 0$. We have then:

Proposition 5. *Let I be an open interval in $\mathbb{R}$, let $u \in C^\infty(I \times \Sigma; \mathbb{R})$ and suppose $\Delta_C u = 0$. Then there exist $a_0, b_0, a_1, b_1, a_2, b_2, \dots \in \mathbb{R}$ such that $u = \sum_{i=0}^\infty (a_i r^{\alpha_i} + b_i r^{\beta_i}) v_i$ where the series converge in the local C^∞ -topology.*

We give a proof for the sake of clarity:

Proof. We put $\log I = \{t \in \mathbb{R} : e^t \in I\}$. For each $t \in \log I$ and $i \in \{0, 1, 2, \dots\}$ we define $u_i(t)$ as the inner product of $u|_{\{e^t\} \times \Sigma}$ and v_i in $L^2(\Sigma; \mathbb{R})$. Then $t \mapsto u_i(t)$ is a smooth function on $\log I$. Since $\Delta_C u = 0$ we get $(\partial_t^2 + (m - 2)\partial_t - \lambda_i)u_i = 0$. Hence we find $a_i, b_i \in \mathbb{R}$ such that $u_i = a_i e^{\alpha_i t} + b_i e^{\beta_i t}$. It suffices therefore to prove that $\sum_{i=0}^\infty u_i v_i$ converges to u in the local C^∞ -topology. We denote by $d\mu_\Sigma$ the Riemannian measure on Σ with respect to g_Σ , and by $\|\bullet\|$ the norm of $L^2(\Sigma; \mathbb{R})$. We have then

$$\int_{\log I \times \Sigma} |u|^2 dt d\mu_\Sigma = \int_{\log I} \sum_{i=0}^\infty \|u_i(t)\|^2 dt = \sum_{i=0}^\infty \int_{\log I} \|u_i(t)\|^2 dt < \infty.$$

Putting $w_n = \sum_{i=0}^n u_i v_i$ we get $\int_{\log I \times \Sigma} |u - w_n|^2 dt d\mu_\Sigma = \sum_{i=n+1}^\infty \int_{\log I} \|u_i(t)\|^2 dt$ which tends to 0 as $n \rightarrow \infty$. Thus w_n tends to u in the L^2 -topology. Applying elliptic regularity to $u - w_n$ we see that w_n tends to u in the local C^∞ -topology. $\square$

Let $u \in C^\infty(C; \mathbb{R})$ and $\alpha \in \mathbb{R}$. Then we shall write $u = O(r^\alpha)$ as $r \rightarrow \infty$ if there exists $R > 0$ such that

$$\sup_{(R, \infty) \times \Sigma} |r^{-\alpha+k} \nabla^k u| < \infty \text{ for every } k = 0, 1, 2, 3, \dots$$

where ∇ denotes the Levi-Civita connexion with respect to g_C , and $|\bullet|$ denotes the norm with respect to g_C .

By a result of Simon [23, Part I, Lemma 5.9] we have:

Proposition. *Suppose $q \in (2 - m, \infty) \setminus \Lambda$ and $f \in C^\infty((R, \infty) \times \Sigma; \mathbb{R})$ with $f = O(r^{q-2})$. Then there exists $u \in C^\infty((R, \infty) \times \Sigma; \mathbb{R})$ with $u = O(r^q)$ such that $\Delta_C u = f$.*

We give a corollary to this:

Corollary 2. *Suppose $R > 0$, $u \in C^\infty((R, \infty) \times \Sigma; \mathbb{R})$, $p, q \in \mathbb{R}$, $2 - m < q < p$, $q \notin \Lambda$, $u = O(r^p)$ and $\Delta_C u = O(r^{q-2})$. Then there exists $(f_\lambda)_{\lambda \in \Lambda} \in \bigoplus_{\lambda \in \Lambda} E_\lambda$ such that $u = \sum_{\lambda \in \Lambda \cap (q, p]} f_\lambda r^\lambda + O(r^q)$ where $\Lambda \cap (q, p]$ is possibly empty and in that case we set $\bigoplus_{\lambda \in \emptyset} E_\lambda = \{0\}$.*

Proof. Applying the proposition above to $\Delta_C u$ in place of f we find $u' \in C^\infty((R, \infty) \times \Sigma; \mathbb{R})$ with $u' = O(r^q)$ such that $\Delta_C u' = \Delta_C u$. By Proposition 5 we can find some $a_0, b_0, a_1, b_1, a_2, b_2, \dots \in \mathbb{R}$ such that $u - u' = \sum_{i=0}^\infty (a_i r^{\alpha_i} + b_i r^{\beta_i}) v_i$.

Since $u' = O(r^q)$ we get $u = \sum_{i=0}^\infty (a_i r^{\alpha_i} + b_i r^{\beta_i}) v_i + O(r^q)$. Since $u = O(r^p)$ and $p > q$ we get $a_i = 0$ if $\alpha_i > p$. Since $\alpha_0 \leq \alpha_1 \leq \alpha_2 \leq \dots$ tend to ∞ we can find a unique integer $i(p)$ such that $\alpha_{i(p)+1} > p$ and $\alpha_{i(p)} \leq p$. Since $a_i = 0$ for every $i > i(p)$ we see that $\sum_{i=0}^\infty b_i r^{\beta_i} v_i$ converges in the local C^∞ -topology. We put $w = \sum_{i=0}^\infty b_i r^{\beta_i} v_i$.

Since $0 \geq 2 - m = \beta_0 \geq \beta_1 \geq \beta_2 \geq \dots$ it follows that for every $\rho > R$ we have

$$\int_\rho^{2\rho} \int_\Sigma |w|^2 \frac{dr}{r} d\mu_\Sigma \leq \left(\frac{\rho}{R}\right)^{2-m} \int_R^{2R} \int_\Sigma |w|^2 \frac{dr}{r} d\mu_\Sigma.$$

Applying elliptic regularity to w we get $w = O(r^{2-m})$. Since $u = \sum_{i=0}^{i(p)} a_i r^{\alpha_i} v_i + w + O(r^q)$ and $q > 2 - m$ we get $u = \sum_{i=0}^{i(p)} a_i r^{\alpha_i} v_i + O(r^q)$, completing the proof. $\square$

We suppose now that C is a smooth special Lagrangian cone in $(\mathbb{R}^{2m} \setminus \{0\}, \omega_0, J_0, \Omega_0)$ and $\Sigma = C \cap S^{2m-1}$. For each $\rho > 0$ we put $B_\rho = \{y \in \mathbb{R}^n : |y| < \rho\}$. We denote by $i_C : C \rightarrow \mathbb{R}^{2m}$ the inclusion map of C into $\mathbb{R}^{2m}$.

Joyce [13, Definition 7.1] defines special Lagrangian submanifolds of $(\mathbb{R}^{2m}, \omega_0, J_0, \Omega_0)$ asymptotic to C with multiplicity 1 at infinity with some rate < 2 . We extend it to varifolds as follows. We denote by $\mathcal{W}$ the set of all special Lagrangian integral varifolds W with $\partial \vec{W} = 0$ in $(\mathbb{R}^{2m}, \omega_0, J_0, \Omega_0)$ asymptotic at infinity to C with multiplicity 1. For each $\lambda < 2$ we denote by $\mathcal{W}_\lambda$ the set of all $W \in \mathcal{W}$

such that we can find a compact subset K of W , an $R > 0$ and a diffeomorphism $f : C \setminus \overline{B_R} \rightarrow \text{Spt } \|W\| \setminus K$ such that $f - i_C = O(r^{\lambda-1})$.

We suppose that C is Jacobi-integrable in the sense of Joyce [13, Definition 6.7]. In a way similar to Simon [23, Part II, §§5 and 6], then, we can prove that if $W \in \mathcal{W}$ then there exists $\lambda < 2$ such that $W \in \mathcal{W}_\lambda$.

Joyce [13, Theorem 4.3] proves a version of Weinstein's theorem [27, Corollary 6.2], which we shall recall next. We denote by T^*C the cotangent bundle over C , by 0_C the zero-section of T^*C , and by ω_C the canonical symplectic form on T^*C . We regard $(0, \infty)$ as a multiplicative group acting upon C and $\mathbb{R}^{2m}$ as re-scaling. We can lift the $(0, \infty)$ -action uniquely to T^*C so that for each $t \in (0, \infty)$ we shall have $t^*\omega_C = t^2\omega_C$ (on the left-hand side we regard t as a map of C into itself). We have then:

Lemma 1. *There exist a neighbourhood UC of $\text{Im } 0_C$ in T^*C invariant under $(0, \infty)$, and a diffeomorphism Φ_C of UC into $\mathbb{R}^{2m}$ equivariant under $(0, \infty)$ with $\Phi_C \circ 0_C = i_C$ and $\Phi_C^*\omega_0 = \omega_C$.*

Let $W \in \mathcal{W}_\lambda$. Then we can take a compact subset K of W and a closed 1-form w on $C \setminus \overline{B_R}$ such that $\text{Spt } \|W\| \setminus K_W = \Phi_C(\text{Graph } w)$. We denote by π_Σ denotes the projection of C onto Σ . We may write $w = \pi_\Sigma^*\eta_W + dh_W$ for some 1-form η_W on Σ and some $h_W : C \setminus \overline{B_R} \rightarrow \mathbb{R}$. By results of Joyce [13, Equations (7.7) and (7.8)] we have:

Lemma 2. *If $\alpha < 2$ and $h_W = O(r^\alpha)$ then we have $\Delta_C h_W = O(r^{2(\alpha-2)})$.*

We can extend a result of Joyce [13, Theorem 7.11] as follows:

Lemma 3. *Let $W \in \mathcal{W}_\lambda$, $\lambda' < \lambda < 2$ and $[\lambda', \lambda) \cap \Lambda = \emptyset$. Then we have $W \in \mathcal{W}_{\lambda'}$.*

Proof. For each integer $n \geq 0$ we put $\lambda(n) = 2^n(\lambda-2)+2$. We can take a unique integer ν such that $\lambda(\nu+1) < \lambda' \leq \lambda(\nu)$. By an induction on $n = 0, 1, \dots, \nu$, we shall prove $h_W = O(r^{\lambda(n)})$.

By the property of h_W we have $h_W = O(r^\lambda) = O(r^{\lambda(0)})$. If $\nu = 0$ we can then complete the induction automatically. We suppose therefore $\nu > 0$. Suppose also that we have $h_W = O(r^{\lambda(n)})$ for some $n = 0, 1, \dots, \nu-1$. By Lemma 2 we have $\Delta h_W = O(r^{2(\lambda(n)-2)}) = O(r^{\lambda(n+1)-2})$. Since $n < \nu$ we get $\lambda(n+1) \geq \lambda'$ and so $[\lambda(n+1), \lambda) \cap \Lambda = \emptyset$. Applying Corollary 2 to $h_W, \lambda, \lambda(n+1)$ in place of u, p, q respectively, we get $h_W = O(r^{\lambda(n+1)})$, completing the induction.

We have thus proved $h_W = O(r^{\lambda(n)})$ for every $n = 0, 1, \dots, \nu$. Putting $n = \nu$ we get $h_W = O(r^{\lambda(\nu)})$. By Lemma 2 we have $\Delta_C h_W = O(r^{\lambda(\nu+1)-2})$. By the definition of ν we have $\lambda(\nu+1) \leq \lambda'$ and so $\Delta_C h_W = O(r^{\lambda'-2})$. Applying Corollary 2 to h_W, λ, λ' in place of u, p, q respectively, we get $h_W = O(r^{\lambda'})$, completing the proof. $\square$

Joyce [14, Definition 3.6] defines the stability of C . We have:

Theorem 6. *Let C be stable in the sense of Joyce. Then there exists $b \in \mathbb{R}^{2m}$ such that $W - b \in \mathcal{W}_0$.*

Proof. By the stability of C we have $\Lambda \cap (1, 2) = \emptyset$ and $E_1 = \{b \cdot x : b \in \mathbb{R}^{2m}\}$. We take $\lambda \in (1, 1 + \epsilon)$. Let h_W be as above. Then by Lemma 3 we have $h_W = O(r^\lambda)$. By Lemma 2, therefore, we have $\Delta_C h_W = O(r^{2(\lambda-2)})$. Applying Corollary 2 to $h_W, \lambda, 2(\lambda - 2)$ in place of u, p, q we get $h_W = b \cdot x|_C + O(r^{2(\lambda-2)+2})$.

We may suppose that for each $t \in [0, 1]$ there exist a compact subset K_t of $\mathbb{R}^{2m}$ and a 1-form w_t on $C \setminus \overline{B_R}$ such that $(\text{Spt } \|W\| - tb) \setminus K_t = \Phi_C(\text{Graph } w_t)$. We put $\beta = (b \cdot x) \circ \Phi_C$. We have then a function $\beta : UC \rightarrow \mathbb{R}$. Notice that $\Phi_C^{-1}(W - tb)$ is the image of the time-one map of the flow generated by $d\beta \lrcorner \omega_C$. Then we have $\partial w_t / \partial t = -w_t^* d\beta$ and so $w_1 = w_0 - d \int_0^1 w_t^* \beta dt = \pi_\Sigma^* \eta_W + d(h_W - \int_0^1 w_t^* \beta dt)$. Hence we get $h_{W-b} = h_W - \int_0^1 w_t^* \beta dt = h_W - \beta|_C - \int_0^1 (w_t^* \beta - \beta|_C) dt = O(r^{2\lambda-2}) + O(r^{\lambda-1}) = O(r^{2\lambda-2})$. By results of Joyce [13, Equations (7.7) and (7.8)] we have $\Delta_C h_{W-b} = O(r^{4\lambda-8})$. Applying Corollary 2 to $h_{W-b}, 2\lambda - 2, 4\lambda - 8$ in place of u, p, q we get $c \in \mathbb{R}$ such that $h_{W-b} = c + O(r^{4\lambda-6}) = O(r^0)$ as we may suppose $4\lambda - 6 < 0$. This completes the proof. $\square$

Let $\lambda < 2$ and $W \in \mathcal{W}_\lambda$. Take a compact subset K_W of $\mathbb{R}^{2m}$, an $R > 0$ and a diffeomorphism $f_W : C \setminus \overline{B_R} \rightarrow \text{Spt } \|W\| \setminus K_W$ with $f_W^* i_C = O(r^{\lambda-1})$. Then we have a Riemannian metric $f_W^* g_0$ over $C \setminus \overline{B_R}$. With respect to $f_W^* g_0$ we can define the Laplacian $\Delta_W : C^\infty(C \setminus \overline{B_R}; \mathbb{R}) \rightarrow C^\infty(C \setminus \overline{B_R}; \mathbb{R})$. We have then:

Proposition. *If $u \in C^\infty(C \setminus \overline{B_R}; \mathbb{R})$ and $u = O(r^\alpha)$ then we have $\Delta_W u = \Delta_C u + O(r^{\alpha+\lambda-4})$.*

Proof. Since $f_W - i_C = O(r^{\lambda-1})$ we get $f_W^* g_0 = i_C^* g_0 + O(r^{\lambda-2})$. We denote by ∇_W and ∇_C the Levi-Civita connexions over $C \setminus \overline{B_R}$ with respect to $f_W^* g_0$ and $i_C^* g_0$ respectively. We have then $\nabla_W = \nabla_C + O(r^{\lambda-3})$ and so $\Delta_W u = \Delta_C u + O(r^{\alpha+\lambda-4})$ as we want. $\square$

We give a corollary to this:

Corollary 3. *If $u \in C^\infty(C \setminus \overline{B_R}; \mathbb{R})$, $\Delta_W u = 0$ and $u = O(r^0)$ then we have $u = c + O(r^{2-m})$ for some $c \in \mathbb{R}$.*

Proof. By the proposition above we have $\Delta_C u = O(r^{0+\lambda-4}) = O(r^{\lambda-4})$. If $\lambda - 2 < 2 - m$ then we can complete the proof by applying Corollary 2 to $0, \lambda - 2$ in place of p, q respectively.

We suppose therefore $\lambda - 2 \geq 2 - m$. We take $\lambda' \in (\lambda, 2)$ such that $\lambda' - 2 > 2 - m$. Applying Corollary 2 to $0, \lambda' - 2$ in place of p, q respectively, we get $u = c + O(r^{\lambda'-2})$ for some $c > 0$. We have thus improved the decay order estimate for u , and so we can complete the proof in a way similar to the proof of Lemma 3. $\square$

We suppose now

$$C = \{(z_1, \dots, z_m) \in \mathbb{C}^m \setminus \{0\} : |z_1| = \dots = |z_m|, z_1 \cdots z_m \in (0, \infty)\}.$$

This is an extension of (1) in §1 to dimension m . Harvey and Lawson [9, Chapter III.3.A, Theorem 3.1] prove that C is a special Lagrangian submanifold of $(\mathbb{R}^{2m} \setminus \{0\}, \omega_0, J_0, \Omega_0)$.

We define a T^{m-1} -action on $\mathbb{C}^m$ as follows. We write $T^{m-1} = S^1 \times \dots \times S^1$ and $S^1 = \{t \in \mathbb{C} : |t| = 1\}$. For each $j \in \{2, \dots, m\}$ we define the j -th S^1 -action on $\mathbb{C}^m$ by setting $t \cdot (z_1, \dots, z_j, \dots, z_m) = (tz_1, \dots, t^{-1}z_j, \dots, z_m)$ for each $(z_1, \dots, z_m) \in \mathbb{C}^m$.

We also define a map $\mu_j : \mathbb{C}^m \rightarrow \mathbb{R}$ by $2\mu_j(z_1, \dots, z_m) = |z_1|^2 - |z_j|^2$. We shall identify $\mathbb{R}$ with the Lie algebra of S^1 so that μ_j will be the moment map on $(\mathbb{C}^m, \omega_0)$ with respect to the S^1 -action. We have then:

Theorem. *If $W \in \mathcal{W}_0$ and $j \in \{2, \dots, m\}$ then we have $\text{grad}_{TW} \mu_j = 0$ almost everywhere on $\mathbb{R}^{2m}$ with respect to $\|W\|$.*

Proof. We can take a compact subset K_W of $\mathbb{R}^{2m}$, an $R > 0$ and a diffeomorphism $f_W : C \setminus \overline{B_R} \rightarrow \text{Spt } \|W\| \setminus K_W$ with $f_W - i_C = O(r^{-1})$. Since $\mu_j = O(r^2)$ and $\mu_j \circ i_C = 0$ we get $f_W^* \mu_j = \mu_j \circ f_W = \mu_j \circ f_W - \mu_j \circ i_C = \int_0^1 d\mu_j|_{(1-t)f_W + ti_C} (f_W - i_C) dt = O(r^{2-1}r^{-1}) = O(r^0)$ and so $f_W^* \mu_j = O(r^0)$. By a result of Joyce [14, Lemma 3.4] we have $\Delta_W f_W^* \mu_j = 0$ for each $j = 2, \dots, m$. By Corollary 3 we can find $c_j \in \mathbb{R}$ such that $f_W^* \mu_j - c_j = O(r^{2-m})$. Putting $\mu'_j = \mu_j - c_j$ we get $f_W^* \mu'_j = O(r^{2-m})$. We have clearly $\text{grad } \mu'_j = \text{grad } \mu_j$.

Take a smooth function $\chi : \mathbb{R} \rightarrow [0, 1]$ with $\chi = 1$ on B_1 and $\chi = 0$ on B_2 . Let $R > 0$, and define a function $\chi_R : \mathbb{R}^{2m} \rightarrow [0, 1]$ by setting $\chi_R(x) = \chi(|x|/R)$. Since W has first variation 0 in B_{2R} we get then

$$\int_{B_{2R}} \text{div}_{TW}(\chi_R \mu'_j \text{grad } \mu_j) d\|W\| = 0.$$

Also by a result of Joyce [14, Lemma 3.4] we have $\text{div}_{TW} \text{grad } \mu_j = 0$ and so

$$\int_{B_{2R}} \mu'_j (\text{grad } \chi_R, \text{grad}_{TW} \mu_j) + \chi_R |\text{grad}_{TW} \mu_j|^2 d\|W\| = 0. \quad (12)$$

Notice that $(\text{grad } \chi_R, \text{grad}_{TW} \mu_j) = (d\chi_R, d\mu_j|_{\text{Spt } \|W\| \setminus K_W})$ on $\text{Spt } \|W\| \setminus K_W$. Take R sufficiently large so that $K_W \subset B_R$. Then we have $\text{grad } \chi_R = 0$ on K_W and so $(\text{grad } \chi_R, \text{grad}_{TW} \mu_j) = (d\chi_R, d\mu_j|_{\text{Spt } \|W\| \setminus K_W})$ on $\text{Spt } \|W\|$. We have therefore

$$\int_{B_{2R}} -\mu'_j (\text{grad } \chi_R, \text{grad}_{TW} \mu_j) d\|W\| \leq \sup_{\text{Spt } \|W\| \setminus K_W} |\mu'_j (d\chi_R, d\mu_j)| \int_{B_{2R}} d\|W\|$$

and so by (12) we have

$$\int_{B_{2R}} \chi_R |\text{grad}_{TW} \mu_j|^2 d\|W\| \leq \sup_{\text{Spt } \|W\| \setminus K_W} |\mu'_j (d\chi_R, d\mu_j)| \int_{B_{2R}} d\|W\|. \quad (13)$$

Since $f_W : C \setminus \overline{B_R} \rightarrow \text{Spt } \|W\| \setminus K_W$ is a diffeomorphism we get

$$\sup_{\text{Spt } \|W\| \setminus K_W} |\mu'_j (d\chi_R, d\mu_j)| = \sup_{C \setminus \overline{B_R}} |f_W^* \mu'_j (d\chi_R, df_W^* \mu_j)|. \quad (14)$$

Since $\chi_R = \chi(r/R)$ we get $|d\chi_R| \leq kR^{-1}$ for some $k > 0$ independent of R . Since $f_W^* \mu'_j = O(r^{2-m})$ we get

$$\sup_{C \setminus \overline{B_R}} |f_W^* \mu'_j(d\chi_R, df_W^* \mu_j)| \leq kR^{2-m} R^{-1} R^{1-m} = kR^{2-2m} \quad (15)$$

for some $k > 0$ independent of R . By (13)–(15) we have

$$\int_{B_{2R}} \chi_R |\text{grad}_{TW} \mu_j|^2 d\|W\| \leq kR^{2-2m} \int_{B_{2R}} d\|W\|.$$

On the other hand, by the monotonicity formula, we have $\int_{B_{2R}} d\|W\| \leq kR^m$ for some $k > 0$ depending only on m and $T_\infty W$. We have therefore

$$\int_{B_{2R}} \chi_R |\text{grad}_{TW} \mu_j|^2 d\|W\| \leq kR^{2-m}, k > 0 \text{ independent of } R.$$

Letting $R \rightarrow \infty$ we get $\int_{\mathbb{R}^{2m}} |\text{grad}_{TW} \mu_j|^2 d\|W\| = 0$, completing the proof. $\square$

We give a corollary to the theorem above. We define a map $f : \mathbb{C}^m \rightarrow \mathbb{C}$ by setting $f(z_1, \dots, z_m) = i^{m+1} z_1 \cdots z_m$ for each $(z_1, \dots, z_m) \in \mathbb{C}^m$. We put $\text{Im } f = (f - \bar{f})/2i : \mathbb{C}^m \rightarrow \mathbb{R}$ and $F = (\mu_2, \dots, \mu_m, \text{Im } f) : \mathbb{C}^m \rightarrow \mathbb{R}^m$. We have then:

Corollary 4. *If $W \in \mathcal{W}_0$ then we have $T_y W = \text{Ker } dF|_y$ for $\|W\|$ -almost every $y \in \mathbb{R}^{2m}$.*

This follows readily from the proof of Harvey and Lawson [9, Chapter III.3.A, Theorem 3.1].

We have moreover:

Corollary 5. *For every $W \in \mathcal{W}_0$ there exists $c \in \mathbb{R}^m$ such that $F = c$ on $\text{Spt } \|W\|$.*

Proof. By Corollary 4 we have $dF|_{\text{Spt } \|W\| \setminus K_W} = 0$, and $F|_{\text{Spt } \|W\| \setminus K_W}$ is therefore locally constant. Since $\text{Spt } \|W\| \setminus K_W \cong C \setminus \overline{B_R} \cong (R, \infty) \times T^{m-1}$ we see that $\text{Spt } \|W\| \setminus K_W$ is connected, and $F|_{\text{Spt } \|W\| \setminus K_W}$ is therefore constant; i.e. we have $F|_{\text{Spt } \|W\| \setminus K_W} = c$ for some $c \in \mathbb{R}^m$.

Put $\phi = |F - c|^2$. Then we have $\phi = 0$ on $\text{Spt } \|W\| \setminus K_W$ and so $\text{Spt } \phi \cap \text{Spt } \|W\| \subset K_W$. By Corollary 4 we have $\text{grad}_{TW} \phi = 0$ almost everywhere on $\mathbb{R}^{2m}$ with respect to $\|W\|$.

We shall now use a result of Michael and Simon [20, Theorem 2.1], who prove a Poincaré–Sobolev inequality for varifolds; we refer also to Simon [24, Theorem 18.6], who uses varifolds more explicitly. We are going to use the following version:

Lemma. *Let W be a stationary integral varifold of dimension m in $(\mathbb{R}^n, g_0)$. Suppose that we have a smooth function $\phi : \mathbb{R}^n \rightarrow [0, \infty)$ with $\text{Spt } \phi \cap \text{Spt } \|W\|$ compact and $\text{grad}_{TW} \phi = 0$ almost everywhere on $\mathbb{R}^n$ with respect to $\|W\|$. Then we have $\phi = 0$ on $\text{Spt } \|W\|$.*

We give a proof for the sake of clarity:

Proof. We define $r : \mathbb{R}^n \rightarrow [0, \infty)$ by setting $r(y) = |y|$. We can then define $\partial_r = \partial/\partial r$ as a smooth vector field on $\mathbb{R}^n \setminus \{0\}$. It is easy to see that $r\partial_r$ extends smoothly to $\mathbb{R}^n$. Let ϕ be as above, and let χ be a compactly-supported smooth function on $\mathbb{R}^n$ with $\chi = 1$ on $\text{Spt } \phi \cap \text{Spt } \|W\|$. Then we can define $\phi r\partial_r$ as a smooth vector field on $\mathbb{R}^n$. Since W has first variation 0 we get

$$\int_{\mathbb{R}^n} \text{div}_{TW} \chi \phi r \partial_r d\|V\| = 0.$$

Since $\chi = 1$ on $\text{Spt } \phi \cap \text{Spt } \|W\|$ we get $\int_{\text{Spt } \phi \cap \text{Spt } \|W\|} \text{div}_{TW} \phi r \partial_r d\|W\| = 0$. Since $\text{grad}_{TW} \phi = 0$ on $\text{Spt } \|W\|$ we get $\int_{\text{Spt } \phi \cap \text{Spt } \|W\|} \phi \text{div}_{TW} r \partial_r d\|W\| = 0$. Since $\text{div}_{TW} r \partial_r = m$ we get $\int_{\text{Spt } \phi \cap \text{Spt } \|W\|} m \phi d\|W\| = 0$. Since $\phi \geq 0$ we get $\phi = 0$ on $\text{Spt } \phi \cap \text{Spt } \|W\|$, completing the proof. $\square$

Hence we get $\phi = |F - c|^2 = 0$ on $\text{Spt } \|W\|$, completing the proof of Corollary 5. $\square$

We suppose now $m = 3$. For the fibres of $F : \mathbb{C}^3 \rightarrow \mathbb{R}^3$ we have an explicit description of the topology and asymptotic behaviour at infinity, which we shall use next. The behaviour of $F : \mathbb{C}^m \rightarrow \mathbb{R}^m$ is rather complicated if $m > 3$, which we shall not discuss.

We put $Y = \{(a, 0, 0) \in \mathbb{R}^3 : a \geq 0\} \cup \{(0, a, 0) \in \mathbb{R}^3 : a \geq 0\} \cup \{(-a, -a, 0) \in \mathbb{R}^3 : a \geq 0\}$. We note that if $c \in \mathbb{R}^3 \setminus Y$ then $F^{-1}(c)$ has no fixed point with respect to the T^2 -action

$$(e^{i\theta}, e^{i\phi}) \cdot (z_1, z_2, z_3) = (e^{i\theta} z_1, e^{i\phi} z_2, e^{-i\theta - i\phi} z_3).$$

We have:

Proposition 6. *Let $W \in \mathcal{W}$ and suppose $\text{Spt } \|W\| \subset F^{-1}(c)$ for some $c \in \mathbb{R}^3$. Then we have $c \in Y$.*

For the proof we shall use:

Lemma 4. *If $c \in \mathbb{R}^3 \setminus Y$ then $F^{-1}(c)$ is a submanifold of $\mathbb{R}^6$ diffeomorphic to $\mathbb{R} \times S^1 \times S^1$ and asymptotic to $C \cup -C$ with multiplicity 1 at infinity.*

Proof. We put $c = (c_1, c_2, c_3)$. We may suppose $c_1 \geq c_2 \geq 0$ without loss of generality. Since $(c_1, c_2, c_3) \in \mathbb{R}^3 \setminus Y$ we get $c_3 \neq 0$ or $c_2 > 0$. For each $t \in \mathbb{R}$ we can find a unique $\phi_c(t) \in [0, \infty)$ such that

$$(\phi_c(t) + c_1)(\phi_c(t) + c_2)\phi_c(t) = |t + ic_3|^2.$$

It is easy to see that $\phi_c(t)$ depends smoothly on t . We put $\psi_c(t) = \sqrt{\phi_c(t) + c_1} \sqrt{\phi_c(t) + c_2}$. Since $c_3 \neq 0$ or $c_2 > 0$ we get $\psi_c(t) > 0$ for every $t \in \mathbb{R}$, and so we can define

$\psi_c^{-1}(t) > 0$ for every $t \in \mathbb{R}$. Define a smooth map $\Phi_c : \mathbb{R} \times S^1 \times S^1 \rightarrow f^{-1}(c)$ by setting

$$\Phi_c(t, u, v) = (\sqrt{\phi_c(t) + c_1 u}, \sqrt{\phi_c(t) + c_2 v}, \frac{t + ic_3}{\psi_c(t)uv}),$$

and a smooth map $\Psi_c(t) : F^{-1}(c) \rightarrow \mathbb{R} \times S^1 \times S^1$ by setting

$$\Psi_c(z_1, z_2, z_3) = (\operatorname{Re} z_1 z_2 z_3, \frac{z_1}{\phi_c(\operatorname{Re} z_1 z_2 z_3)}, \frac{z_2}{\phi_c(\operatorname{Re} z_1 z_2 z_3)}).$$

Then $\Psi_c \circ \Phi_c$ is clearly the identity map of $\mathbb{R} \times S^1 \times S^1$. It is also easy to see that $\Phi_c \circ \Psi_c$ is the identity map of $F^{-1}(c)$, and so $F^{-1}(c)$ is a submanifold of $\mathbb{C}^3$ diffeomorphic to $\mathbb{R} \times S^1 \times S^1$. It is also easy to see that $\Phi_c \circ \Psi_c$ is asymptotic to $C \cup -C$ with multiplicity 1 at infinity. $\square$

Proof of Proposition 6. We give a proof by contradiction, and so suppose $c \notin Y$. By Lemma 4, then, $F^{-1}(c)$ will be a connected submanifold of $\mathbb{R}^6$. By a constancy theorem of Allard [1, Theorem 4.6(3)] or Simon [24, Theorem 41.1], therefore, Θ_W will be constant on $F^{-1}(c)$. On the other hand we have $\Theta_W = 1$ near infinity on $\operatorname{Spt} \|W\|$ and so we shall have $\Theta_W = 1$ on $F^{-1}(c)$, which implies $\|W\| = |F^{-1}(c)|$. By Lemma 4, however, $F^{-1}(c)$ is asymptotic at infinity to $C \cup -C$ with multiplicity 1, which contradicts that W is asymptotic at infinity to C with multiplicity 1. This completes the proof. $\square$

Let $c \in Y \subset \mathbb{R}^2 \times \{0\}$. Then we have $z_1 z_2 z_3 \in \mathbb{R}$ on $F^{-1}(c)$. For each $\epsilon > 0$ we put $L_c^\epsilon = F^{-1}(c) \cap \{z_1 z_2 z_3 > \epsilon\}$. We also put $L_c = F^{-1}(c) \cap \{z_1 z_2 z_3 \geq 0\}$. By the definition of C we have $L_0 = C$. By (11) we have $L_1 = L_{(2,0,0)}$, $L_2 = L_{(0,2,0)}$ and $L_3 = L_{(0,-2,-2)}$.

We have:

Proposition 7. *Let $W \in \mathcal{W}$ and suppose $\operatorname{Spt} \|W\| \subset F^{-1}(c)$ for some $c \in Y$. Then we have $\|W\| = |sL|$ for some $s > 0$ and $L \in \{C, L_1, L_2, L_3\}$.*

For the proof we shall use:

Lemma 5. *L_1, L_2, L_3 are submanifolds of $\mathbb{C}^3$ diffeomorphic to $S^1 \times \mathbb{R}^2$ and asymptotic to C with multiplicity 1 at infinity.*

Proof of Proposition 7. We give the proof only for L_1 ; we can do it likewise for L_2 and L_3 . By definition we have $L_1 = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : |z_1|^2 - 1 = |z_2|^2 = |z_3|^2, z_1 z_2 z_3 \in [0, \infty)\}$. Define a smooth map $\phi : L_1 \rightarrow S^1 \times \mathbb{C}$ by setting $\phi(z_1, z_2, z_3) = (z_1 / \sqrt{|z_2|^2 + 1}, z_2)$, and a smooth map $\psi : S^1 \times \mathbb{C} \rightarrow L_1$ by setting $\psi(e^{i\theta}, z) = (\sqrt{|z|^2 + 1}e^{i\theta}, z, e^{-i\theta}\bar{z})$. Then $\phi \circ \psi$ is clearly the identity map of $S^1 \times \mathbb{C}$. It is also easy to see that $\psi \circ \phi$ is the identity map of L_1 . This implies that L_1 is a submanifold of $\mathbb{C}^3$ diffeomorphic to $S^1 \times \mathbb{C}$. It is also easy to see that L_1 is asymptotic to C with multiplicity 1 at infinity. $\square$

Proof of Proposition 7. We have only to prove $\|W\| = |L_c|$. For each $\epsilon > 0$ we put $U_\epsilon^+ = \{\operatorname{Re} z_1 z_2 z_3 > \epsilon\}$ and $U_\epsilon^- = \{\operatorname{Re} z_1 z_2 z_3 < -\epsilon\}$. In a way similar to the proof of Lemma 5 we can take $\epsilon_0 > 0$ such that if $\epsilon \in (0, \epsilon_0)$ then $F^{-1}(c) \cap U_\epsilon^+$ and $F^{-1}(c) \cap U_\epsilon^-$ will be connected submanifolds of $\mathbb{R}^6$. By the constancy theorem, therefore, Θ_W is constant on $F^{-1}(c) \cap U_\epsilon^+$ and $F^{-1}(c) \cap U_\epsilon^-$, respectively.

In a way similar to the proof of Lemma 5 we can prove that $F^{-1}(c) \cap U_\epsilon^+$ is asymptotic at infinity to C with multiplicity 1 and $F^{-1}(c) \cap U_\epsilon^-$ is asymptotic at infinity to $-C$ with multiplicity 1. Notice also that W is asymptotic at infinity to C with multiplicity 1. Then we see that on $F^{-1}(c) \cap U_\epsilon^+$ we have $\Theta_W = 1$ and on $F^{-1}(c) \cap U_\epsilon^-$ we have $\Theta_W = 0$. This holds for every $\epsilon > 0$, and so $\|W\| = |F^{-1}(c) \cap \{\operatorname{Re} z_1 z_2 z_3 \geq 0\}| = |L_c|$ as we want. $\square$

We shall now complete the proof of Theorem 5. We suppose that W is as in Theorem 5. By Theorem 6, then, we can find $b \in \mathbb{R}^6$ such that $W - b \in \mathcal{W}_0$. By Corollary 5, therefore, we can find $c \in \mathbb{R}^3$ such that $\operatorname{Spt} \|W - b\| \subset F^{-1}(c)$. By Proposition 6, therefore, we have $c \in Y$. By Proposition 7, therefore, we have $\|W - b\| = |sL|$ for some $a > 0$ and $L \in \{C, L_1, L_2, L_3\}$ as we want.

5 Proof of Theorem 1

As in §1 the main steps in the proof of Theorem 1 are the bubbling-off analysis in §3 and the classification result in §4. In order to complete the proof, however, we have to do some technical things in addition. We begin with a review of the theory of Joyce [13, 14, 15, 16, 17].

We suppose that (M, ω, J, Ω) is an almost Calabi–Yau manifold of dimension 3 over $\mathbb{C}$. We suppose that M is compact. We take $a \in H_3(M; \mathbb{Z})$. As in §2 we denote by $\mathcal{V}$ the space of all special Lagrangian varifolds V in (M, ω, J, Ω) with $\operatorname{Spt} \|V\|$ compact, $\partial \vec{V} = 0$ and $[\vec{V}] = a$.

We denote by $\mathcal{X}$ the set of all elements of $\mathcal{V}$ with support connected, and singularity only at one point and modelled on C with multiplicity 1. We take $X \in \mathcal{X}$. We denote by x the singular point of X . We take a linear isomorphism $\gamma : \mathbb{R}^6 \rightarrow T_x M$ with $\gamma^* g|_x = g_0$, $\gamma^* J|_x = J|_0$, $\gamma^* \Omega|_x = \Omega_0$ and $\gamma(C)$ a multiplicity 1 smooth tangent cone to X at x . We may regard X as a submanifold of $M \setminus \{x\}$, which we shall denote by X' .

As in §2 we define $\psi : M \rightarrow (0, \infty)$ by (2). We have then $\omega_0 = \psi^2(x) \gamma^* \omega|_x$. By a theorem of Darboux we can find a real number $\delta > 0$ and a diffeomorphism Γ of B_δ into M with $\Gamma(0) = x$, $d\Gamma|_0 = \psi(x) \gamma$ and $\Gamma^* \omega|_x = \omega_0$.

Joyce [14, §3.2] proves that C is rigid and so special Lagrangian Jacobi-integrable, which implies that x is an isolated conical singularity in the sense of Joyce [13, Definition 3.6]; for the details we refer to Joyce [13, Theorem 6.8], Allard–Almgren [2], Simon [23, Part II, §6] and Adams–Simon [4].

We define T^*C , ω_C , U_C and Φ_C as in Lemma 1. From results of Joyce [13, Theorem 4.4 and Lemma 4.5] we see that making $\delta > 0$ smaller if necessary we can take an embedding f_X of $C \cap B_\delta$ into X' , a function $h_X : C \cap B_\delta \rightarrow \mathbb{R}$

and an $\alpha > 2$ with $h_X = O(r^\alpha)$ such that $f_X^{-1}(X') = \Phi_C(\text{Graph } dh_X)$. We put $Z = X' \setminus \text{Im } f_X$ where $\text{Im } f_X = f_X(C \cap B_\delta)$. It is clear that Z is a compact submanifold of X' of co-dimension 0 with boundary diffeomorphic to Σ .

We denote by T^*X' the cotangent bundle over X' , by 0_C the zero-section of T^*X' , and by ω_X the canonical symplectic form on T^*X' . Since f_X maps $C \cap B_\delta$ diffeomorphically onto $X' \setminus Z$ we get a vector-bundle isomorphism $T^*f_X : T^*(X' \setminus Z) \rightarrow T^*(C \cap B_\delta)$ covering $f_X^{-1} : X' \setminus Z \rightarrow C \cap B_\delta$. By a result of Joyce [13, Theorem 4.6] we can find a neighbourhood UX of $\text{Im } 0_X$ in T^*X' and a diffeomorphism Φ_X of UX into M with $\Phi_X \circ 0_X = i_X$, $\Phi_X^*\omega = \omega_X$ and

$$\Phi_X|_{UX \setminus T^*Z} = \Gamma \circ \Phi_C \circ (+dh_X) \circ T^*f_X$$

where $+dh_X$ denotes the fibrewise translation of T^*C by dh_X .

For each $y \in M$ we denote by $P|_y$ the set of all linear isomorphisms $\phi : \mathbb{R}^6 \rightarrow T_y M$ with $\phi^*g|_y = g_0$, $\phi^*J|_y = J_0$ and $\phi^*\Omega|_y = \Omega_0$. We put $P = \bigcup_{y \in M} P|_y$. It is clear that P is a principal bundle over M with structure group SU_3 . We have $(x, \gamma) \in P$. By a result of Joyce [14, Theorem 5.2] we can take a neighbourhood $U_{x, \gamma}$ of (x, γ) in P such that for all $p = (y, \phi) \in U_{x, \gamma}$ we can construct embeddings Γ_p of B_δ into M depending smoothly on p with $\Gamma_p(0) = y$, $d\Gamma_p|_0 = \phi$ and $\psi^2(y)\Gamma_p^*\omega = \omega_0$, and embeddings Φ_p of UX into M depending smoothly on p with $\Phi_p \circ 0_X = i_X$, $\Phi_p^*\omega = \omega_X$ and

$$\Phi_p|_{UX \setminus T^*Z} = \Gamma_p \circ \Phi_C \circ (+dh_X) \circ T^*f_X.$$

Joyce [14, Definition 5.6] defines a topology on $\mathcal{X}$. On the other hand $\mathcal{X}$ has also the topology induced from $\mathcal{V}$. By a result of Joyce [14, Theorem 5.3], Allard's regularity theorem and Theorem 3, the two topologies on $\mathcal{X}$ are the same.

By the stability of C we can apply a result of Joyce [14, Corollary 6.11], who proves that we can take a neighbourhood $\mathcal{Y}$ of X in $\mathcal{X}$ which has a structure of manifold, and $T_X \mathcal{Y}$ is isomorphic to the image of $H_c^1(X'; \mathbb{R}) \rightarrow H^1(X'; \mathbb{R})$. We can parametrize $\mathcal{Y}$ as follows. Since $\text{Spt } \|X\|$ and Σ are connected we see that X' is also connected, and so $H_c^1(X'; \mathbb{R}) \rightarrow H^1(X'; \mathbb{R})$ is injective. We denote by Ω_Z^1 the space of all 1-forms on X' supported in Z ; we can then define the natural projection $\pi_Z : \Omega_Z^1 \rightarrow H_c^1(X'; \mathbb{R})$; we can take a linear subspace H_Z^1 of Ω_Z^1 such that π_Z maps H_Z^1 isomorphically onto $H_c^1(X'; \mathbb{R})$. We take a real number $\mu \in (2, 3)$ as in Joyce [14, Definition 3.7] and define a function space $C_\mu^\infty(X'; \mathbb{R})$ as in [14, Definition 4.2]. From the proof of Joyce [14, Corollary 6.11] we get:

Lemma 6. *There exist a neighbourhood $\mathcal{Z}$ of 0 in H_Z^1 and an embedding $\mathcal{H}$ of $\mathcal{Z}$ into $\mathcal{X}$ with the following property: for all $\zeta \in \mathcal{Z}$ there exist $p(\zeta) \in P$ and $h_\zeta \in C_\mu^\infty(X'; \mathbb{R})$ depending smoothly on ζ with $\mathcal{H}(\zeta) = \Phi_{p(\zeta)}(\text{Graph}(\zeta + dh_\zeta))$.*

We define L_1 , L_2 and L_3 by (11) in §4. We take $L \in \{L_1, L_2, L_3\}$. We can then take a compact submanifold K of L of co-dimension 0 with boundary diffeomorphic to Σ , a real number $R > 0$ and a diffeomorphism f_L of $C \setminus \overline{B_R}$

onto $L \setminus K$ with $f_L - i_C = O(r^{-1})$. Making $R > 0$ larger if necessary we can find a 1-form η_L on $C \setminus \overline{B_R}$ with $\Phi_C(\text{Graph } \eta_L) = L \setminus K$.

We denote by T^*L the cotangent bundle over L , by 0_L the zero-section of T^*L , and by ω_L the canonical symplectic form on T^*L . Since f_L maps $C \setminus \overline{B_R}$ diffeomorphically onto $L \setminus K$ we get a vector-bundle isomorphism $T^*f_L : T^*(L \setminus K) \rightarrow T^*(C \setminus \overline{B_R})$ covering $f_L^{-1} : L \setminus K \rightarrow C \cap \overline{B_R}$. By a result of Joyce [13, Theorem 7.5] we can find a neighbourhood UL of $\text{Im } 0_L$ in T^*L and a diffeomorphism Φ_L of UL into $\mathbb{R}^6$ with $\Phi_L^* \omega_0 = \omega_L$, $\Phi_L \circ 0_L = i_L$ and

$$\Phi_L|_{UL \setminus T^*K} = \Phi_C \circ (+\eta_L) \circ T^*f_L$$

where $+\eta_L$ denotes the fibrewise translation of T^*C by η_L .

We have a homomorphism $f_X^* : H^1(X'; \mathbb{R}) \rightarrow H^1(C \cap B_\delta; \mathbb{R}) \cong H^1(\Sigma; \mathbb{R})$. We denote by $[\eta_L]$ the de Rham cohomology class of η_L . We have then $[\eta_L] \in H^1(C \setminus \overline{B_R}; \mathbb{R}) \cong H^1(\Sigma; \mathbb{R})$. By Hodge theory we can take a harmonic 1-form η'_L on Σ with $[\eta'_L] = [\eta_L]$. We denote by π_Σ the projection of C onto Σ . We put $\tilde{\eta}_L = \pi_\Sigma^* \eta'_L$.

We denote by $\text{Im } f_X^*$ the image of the linear map $f_X^* : H^1(X'; \mathbb{R}) \rightarrow H^1(C \cap B_\rho; \mathbb{R}) \cong H^1(\Sigma; \mathbb{R})$. Joyce [17, Lemma 10.1] proves that $\text{Im } f_X^*$ is of dimension 1 over $\mathbb{R}$. Joyce [17, Equation (77)] proves also that if $L \neq L' \in \{L_1, L_2, L_3\}$ then $[\eta_L]$ and $[\eta_{L'}]$ are linearly independent. Hence we get:

Lemma 7. *There exists at most one $L \in \{L_1, L_2, L_3\}$ with $[\eta_L] \in \text{Im } f_X^*$.*

As in §1 we suppose $X \in \partial \mathcal{N}$. By Corollary 6 below, then, we shall have $L \in \{L_1, L_2, L_3\}$ with $[\eta_L] \in \text{Im } f_X^*$. Hence we can find a closed 1-form ξ_L on X' with $f_X^*[\xi_L] = [\eta_L]$. By a result of Joyce [17, Theorem 10.4] there exist a real number $\tau > 0$, an open submanifold $\hat{K}$ of L with $K \subset \hat{K}$ and an open submanifold $\hat{Z}$ of X with $Z \subset \hat{Z}$ such that for all $t \in (0, \tau)$ we can find $u_C : C \cap A_{tR, \delta} \rightarrow \mathbb{R}$ with $|du_C|_{[tR, \delta]}^{1, \text{cyl}} = o(t)$, $u_L : \hat{K} \rightarrow \mathbb{R}$ with $|du_L|_{C^1(\hat{K})} = o(t)$, and $u_X : \hat{Z} \rightarrow \mathbb{R}$ with $|du_X|_{C^1(\hat{Z})} = o(t)$ such that on $f_L^{-1}(\hat{K}) \cap A_{tR, \delta}$ we have $t^2(f_L^* du_L + \eta_L) = du_C + t^2 \tilde{\eta}_L$, on $f_X^{-1}(\hat{Z}) \cap A_{tR, \delta}$ we have $f_X^*(du_X + t^2 \xi_L) + dh_X = du_C + t^2 \tilde{\eta}_L$, and

$$\Gamma(t\Phi_L(\text{Graph } du_L)) \cup \Gamma \circ \Phi_C(\text{Graph}(du_C + t^2 \tilde{\eta}_L)) \cup \Phi_X(\text{Graph}(du_X + t^2 \xi_L))$$

will be a compact special Lagrangian submanifold of (M, ω, J, Ω) . From the proof Joyce [17, Theorem 10.4] it follows readily that we can extend the result as follows:

Lemma 8. *There exist a real number $\tau > 0$, a neighbourhood $U_{x, \gamma}$ of (x, γ) in P , a neighbourhood $\mathcal{Z}$ of 0 in H_Z^1 , an open submanifold $\hat{K}$ of L with $K \subset \hat{K}$ and an open submanifold $\hat{Z}$ of X with $Z \subset \hat{Z}$ such that for all $(t, p, \zeta) \in (0, \tau) \times U_{x, \gamma} \times \mathcal{Z}$ we can find $u_C \in C^\infty(C \cap A_{tR, \delta}; \mathbb{R})$ with $|u_C|_{[tR, \delta]}^{1, \text{cyl}} = o(t)$, $u_L \in C^\infty(\hat{K}; \mathbb{R})$ with $|u_L|_{C^1(\hat{K})} = o(t)$, and $u_X \in C^\infty(\hat{Z}; \mathbb{R})$ with $|u_X|_{C^1(\hat{Z})} = o(t)$ such that on*

$f_L^{-1}(\widehat{K}) \cap A_{tR,\delta}$ we have $t^2(f_L^* du_L + \eta_L) = du_C + t^2 \tilde{\eta}_L$, on $f_X^{-1}(\widehat{Z}) \cap A_{tR,\delta}$ we have $f_X^*(du_X + \zeta + dh_\zeta + t^2 \xi_L) + dh_X = du_C + t^2 \tilde{\eta}_L$, and if we put

$$G(t, p, \zeta) = \Gamma_p(t\Phi_L(\text{Graph } du_L)) \cup \Gamma_p \circ \Phi_C(\text{Graph}(du_C + t^2 \tilde{\eta}_L)) \\ \cup \Phi_p(\text{Graph}(du_X + \zeta + dh_\zeta + t^2 \xi_L))$$

then $G(t, p, \zeta)$ will be a compact special Lagrangian submanifold of (M, ω, J, Ω) .

Thus we get a map $G : (0, \tau) \times U_{x,\gamma} \times \mathcal{Z} \rightarrow \mathcal{N}$.

We take $(t, p, \zeta) \in (0, \tau) \times U_{x,\gamma} \times \mathcal{Z}$. We denote by $i_{t,p,\zeta}$ the inclusion map of $G(t, p, \zeta)$ into M . We denote by $T^*G(t, p, \zeta)$ the cotangent bundle over $G(t, p, \zeta)$, and by $0_{t,p,\zeta}$ the zero-section of $T^*G(t, p, \zeta)$. We shall define a neighbourhood $U_{t,p,\zeta}$ of $\text{Im } 0_{t,p,\zeta}$ and an embedding $\Phi_{t,p,\zeta}$ of $U_{t,p,\zeta}$ into M with $\Phi_{t,p,\zeta} \circ 0_{t,p,\zeta} = i_{t,p,\zeta}$.

We put $\phi_L^{t,p,\zeta} = \Gamma_p \circ t \circ \Phi_L \circ du_L$ where du_L is regarded as a map of $\widehat{K}$ into $T^*\widehat{K}$. Thus we get an embedding $\phi_L^{t,p,\zeta} : \widehat{K} \rightarrow G(t, p, \zeta)$. We put $G_L^{t,p,\zeta} = \text{Im } \phi_L^{t,p,\zeta}$. Since $\phi_L^{t,p,\zeta}$ maps $\widehat{K}$ diffeomorphically onto $G_L^{t,p,\zeta}$ we get a vector bundle isomorphism $T^*\phi_L^{t,p,\zeta} : T^*G_L^{t,p,\zeta} \rightarrow T^*\widehat{K}$. On a neighbourhood of $\text{Im}(0_{t,p,\zeta} \circ \phi_L^{t,p,\zeta})$ in $T^*G(t, p, \zeta)$ we set $\Phi_{t,p,\zeta} = \Gamma_p \circ t \circ \Phi_L \circ (+du_L) \circ T^*\phi_L^{t,p,\zeta}$.

We put $\phi_C^{t,p,\zeta} = \Gamma_p \circ \Phi_C \circ (du_C + t^2 \tilde{\eta}_L)$ where $du_C + t^2 \tilde{\eta}_L$ is regarded as a map of $C \cap A_{tR,\delta}$ into T^*C . Thus we get an embedding $\phi_C^{t,p,\zeta} : C \cap A_{tR,\delta} \rightarrow G(t, p, \zeta)$. We put $G_C^{t,p,\zeta} = \text{Im } \phi_C^{t,p,\zeta}$. Since $\phi_C^{t,p,\zeta}$ maps $C \cap A_{tR,\delta}$ diffeomorphically onto $G_C^{t,p,\zeta}$ we get a vector bundle isomorphism $T^*\phi_C^{t,p,\zeta} : T^*G_C^{t,p,\zeta} \rightarrow T^*(C \cap A_{tR,\delta})$. On a neighbourhood of $\text{Im}(0_{t,p,\zeta} \circ \phi_C^{t,p,\zeta})$ in $T^*G(t, p, \zeta)$ we set $\Phi_{t,p,\zeta} = \Gamma_p \circ \Phi_C \circ (+du_C) \circ T^*\phi_C^{t,p,\zeta}$.

We put $\phi_X^{t,p,\zeta} = \Phi_p \circ (du_X + \zeta + dh_\zeta + t^2 \xi_L)$ where $du_X + \zeta + dh_\zeta + t^2 \xi_L$ is regarded as a map of $\widehat{Z}$ into $T^*\widehat{Z}$. Thus we get an embedding $\phi_X^{t,p,\zeta} : \widehat{Z} \rightarrow G(t, p, \zeta)$. We put $G_X^{t,p,\zeta} = \text{Im } \phi_X^{t,p,\zeta}$. Since $\phi_X^{t,p,\zeta}$ maps $\widehat{Z}$ diffeomorphically onto $G_X^{t,p,\zeta}$ we get a vector bundle isomorphism $T^*\phi_X^{t,p,\zeta} : T^*G_X^{t,p,\zeta} \rightarrow T^*\widehat{Z}$. On a neighbourhood of $\text{Im}(0_{t,p,\zeta} \circ \phi_X^{t,p,\zeta})$ in $T^*G(t, p, \zeta)$ we set $\Phi_{t,p,\zeta} = \Phi_p \circ (+du_X) \circ T^*\phi_X^{t,p,\zeta}$. Thus we get:

Lemma 9. *There exists a neighbourhood $U_{t,p,\zeta}$ of $\text{Im } 0_{t,p,\zeta}$ and an embedding $\Phi_{t,p,\zeta}$ of $U_{t,p,\zeta}$ into M with $\Phi_{t,p,\zeta} \circ 0_{t,p,\zeta} = i_{t,p,\zeta}$.*

We shall now take a neighbourhood $\mathcal{U}$ of X in $\mathcal{V}$ and define a map $F : \mathcal{U} \cap \mathcal{N} \rightarrow (0, \tau) \times \mathcal{Z}$. By Allard's regularity theorem we may suppose that for every $N \in \mathcal{U} \cap \mathcal{N}$ we can find a unique 1-form β_X on Z such that $N \setminus \Phi_X(T^*(C \cap B_\delta)) = \Phi_X(\text{Graph } \beta_X)$. Since $0 \neq [\eta_L] \in \text{Im } f_X^*$ and $\text{Im } f_X^*$ is of dimension 1 we can find a unique $s \in \mathbb{R}$ with $s[\eta_L] = f_X^*[\beta_X]$. We have then a compactly-supported 1-form $\beta_X - s\xi_L$ on Z . As $[\beta_X - s\xi_L]|_{\partial Z} = 0$ we may write $[\beta_X - s\xi_L] \in H_c^1(X'; \mathbb{R})$. As π_Z maps H_Z^1 isomorphically onto $H_c^1(X'; \mathbb{R})$ we may write $[\beta_X - s\xi_L] \in H_Z^1$. We set $F(N) = (s, [\beta_X - s\xi_L])$.

It is easy to see that for each $(t, \zeta) \in (0, \tau) \times \mathcal{Z}$ we have $F \circ G(t, \zeta) = (t^2, \zeta)$. Hence we see that $G : (0, \tau) \times \mathcal{Y} \rightarrow \mathcal{N}$ is injective. By an argument of Joyce [17,

§8.2], moreover, $F : \mathcal{U} \cap \mathcal{N} \rightarrow (0, \tau) \times \mathcal{Z}$ is a local diffeomorphism. Hence we see that $G(t, p, \zeta)$ is independent of $p \in U_{x, \gamma}$ as we may suppose that $U_{x, \gamma}$ is connected. We may therefore write $G : (0, \tau) \times \mathcal{Z} \rightarrow \mathcal{N}$.

We can extend G continuously to $[0, \tau) \times \mathcal{Z}$ by setting $G(0, 0) = 0$ for each $0 \in \mathcal{Z}$. We can re-state Theorem 1 as follows:

Theorem. *G maps $[0, \tau) \times \mathcal{Z}$ onto a neighbourhood of X in $\mathcal{V}$.*

We are going now to prove the theorem above.

Using the results of §§3–4 we shall determine a neighbourhood of X in $\mathcal{V}$:

Theorem 7. *Let $\epsilon > 0$, let $\widehat{K}$ be an open submanifold of L with $K \subset \widehat{K}$, and let $\widehat{Z}$ be an open submanifold of X with $Z \subset \widehat{Z}$. Then there exists a neighbourhood $\mathcal{U}$ of X in $\mathcal{V}$ such that for every $V \in \mathcal{U} \setminus \mathcal{X}$ we can find an $L \in \{L_1, L_2, L_3\}$, a real number $s > 0$, a point $p \in P$, a 1-form β_C on $C \cap A_{sR, \delta}$ with $|\beta_C|_{[sR, \delta]}^{1, \text{cy}1} < \epsilon$, a 1-form β_L on $\widehat{K}$ with $|\beta_L|_{C^1(\widehat{K})} < \epsilon$, and a 1-form β_X on $\widehat{Z}$ with $|\beta_X|_{C^1(\widehat{Z})} < \epsilon$ such that on $f_L^{-1}(\widehat{K}) \cap A_{sR, \delta}$ we have $s^2(f_L^* \beta_L + \eta_L) = \beta_C$, on $f_X^{-1}(\widehat{Z}) \cap A_{sR, \delta}$ we have $f_X^* \beta_X + dh_X = \beta_C$, and if we put*

$$N = |\Gamma_p(s\Phi_L(\text{Graph } \beta_L)) \cup \Gamma_p \circ \Phi_C(\text{Graph } \beta_C) \cup \Phi_X(\text{Graph } \beta_X)|$$

then we have $\|V\| = |N|$ where $|N| = \mathcal{H}^3 \llcorner N$ where $\mathcal{H}^3$ denotes the Hausdorff 3-dimensional measure on (M, g) .

Remark. Since $s^2(f_L^* \beta_L + \eta_L) = \beta_C$ on $f_L^{-1}(\widehat{K}) \cap A_{sR, \delta}$ and $f_X^{-1}(\widehat{Z}) \cap A_{sR, \delta}$ on $f_X^* \beta_X + dh_X = \beta_C$ we see that N will be a compact submanifold of M .

Proof. Let $(V_n)_{n=1}^\infty$ be a sequence in $\mathcal{V} \setminus \mathcal{X}$ converging to X . Then we have only to prove that for each $\nu = 1, 2, 3, \dots$ we can find an integer $n > \nu$ such that with V_n in place of V the property above will hold.

We consider $\gamma^* \exp_x^* V_n$. To $\gamma^* \exp_x^* V_n$ in place of V_n we can apply Theorem 4, and so we can find a sequence $(\delta_n)_{n=1}^\infty$ in $(0, \infty)$ converging to 0, a sequence $(y_n)_{n=1}^\infty$ in B_ρ converging to 0, and a subsequence of $(\delta_n^{-1}(V_n - y_n))_{n=1}^\infty$ converging to some special Lagrangian varifold W in $(\mathbb{R}^6, \omega_0, J_0, \Omega_0)$ asymptotic at infinity to some $C' \in \mathcal{C}_\epsilon$ with multiplicity 1 and satisfying $\mathcal{E}(W - b) > 0$ for each $b \in \mathbb{R}^6$. We may suppose that $(\delta_n^{-1}(V_n - y_n))_{n=1}^\infty$ itself converges to W .

Since C is rigid in the sense of Joyce we can find $a \in SU_3$ with $aC = C'$. To $a^{-1}W$ we can apply Theorem 5 and so we can find $\lambda > 0$, $L \in \{C, L_1, L_2, L_3\}$ and $b \in \mathbb{R}^6$ such that $\|W\| = |\lambda L + b|$. Since $\mathcal{E}(W - b) > 0$ we get $\mathcal{E}(L) > 0$ and so $L \neq C$, which implies $L \in \{L_1, L_2, L_3\}$.

We define an embedding E_n of B_ρ into M by setting $E_n(y) = \exp_x \gamma a(y + \delta_n b)$. We put $y(n) = E_n(0) = \exp_x \gamma a(\delta_n b)$. It is easy to see that $y(n)$ tends as $n \rightarrow \infty$ to $\exp_x \gamma a(0) = x$. We put also $\phi(n) = dE_n|_0 = \gamma \circ a$. It is easy to see that $\phi(n)$ is close to γ .

We put $s_n = \delta_n \lambda$. From Allard's regularity theorem and Theorem 2 we see that for n sufficiently large we can find an embedding $e_{L, n}$ of $\widehat{K}$ into $\mathbb{R}^6$ with $|e_{L, n} - i_L|_{C^1(\widehat{K})} < \epsilon$, an embedding $e_{C, n}$ of $C \cap A_{s_n R, \sigma}$ into $\mathbb{R}^6$, with

$|e_{C,n} - i_C|_{[s_n R, \sigma]}^{1, \text{cyl}} < \epsilon$ and an embedding of $\widehat{Z}$ into X with $|e_{X,n} - i_X|_{C^1(\widehat{Z})} < \epsilon$ such that

$$\|V_n\| = |E_n(s_n \text{Im } e_{L,n}) \cup E_n(\text{Im } e_{C,n}) \cup \text{Im } e_{X,n}|.$$

We put $p(n) = (y(n), \phi(n))$. We have then $\Gamma_{p(n)}(0) = E_n(0)$ and $d\Gamma_{p(n)}|_0 = dE_n|_0$. Hence we can find an embedding $f_{L,n}$ of $\widehat{K}$ into $\mathbb{R}^6$ with $|f_{L,n} - i_L|_{C^1(\widehat{K})} < \epsilon$, an embedding $f_{C,n}$ of $C \cap A_{s_n R, \sigma}$ into $\mathbb{R}^6$, with $|f_{C,n} - i_C|_{[s_n R, \sigma]}^{1, \text{cyl}} < \epsilon$ and an embedding of $\widehat{Z}$ into X with $|f_{X,n} - i_X|_{C^1(\widehat{Z})} < \epsilon$ such that

$$\|V_n\| = |\Gamma_{p(n)}(s_n \text{Im } f_{L,n}) \cup \Gamma_{p(n)}(\text{Im } f_{C,n}) \cup \text{Im } f_{X,n}|.$$

Hence we can find a 1-form $\beta_{L,n}$ on $\widehat{K}$ with $|\beta_{L,n}|_{C^1(\widehat{K})} < \epsilon$, a 1-form $\beta_{C,n}$ on $C \cap A_{s_n R, \delta}$ with $|\beta_{C,n}|_{[s_n R, \delta]}^{1, \text{cyl}} < \epsilon$, and a 1-form $\beta_{X,n}$ on $\widehat{Z}$ with $|\beta_{X,n}|_{C^1(\widehat{Z})} < \epsilon$ such that

$$\|V\| = |\Gamma_p(s_n \Phi_L(\text{Graph } \beta_L)) \cup \Gamma_p \circ \Phi_C(\text{Graph } \beta_C) \cup \Phi_X(\text{Graph } \beta_X)|.$$

This implies that on $f_L^{-1}(\widehat{K}) \cap A_{s_n R, \delta}$ we have $s^2(f_L^* \beta_{L,n} + \eta_L) = \beta_{C,n}$, on $f_X^{-1}(\widehat{Z}) \cap A_{s_n R, \delta}$ we have $f_X^* \beta_{X,n} + dh_X = \beta_{C,n}$. This contradicts our hypothesis, and so completes the proof. $\square$

We give a corollary to Theorem 7:

Corollary 6. *Suppose that for every neighbourhood $\mathcal{U}$ of X in $\mathcal{V}$ we have $\mathcal{U} \setminus \mathcal{X} \neq \emptyset$. Then there exists an $L \in \{L_1, L_2, L_3\}$ with $[\eta_L] \in \text{Im } f_X^*$.*

Proof. Let $\epsilon > 0$, and let $\mathcal{U}$ be as in Theorem 7. Then we have by assumption $\mathcal{U} \setminus \mathcal{X} \neq \emptyset$, and so we can find some $V \in \mathcal{U} \setminus \mathcal{X}$. Let s, β_L, β_C and β_X be as in Theorem 7. Then we have $s^2[f_L^* \beta_L + \eta_L] = [f_X^* \beta_X + dh_X] \in H^1(\Sigma; \mathbb{R})$ and so $[f_L^* \beta_L + \eta_L] \in \text{Im } f_X^*$. Notice also that $[\beta_L]$ tends to 0 as $\epsilon \rightarrow 0$. Then we get $[\eta_L] \in \text{Im } f_X^*$, completing the proof. $\square$

We shall give another corollary to Theorem 7. By assumption we have $X \in \partial \mathcal{N}$. By Corollary 6, therefore, we can find $L \in \{L_1, L_2, L_3\}$ with $[\eta_L] \in \text{Im } f_X^*$. We take $\epsilon > 0$ and suppose that $\widehat{K}$ and $\widehat{Z}$ are as in Theorem 7. Making $\mathcal{K}$ and $\widehat{Z}$ smaller if necessary we can construct $G : (0, \tau) \times U_{x, \gamma} \times \mathcal{Z} \rightarrow \mathcal{N}$ as in Lemma 8. We define $\Phi_{t,p,\zeta}$ as in Lemma 9. We have then:

Corollary 7. *There exists a neighbourhood $\mathcal{U}$ of X in $\mathcal{V}$ such that for every $V \in \mathcal{U} \setminus \mathcal{X}$ there exist $t > 0$, $p \in P$, $\zeta \in H_Z^1$ and $w \in C^\infty(G(t, p, \zeta); \mathbb{R})$ with $|dw|_{C^1(G(t,p,\zeta))} < \epsilon$ such that $\|V\| = |\Phi_{t,p,\zeta}(\text{Graph } dw)|$.*

Proof. Let $\mathcal{U}$ be as in Theorem 7, and let $V \in \mathcal{U} \setminus \mathcal{X}$. Let β_C and β_X be as in Theorem 7. Then we have $[\beta_C] = [dh_X + f_X^* \beta_X] = [f_X^* \beta_X]$ and so $[\beta_C] \in \text{Im } f_X^*$. Let s and β_L be as in Theorem 7. Then we have $s^2[f_L^* \beta_L + \eta_L] = [\beta_C]$. We have also $|\beta_L|_{C^1(\widehat{K})} < \epsilon$ and $[\eta_L] \neq 0$. We may therefore suppose $[\beta_C] \neq 0$. Notice that $\text{Im } f_X^*$ is generated by $[\eta_L]$ and $[\beta_C]$ is sufficiently close to $s^2[\eta_L]$. Then we can find a unique $t > 0$ with $t^2[\eta_L] = [\beta_C]$ and t sufficiently close to s .

Notice that $f_X^*[\beta_X - t^2\xi_L] = [\beta_C] - t^2[\eta_L] = 0$. Notice also that we have an exact sequence $H_c^1(X'; \mathbb{R}) \rightarrow H^1(X'; \mathbb{R}) \rightarrow H^1(\Sigma; \mathbb{R})$. Then we get $\beta_X - t^2\xi_L \in \text{Im}(H_c^1(X'; \mathbb{R}) \rightarrow H^1(X'; \mathbb{R}))$. Hence we can find $a : X' \rightarrow \mathbb{R}$ with $\beta_X - t^2\xi_L - da$ compactly-supported. We denote its compactly-supported de Rham cohomology class by $[v_X - t^2\xi_L - da]_c$. Let π_Z and H_Z^1 be as in Lemma 6. Then π_Z maps H_Z^1 isomorphically onto $H_c^1(X'; \mathbb{R})$. Hence we can find a unique $\zeta \in H_Z^1$ with $[\zeta]_c = [\beta_X - t^2\xi_L - da]_c$ where $[\zeta]_c$ denotes compactly-supported de Rham cohomology class of ζ . Let $\phi_X^{t,p,\zeta}$ be as in Lemma 9. Then we can find an exact 1-form α_X on $\widehat{Z}$ such that $\text{Spt } \|V\| \cap \Phi_p(T^*Z) = \Phi_{t,p,\zeta}(\text{Graph } \alpha_X)$.

Let p be as in Theorem 7. Then we can find a 1-form α on $G(t, p, \zeta)$ with $|\alpha|_{C^1(G(t,p,\zeta))} < \epsilon$ such that $\|V\| = |\Phi_{t,p,\zeta}(\text{Graph } w)|$. We have to prove that α is an exact 1-form.

Since L and X' are connected we get an exact sequence $0 \rightarrow H^1(G(t, p, \zeta)) \rightarrow H^1(\widehat{K}) \oplus H^1(\widehat{Z})$. It suffices therefore to prove $[\alpha]$ maps to 0 under $H^1(G(t, p, \zeta)) \rightarrow H^1(\widehat{K}) \oplus H^1(\widehat{Z})$.

The map $H^1(G(t, p, \zeta)) \rightarrow H^1(\widehat{Z})$ is given by $\phi_X^{t,p,\zeta}$ and we have $\phi_X^{t,p,\zeta*} \alpha = \alpha_X$, which is an exact 1-form.

Let $\phi_L^{t,p,\zeta}$ be as in Lemma 9, and let $\alpha_L = \phi_L^{t,p,\zeta*} \alpha$. Then we have to only prove $[\alpha|_L] = 0 \in H^1(\widehat{K})$. Let u_L be as in Lemma 8. Then we have

$$t^2(f_L^*(\alpha|_L + du_L) + \eta_L) = \beta_C.$$

on $f_L^{-1}(\widehat{K}) \cap A_{tR,\delta}$. We have also $t^2[\eta_L] = [\beta_C]$ and so $f_L^*[\alpha_L] = 0 \in H^1(\Sigma; \mathbb{R})$. As in the proof of Lemma 5 we can construct a diffeomorphism of L onto $S^1 \times \mathbb{R}^2$ and we may suppose that $\widehat{K}$ is diffeomorphic to $S^1 \times \mathbb{R}^2$. Hence we get $H_c^1(\widehat{K}; \mathbb{R}) = 0$. Notice also that we have an exact sequence $H_c^1(\widehat{K}; \mathbb{R}) \rightarrow H^1(\widehat{K}; \mathbb{R}) \rightarrow H^1(\Sigma; \mathbb{R})$. Then we see that $f_L^* : H^1(\widehat{K}; \mathbb{R}) \rightarrow H^1(\Sigma; \mathbb{R})$ is injective. On the other hand we have $f_L^*[\alpha|_L] = 0$ and so $[\alpha|_L] = 0 \in H^1(\widehat{K}; \mathbb{R})$ as we want. $\square$

We shall now complete the proof of Theorem 1. Let $\mathcal{U}$ be as in Corollary 7, and let $V \in \mathcal{U} \setminus \mathcal{X}$. Then we have only to prove $\text{Spt } \|V\| = G(t, \zeta)$ for some $(t, \zeta) \in (0, \tau) \times \mathcal{Z}$. Let (t, p, ζ) be as in Corollary 7. Then we can find a function $w : G(t, \zeta) \rightarrow \mathbb{R}$ with $\Phi_{t,p,\zeta}(dw) = \text{Spt } \|V\|$. As $\text{Spt } \|V\|$ is special Lagrangian w satisfies an elliptic equation to which we can apply a maximum principle of Hopf [11]. Hence we see that w is constant, and so $\|V\| = |G(t, \zeta)|$.

References

- [1] W. K. Allard, *On the first variation of a varifold*, Ann. of Math. (2) **95** (1972), 417–491.
- [2] W. K. Allard and F. J. Almgren, *On the radial behavior of minimal surfaces and the uniqueness of their tangent cones*, Ann. of Math. (2) **113** (1981), 215–265.

- [3] M. F. Atiyah, N. J. Hitchin, and I. M. Singer, *Self-duality in four-dimensional Riemannian geometry*, Proc. Roy. Soc. London Ser. A **362** (1978), 425–461.
- [4] D. Adams and L. Simon, *Rates of asymptotic convergence near isolated singularities of geometric extrema*, Indiana Univ. Math. J. **37** (1988), 225–254.
- [5] R. Bott, *Nondegenerate critical manifolds*, Ann. of Math, (2) **60** (1954), 248–261.
- [6] S. K. Donaldson, *An application of gauge theory to four-dimensional topology*, J. Differential Geom. **18** (1983), 279–315.
- [7] H. Federer, *Geometric measure theory*, Springer, 1969.
- [8] D. S. Freed and K. K. Uhlenbeck, *Instantons and four-manifolds*, Springer, 1984.
- [9] R. Harvey and H. B. Lawson, *Calibrated geometries*, Acta Math. **148** (1982), 47–157.
- [10] M. Haskins, *The geometric complexity of special Lagrangian T^2 -cones*, Invent. Math. **157** (2004), 11–70.
- [11] E. Hopf, *Elementare Bemerkungen über die Lösungen partieller Differentialgleichungen zweiter Ordnung vom elliptischen Typus*, Sitz. Ber. Preuss. Akad. Wiss. **19** (1927), 147–152.
- [12] Y. Imagi, *A uniqueness theorem for gluing special Lagrangian submanifolds in Calabi–Yau manifolds*, preprint, arXiv:11052628.
- [13] D. D. Joyce, *Special Lagrangian submanifolds with isolated conical singularities I. Regularity*, Ann. Global Anal. Geom. **25** (2003), 201–258.
- [14] D. D. Joyce, *Special Lagrangian submanifolds with isolated conical singularities II. Moduli spaces*, Ann. Global Anal. Geom. **25** (2003), 301–352.
- [15] D. D. Joyce, *Special Lagrangian submanifolds with isolated conical singularities III. Desingularization, the unobstructed case*, Annal. Global Anal. Geom. **26** (2004), 1–58.
- [16] D. D. Joyce, *Special Lagrangian submanifolds with isolated conical singularities IV. Desingularization, obstructions and families*, Annal. Global Anal. Geom. **26** (2004), 117–174.
- [17] D. D. Joyce, *Special Lagrangian submanifolds with isolated conical singularities V. Survey and applications*, J. Differential Geom. **63** (2003), 279–347.
- [18] S. Łojasiewicz, *Ensembles semi-analytiques*, IHES, 1965.

- [19] R. C. Mclean, *Deformations of calibrated submanifolds*, Comm. Anal. Geom. **6** (1998), 705–747.
- [20] J. H. Michael and L. Simon, *Sobolev and mean-value inequalities on generalized submanifolds of $\mathbb{R}^n$* , Comm. Pure Appl. Math. **26** (1973), 361–379.
- [21] Y.-G. Oh, *Second variation and stabilities of minimal Lagrangian submanifolds in Kähler manifolds*, Invent. Math. **101** (1990), 501–519.
- [22] L. Simon, *Asymptotics for a class of nonlinear evolution equations, with applications to geometric problems*, Ann. of Math. (2) **118** (1983), 525–571.
- [23] L. Simon, *Isolated singularities of extrema of geometric variational problems* in *Harmonic mappings and minimal immersions*, Springer, 1985.
- [24] L. Simon, *Lectures on geometric measure theory*, Proceedings of the Centre for Mathematical Analysis, Australian National University, 1983.
- [25] C. H. Taubes, *Self-dual Yang-Mills connections on non-self-dual 4-manifolds*, J. Differential Geom. **17** (1982), 139–170.
- [26] K. K. Uhlenbeck, *Removable singularities in Yang-Mills fields*, Commun. Math. Phys. **83** (1982), 11–29.
- [27] A. Weinstein, *Symplectic manifolds and their Lagrangian submanifolds*, Adv. in Math. **6** (1971), 329–346.