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A note on a theorem of Fukasawa-Gel'fond

by

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§ 1. Introduction

In 1915, G. Pólya [5] showed that an entire function f satisfying $f(\mathbb{N}_0) \subset \mathbb{Z}$ and $\overline{\lim}_{r \to +\infty} \frac{\log |f|_r}{r} < \log 2$, where $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $|f|_r := \max_{|z| \leqq r} |f(z)|$, is a polynomial. Because of the existance of the entire function 2^z, the value $\log 2$ in the above result is best possible. Let $\ell \in \mathbb{N}_0$ and let $f^{(k)}(z)$ for $k \in \mathbb{N}_0$ denote k-th derivative of $f(z)$. Then A. Gel'fond [2], in 1929, proved that an entire function f which satisfies $\overline{\lim}_{r \to +\infty} \frac{\log|f|_r}{r} < (\ell+1)\log\left\{1 + e^{-\ell/(\ell+1)}\right\}$ and $f^{(k)}(\mathbb{N}_0) \subset \mathbb{Z}$ for all $k=0,1,\ldots,\ell$ is a polynomial. A. Selberg [6] showed that the above upper boound can be replaced by $(\ell+1)\log\omega_\ell$ with some $\omega_\ell > 1 + e^{-\ell/(\ell+1)}$ when $\ell \geqq 1$.

In another direction, S. Fukasawa [1], in 1926, studied entire functions satisfying $f(\mathbb{Z}[i]) \subset \mathbb{Z}[i]$, and in 1929, A. Gel'fond [3] refined the result of Fukasawa and obtained: There exists a real number $\alpha > 0$ such that if f is an entire function satisfying $\overline{\lim}_{r \to +\infty} \frac{\log |f|_r}{r^2} < \alpha$ and $f(\mathbb{Z}[i]) \subset \mathbb{Z}[i]$, then f is a polynomial.

Several authors have tried to determine the exact value of α, and finally in 1981, F. Gramain [4], proved a more general theorem to show that the best possible value of α is equal to $\pi/2e$:

Theorem. (F. Gramain) Let $\mathbb{K}$ be any imaginary quadratic number field whose discriminant is $-\Delta$ and let $a := \sqrt{\Delta}/2$ be the area of the fundamental parallelogram of the lattice of integers $\mathcal{O}_{\mathbb{K}}$ in $\mathbb{K}$.

(i) If f is an entire function satisfying

$$f(\mathcal{O}_{\mathbb{K}}) \subset \mathcal{O}_{\mathbb{K}} \tag{1}$$

and

$$\varlimsup_{r\to+\infty} \frac{\log |f|_r}{r^2} < \frac{\pi}{2ea}, \tag{2}$$

then f is a polynomial.

(ii) There exists an entire function f such that $f(\mathcal{O}_{\mathbb{K}}) \subset \mathcal{O}_{\mathbb{K}}$ and

$$\varlimsup_{r\to+\infty} \frac{\log |f|_r}{r^2} = \frac{\pi}{2ea}.$$

In particular, f is not a polynomial.

In this note, we shall prove the following generalization of part (ii) of Gramain's theorem:

Theorem. Let $\mathbb{K}$ and $\mathcal{O}_{\mathbb{K}}$ be as above, then there exists an entire function f such that

$$\frac{1}{k!} f^{(k)}(\mathcal{O}_{\mathbb{K}}) \subset \mathcal{O}_{\mathbb{K}} \quad \text{for all } k = 0, 1, \dots, \ell, \tag{3}$$

and

$$\varlimsup_{r\to+\infty} \frac{\log |f|_r}{r^2} = \frac{(\ell+1)\pi}{2ea}.$$

It follows from our theorem that when the condition (1) in Gramain's theorem is replaced by (3), the upper bound which corresponds to the right-hand side of (2) does not exceed $(\ell+1)\pi/2ea$.

§2. Lemmas

In this section we prepare some notions and lemmas.

Let $\Lambda = \{\zeta_m\}_{m\in\mathbb{N}_0}$ be any homogeneous lattice in $\mathbb{R}^2 = \mathbb{C}$, whoose elements are arranged in the following way: $m<n$ $(m, n\in\mathbb{N}_0)$ if and only if we have either $|\zeta_m| < |\zeta_n|$ or $|\zeta_m| = |\zeta_n|$ with $\arg \zeta_m < \arg \zeta_n$.

Lemma 1. ([4] lemma 2) Let a be the area of the fundamental parallelogram of Λ, then we have for any $n\in\mathbb{N}_0$,

$$\left| |\zeta_n| - \sqrt{\frac{an}{\pi}} \right| \leqq c_1.$$

Here and in the sequel $c_1, c_2, \dots$ denote effectively computable positive constants depending only on Λ.

Lemma 2. ([4] lemma 3) Let $n\in\mathbb{N}$ with $n\geqq 2$ and let $z\in\mathbb{C}$, if we define $\theta\geqq 0$ by $|z| = \theta|\zeta_n|$, then

$$\left| \log \prod_{\substack{j=0 \\ |z-\zeta_j|\geqq 1}}^{n} |z-\zeta_j| - \frac{1}{2} n \log n - n w(\theta) \right| \leqq c_2 \max(1, \theta) \sqrt{n} \log n,$$

where

$$w(\theta) := \begin{cases} \log\theta - \frac{1}{2}\log\frac{\pi}{a} & \text{if } \theta\geqq 1, \\ \frac{\theta^2}{2} - \frac{1}{2} - \frac{1}{2}\log\frac{\pi}{a} & \text{if } \theta\leqq 1. \end{cases}$$

In what follows we assume that k is always an integer with $0\leqq k\leqq \ell$

The following lemma 3 is a generalization of lemma 7 in [4] .

Lemma 3. Let $\Lambda = \{\zeta_m\}_{m\in\mathbb{N}_0}$ and a be as in lemma 1 and let f be an entire function. Define for $n\in\mathbb{N}_0$,

$$P_{n,k}(z) := \prod_{m=0}^{n-1} (z-\zeta_m)^{\ell+1} (z-\zeta_n)^k$$

with the convention that $P_{0,k}(z) := z^k$, and let

$$a_{n,k} := \frac{1}{2\pi i}\int_{C_n} \frac{f(\zeta)}{P_{n,k+1}(\zeta)}\, d\zeta, \tag{4}$$

where C_n is a closed curve containing the points $\zeta_0, \zeta_1, \dots, \zeta_n$ in its interior. Then the following formula holds for all $z\in\mathbb{C}$ contained in the interior of C_N:

$$f(z) = \sum_{n=0}^{N}\sum_{k=0}^{\ell} a_{n,k}P_{n,k}(z) + \frac{P_{N+1,0}(z)}{2\pi i}\int_{C_N} \frac{f(\zeta)}{P_{N+1,0}(\zeta)(\zeta-z)}\, d\zeta. \tag{5}$$

(i)If f satisfies

$$\tau := \varlimsup_{r\to+\infty} \frac{\log|f|_r}{r^2} < \frac{(\ell+1)\pi}{2a}, \tag{6}$$

then the series

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\ell} a_{n,k}P_{n,k}(z) \tag{7}$$

converges uniformly to f on any compact set in $\mathbb{C}$, and the coefficients $a_{n,k}$ satisfy

$$\varlimsup_{n\to+\infty} \frac{\log|a_{n,k}| + \frac{\ell+1}{2} n\log n}{n} \leqq \frac{\ell+1}{2}\left\{ 1+\log\left(\frac{\tau}{\ell+1}\right) \right\}. \tag{8}$$

(ii) If $\{b_{n,k};n\in\mathbb{N}_0,\ 0\leqq k\leqq \ell\}$ is a sequence of complex numbers satisfying

$$\overline{\lim_{n\to+\infty}}\ \frac{\log|b_{n,k}|+\frac{\ell+1}{2}n\log n}{n}=:\lambda<\frac{\ell+1}{2}\left(1+\log\frac{\pi}{a}\right),\tag{9}$$

then the series

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\ell} b_{n,k}P_{n,k}(z)\tag{10}$$

converges uniformly on any compact set in $\mathbb{C}$ and defines an entire function g satisfying

$$\overline{\lim_{r\to+\infty}}\ \frac{\log|g|_r}{r^2}\leqq\frac{\ell+1}{2}\exp\left(\frac{2\lambda}{\ell+1}-1\right).\tag{11}$$

Remark. From the conclusions of both parts of lemma 3, the inequalities (8) and (11) can be replaced by equalities.

Proof. We first prove part (ii) of lemma 3. Let $\rho_n:=|\zeta_n|$ and fix λ' such that $\lambda<\lambda'<\frac{\ell+1}{2}(1+\log\frac{\pi}{a})$, and choose a sufficiently small $\theta\in]0,1[$ satisfying

$$\lambda'+\frac{\ell+1}{2}\left(\theta^2-1-\log\frac{\pi}{a}\right)<0.\tag{12}$$

By the assumption (9), there exists an integer n_1 such that $\log|b_{n,k}|+\frac{\ell+1}{2}n\log n\leqq\lambda' n$ for all $n\geqq n_1$. For any $R\geqq 0$, there exists some integer $n_0\geqq n_1$ such that $\theta\rho_{n_0}\geqq R$. Hence it follows from lemma 2 that

$$\log|b_{n,k}P_{n,k}(z)|\leqq\left\{\lambda'+\frac{\ell+1}{2}\left(\theta^2-1-\log\frac{\pi}{a}\right)\right\}n+o(n)\leqq -C_4 n$$

for all $n>n_0$ and all $z\in\mathbb{C}$ with $|z|\leqq R$. Therefore the series (10) converges uniformly on any compact set in $\mathbb{C}$ and defines an entire function g of which we consider the rate of growth.

Let $z\in\mathbb{C}$ satisfy $|z|=:r>\rho_{n_0}$, then, using lemmas 1 and 2, we get for all n with $\rho_n\geqq r$

$$\log|b_{n,k}P_{n,k}(z)|\leqq\left\{\lambda'+\frac{\ell+1}{2}\left(\frac{r^2}{\rho_n^2}-1-\log\frac{\pi}{a}\right)\right\}n+O(\sqrt{n}\log n),\tag{13}$$

and also for all $n\geqq n_1$ with $\rho_n\leqq r$

$$\log|b_{n,k}P_{n,k}(z)|\leqq\left\{\lambda'+(\ell+1)\left(\log\frac{r}{\rho_n}-\frac{1}{2}\log\frac{\pi}{a}\right)\right\}n+O(r\log r)\tag{14}$$

If we define s_0, s_1 and s_2 by

$$s_0 := \Big| \sum_{0 \leqq n \leqq n_1} \sum_{0 \leqq k \leqq \ell} b_{n,k} P_{n,k}(z) \Big|_r ,$$

$$s_1 := \Big| \sum_{\substack{n \geqq n_1 \\ \rho_n \leqq r}} \sum_{0 \leqq k \leqq \ell} b_{n,k} P_{n,k}(z) \Big|_r ,$$

$$s_2 := \Big| \sum_{\rho_n > r} \sum_{0 \leqq k \leqq \ell} b_{n,k} P_{n,k}(z) \Big|_r ,$$

then we have

$$|g|_r \leqq s_0 + s_1 + s_2 . \tag{15}$$

Making use of lemmas 1 ,2 and (13), we get

$$s_2 \leqq e^{(\ell+1)\pi r^2/2a} \sum_{\rho_n > r} \exp\Big[\{\lambda' - \frac{\ell+1}{2}(1+\log\frac{\pi}{a})\} n + O(\sqrt{n}\log n)\Big] ,$$

and thus, by (12) and the fact that $\frac{\pi}{a}(\lambda' - \frac{\ell+1}{2}\log\frac{\pi}{a}) \leqq \frac{\ell+1}{2}\exp(\frac{2\lambda'}{\ell+1} - 1)$,

$$\log s_2 \leqq \frac{\ell+1}{2} r^2 \exp(\frac{2\lambda'}{\ell+1} - 1) + O(r\log r). \tag{16}$$

Since we have, by (14),

$$s_1 \leqq \sum_{\rho_n \leqq r} \exp\Big[n\{\lambda' + (\ell+1)(\log r - \frac{1}{2}\log n)\} + O(r\log r)\Big] ,$$

we obtain from lemma 1 and the fact that $\max_{x>0} x\Big\{\lambda' + (\ell+1)(\log r - \frac{1}{2}\log x)\Big\}$ $= \frac{\ell+1}{2} r^2 \exp(\frac{2\lambda'}{\ell+1} - 1)$,

$$\log s_1 \leqq \frac{\ell+1}{2} r^2 \exp(\frac{\lambda'}{\ell+1} - 1) + o(r^2). \tag{17}$$

Therefore, from (15), combining the estimates (16), (17) and $\log s_0 \leqq c_3 \log r$ and then letting $\lambda' \to \lambda$, we get the conclusion (11) of part (ii) of lemma 3.

We next prove part (i) of lemma 3. Let τ' with $\tau < \tau' < (\ell+1)\pi/2a$ be fixed and let C_n in (4) be the circle with center 0 and radius $\theta\rho_n$ where $\theta > 1$ is a parameter which will be chosen later. Then, by (4), (6) and lemma 2, we get

$$\log |a_{n,k}| \leqq \tau'(\theta\rho_n)^2 + \log\theta\rho_n - (\ell+1)\Big\{ \frac{1}{2} n\log n + n(\log\theta - \frac{1}{2}\log\frac{\pi}{a})\Big\}$$

for all sufficiently large $n \in \mathbb{N}$, and hence, using lemma 1, we obtain

$$\frac{1}{n}\Big\{ \log|a_{n,k}| + \frac{\ell+1}{2} n\log n\Big\} \leqq \frac{\tau'\theta^2 a}{\pi} - (\ell+1)(\log\theta - \frac{1}{2}\log\frac{\pi}{a}) + o(1).$$

Since the right-hand side of the above inequality attains its minimal value when $\theta^2 = (\ell+1)\pi / 2a\tau'$, we obtain

$$\overline{\lim_{n\to+\infty}} \frac{1}{n} \left\{ \log |a_{n,k}| + \frac{\ell+1}{2} n \log n \right\} \leqq \frac{\ell+1}{2} \left\{ 1 + \log\left(\frac{2\tau'}{\ell+1} \right) \right\},$$

which yields (8).

Thus, by the assumption $\tau < (\ell+1)\pi/2a$ as well as the inequality (8), it follows from part (ii) of lemma 3 that the series (7) converges on any compact set and defines an entire function g(z) satisfying $\overline{\lim_{r\to+\infty}} \frac{\log |g|_r}{r^2} \leqq \tau$ and further, from the formula (5), $f^{(k)}(\zeta) = g^{(k)}(\zeta)$ holds for all $k = 0, 1, \ldots, \ell$ and all $\zeta \in \Lambda$. Hence, to complete the proof of part (i) of lemma 3, it is sufficient to show that an entire function ϕ satisfying

$$\overline{\lim_{r\to+\infty}} \frac{\log |\phi|_r}{r^2} =: \tau < \frac{(\ell+1)\pi}{2a} \tag{18}$$

and $\phi^{(k)}(\Lambda) = \{0\}$ for all $k = 0, 1, \ldots, \ell$, is identically zero. Assume that ϕ is not zero, then the Taylor expansion of ϕ at z=0 has the form $\alpha_h z^h + \alpha_{h+1} z^{h+1} + \ldots$ with some $\alpha_h \neq 0$ $(h \geqq 0)$ and, by Jensen's formula, we have

$$\log |\phi|_r \geqq \log |\alpha_h| + h \log r + (\ell+1) \cdot \sum_{0<|\zeta_j|\leqq r} \log \frac{r}{|\zeta_j|}. \tag{19}$$

Applying lemmas 2 and 3 to the right-hand side of (19), we obtain $\tau \geqq (\ell+1)\pi/2a$, which contradicts the assumption (18).

§ 3. Proof of Theorem

We can deduce the conclusion of our theorem by taking $\Lambda = D = \mathcal{O}_{\mathbb{K}}$ in the following lemma 4:

Lemma 4. Let the lattice $\Lambda \subset \mathbb{R}^2 = \mathbb{C}$ be as in lemma 3 and let D be a subset of $\mathbb{C}$ which has the following property: there exists a constant $\delta > 0$ such that for any $z \in \mathbb{C}$, we can find some $d \in D$ satisfying $|z - d| \leqq \delta$. Then there exists an entire function satisfying $\frac{1}{k!} f^{(k)}(\Lambda) \subset D$ for all $k = 0, 1, \ldots, \ell$ and $\overline{\lim_{r\to+\infty}} \frac{\log |f|_r}{r^2} = \frac{(\ell+1)\pi}{2ea}$.

Proof. We use the same notation as in lemma 3. Since the coefficients of a generalized interpolation series of f at the points of Λ are given by (4), we obtain from the residue theorem

$$a_{n,k} = \sum_{m=0}^{n-1} \sum_{h=0}^{\ell} \frac{f^{(h)}(\zeta_n)}{h!(\ell-h)!} \left[\left(\frac{d}{d\zeta}\right)^{\ell-h} \frac{(\zeta-\zeta_m)^{\ell+1}}{P_{n,k+1}(\zeta)} \right]_{\zeta=\zeta_m} +$$

$$+ \sum_{h=0}^{k} \frac{f^{(h)}(\zeta_n)}{h!(k-h)!} \left[\left(\frac{d}{d\zeta}\right)^{k-h} \frac{1}{P_{n,0}(\zeta)} \right]_{\zeta=\zeta_n} .$$

Thus we have

$$P_{n,0}(\zeta_n)\, a_{n,k} = \sum_{m=0}^{n-1} \sum_{h=0}^{\ell} p_{h,k,m,n} \frac{f^{(h)}(\zeta_m)}{h!} + \sum_{h=0}^{k-1} q_{h,k,n} \frac{f^{(h)}(\zeta_n)}{h!} +$$

$$+ \frac{f^{(k)}(\zeta_n)}{k!} .$$

with some $p_{h,k,m,n}$ and $q_{h,k,n} \in \mathbb{C}$. Therefore, if we define $a_{n,k}$ by choosing $\frac{1}{k!} f^{(k)}(\zeta_n) \in D$ such that

$$| P_{n,0}(\zeta_n) a_{n,k} - 2\delta | \leqq \delta, \tag{20}$$

then we get from lemmas 1 and 2

$$\log | a_{n,k} | = - \frac{\ell+1}{2}(n \log n - n \log \frac{\pi}{a}) + o(n),$$

and thus, from part (ii) of lemma 3, the series (10) converges and defines an entire function f which satisfies the required conditions; and the proof of lemma 4 is completed.

Remark. As in the remark of lemma 8 in [4], we can construct infinitely (even uncountably) many functions f, by choosing $\frac{1}{k!} f^{(k)}(\zeta_n) \in D$ such that, for example $| a_{n,k} P_{n,k}(\zeta_n) - 4\delta | \leqq 3\delta$ instead of (20).

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